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# Chapter 10

## Advance of Mercury's Perihelion

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- 13 • *What does "advance of the perihelion" mean?*
- 14 • *You say Newton does not predict any advance of Mercury's perihelion in*  
15 *the absence of other planets. Why not?*
- 16 • *The advance of Mercury's perihelion is tiny. So why should we care?*
- 17 • *Why pick out Mercury? Doesn't the perihelion of every planet change*  
18 *with Earth-time?*
- 19 • *You are always shouting at me to say whose time measures various*  
20 *motions. Why are you so sloppy about time in analyzing Mercury's orbit?*

## CHAPTER

## 10

## Advance of Mercury's Perihelion

Edmund Bertschinger &amp; Edwin F. Taylor \*

23 *This discovery was, I believe, by far the strongest emotional*  
 24 *experience in Einstein's scientific life, perhaps in all his life.*  
 25 *Nature had spoken to him. He had to be right. "For a few*  
 26 *days, I was beside myself with joyous excitement."* Later, he  
 27 *told Fokker that his discovery had given him palpitations of*  
 28 *the heart. What he told de Haas is even more profoundly*  
 29 *significant: when he saw that his calculations agreed with the*  
 30 *unexplained astronomical observations, he had the feeling that*  
 31 *something actually snapped in him.*

—Abraham Pais

## 10.1 ■ JOYOUS EXCITEMENT

34 *Tiny effect; large significance.*"Perihelion  
precession"?

35 What discovery sent Einstein into "joyous excitement" in November 1915? It  
 36 was his calculation showing that his brand new (not quite completed) theory  
 37 of general relativity gave the correct value for one detail of the orbit of the  
 38 planet Mercury that had not been previously explained, an effect with the  
 39 technical name **precession of Mercury's perihelion**.

Newton:  
Sun-Mercury  
perihelion fixed.

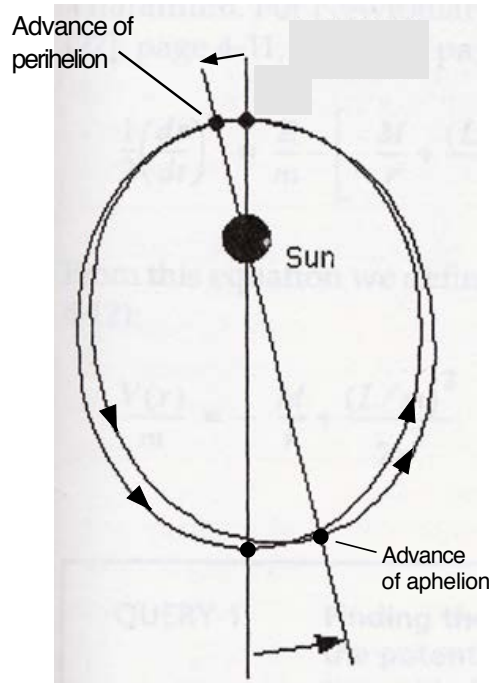
40 Mercury (and every other planet) circulates around the Sun in a  
 41 not-quite-circular orbit. In this orbit it oscillates in and out radially while it  
 42 circles tangentially. A full Newtonian analysis predicts an elliptical orbit.  
 43 Newton tells us that if we consider only the interaction between Mercury and  
 44 the Sun, then the time for one 360-degree trip around the Sun is *exactly* the  
 45 same as the time for one in-and-out radial oscillation. Therefore the orbital  
 46 point closest to the Sun, the so-called **perihelion**, stays in the same place; the  
 47 elliptical orbit does not shift around with each revolution—according to  
 48 Newton. You will begin by verifying his nonrelativistic prediction for the  
 49 simple Sun-Mercury system.

50 However, observation shows that Mercury's orbit does indeed change. The  
 51 perihelion moves forward in the direction of rotation of Mercury; it *advances*

\*Draft of Second Edition of *Exploring Black Holes: Introduction to General Relativity*  
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Advance of Mercury's Perihelion



**FIGURE 1** Exaggerated view of the advance, during one century, of Mercury’s perihelion (and aphelion). The figure shows two elliptical orbits. One of these orbits is the one that Mercury traces over and over again in the year, say, 1900. The other is the elliptical orbit that Mercury traces over and over again in the year, say, 2000. The two are shifted with respect to one another, a rotation called *the advance (or precession) of Mercury’s perihelion*. The unaccounted-for precession in one Earth-century is about 43 arcseconds, less than the thickness of a line in this figure.

Observation:  
perihelion advances.

52 with each orbit (Figure 1). The long (“major”) axis of the ellipse rotates. We  
53 call this rotation of the axis the **advance (or precession) of the**  
54 **perihelion**.

55 The **aphelion** is the point of the orbit farthest from the Sun; it advances  
56 at the same angular rate as the perihelion (Figure 1).

Newton: Influence  
of other planets,  
predicts most of the  
perihelion advance . . .

57 Observation shows that the perihelion of Mercury precesses at the rate of  
58 574 arcseconds (0.159 degree) *per Earth-century*. (One degree equals 3600  
59 arcseconds.) Newton’s mechanics accounts for 531 seconds of arc of this  
60 advance by computing the perturbing influence of the other planets. But a  
61 stubborn 43 arcseconds (0.0119 degree) per Earth-century, called a **residual**,  
62 remains after all these effects are accounted for. This residual (though not its  
63 modern value) was computed from observations by Urbain Le Verrier as early  
64 as 1859 and more accurately later by Simon Newcomb (Box 1). Le Verrier  
65 attributed the residual in Mercury’s orbit to the presence of an unknown inner  
66 planet, tentatively named Vulcan. We know now that there is no planet  
67 Vulcan. (Sorry, Mr. Spock!)

. . . but leaves  
a *residual*.

### Box 1. Simon Newcomb



**FIGURE 2** Simon Newcomb  
Born 12 March 1835, Wallace, Nova Scotia.  
Died 11 July 1909, Washington, D.C.  
(Photo courtesy of Yerkes Observatory)

From 1901 until 1959 and even later, the tables of locations of the planets (so-called **ephemerides**) used by most

astronomers were those compiled by Simon Newcomb and his collaborator George W. Hill.

By the age of five Newcomb was spending several hours a day making calculations, and before the age of seven was extracting cube roots by hand. He had little formal education but avidly explored many technical fields in the libraries of Washington, D. C. He discovered the *American Ephemeris and Nautical Almanac*, of which he said, "Its preparation seemed to me to embody the highest intellectual power to which man had ever attained."

Newcomb became a "computer" (a person who computes) in the American Nautical Almanac office and by stages rose to become its head. He spent the greater part of the rest of his life calculating the motions of bodies in the solar system from the best existing data. Newcomb collaborated with Q. M. W. Downing to inaugurate a worldwide system of astronomical constants, which was adopted by many countries in 1896 and officially by all countries in 1950.

The advance of the perihelion of Mercury computed by Einstein in 1914 would have been compared to entries in the tables of Simon Newcomb and his collaborator.

Einstein correctly predicts residual precession.

Method: Compare in-and-out time with round-and-round time for Mercury.

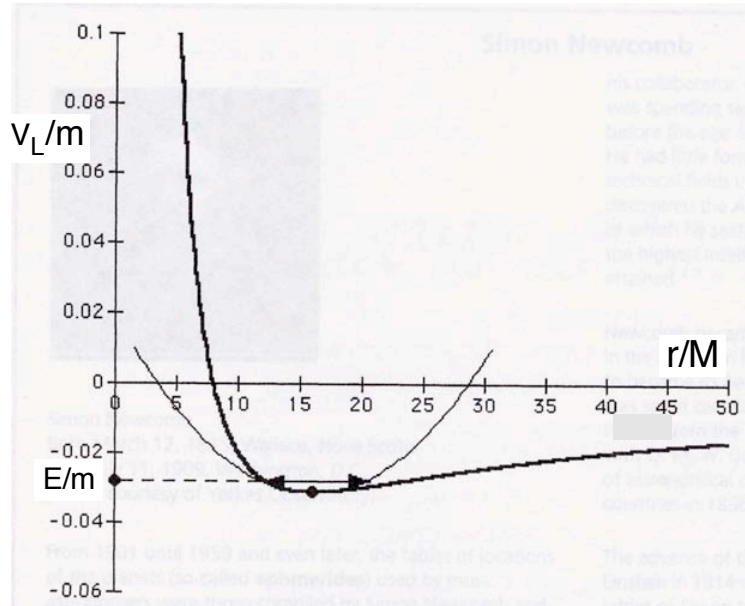
68 Newton's mechanics says that there should be *no residual* advance of the  
69 perihelion of Mercury's orbit and so cannot account for the 43 seconds of arc  
70 per Earth-century which, though tiny, is nevertheless too large to be ignored  
71 or blamed on observational error. But Einstein's general relativity accounted  
72 for the extra 43 arcseconds on the button. Result: joyous excitement!

73 **Preview, Newton:** This chapter begins with Newton's approximations  
74 that lead to his no-precession conclusion (in the absence of other planets).  
75 Mercury moves in a near-circular orbit; Newton calculates the time for one  
76 orbit. The approximation also describes the small radial in-and-out motion of  
77 Mercury as if it were a harmonic oscillator moving back and forth about a  
78 potential energy minimum (Figure 3). Newton calculates the time for one  
79 in-and-out radial oscillation and compares it with the time for one orbit. The  
80 orbital and radial oscillation  $T$ -values are exactly equal (according to Newton),  
81 provided one considers only the Mercury-Sun interaction. He concludes that  
82 Mercury circulates around once in the same time that it oscillates radially  
83 inward and back out again. The result is an elliptical orbit that closes on itself.  
84 In the absence of other planets, Mercury repeats this exact elliptical path  
85 forever—according to Newton.

86 **Preview, Einstein:** In contrast, our general relativity approximation  
87 shows that these two times—the orbital round-and-round and the radial  
88 in-and-out  $T$ -values—are *not quite equal*. The radial oscillation takes place  
89 more slowly, so that by the time Mercury returns to its inner limit, the

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**FIGURE 3** Newton's effective potential, equation (5) (heavy curve), on which we superimpose the parabolic potential of the simple harmonic oscillator (thin curve) with the shape given by equation (3). Near the minimum of the effective potential, the two curves closely conform to one another.

90 circular motion has carried it farther around the Sun than it was at the  
 91 preceding minimum  $r$ -coordinate. From this difference Einstein reckons the  
 92 residual angular rate of advance of Mercury's perihelion around the Sun and  
 93 shows that this predicted difference is close to the observed residual advance.  
 94 Now for the details.

**Comment 1. Relaxed about Newton's time and coordinate  $T$**

95 In this chapter we speak freely about Newton's time or Einstein's change in  
 96 global  $T$ -value, without worrying about which we are talking about. We get away  
 97 with this sloppiness for two reasons: (1) All observations are made from Earth's  
 98 surface. Every statement about time should in principle be followed by the  
 99 phrase, "as observed on Earth." (2) For this system, the effects of spacetime  
 100 curvature on the rates of local clocks are so small that all time or  $T$ -measures  
 101 give essentially the same rate of precession, as summarized in Section 10.11.  
 102

**10.2.3 ■ NEWTON'S SIMPLE HARMONIC OSCILLATOR**

104 *Assume radial oscillation is sinusoidal.*

105 Why does the planet oscillate in and out radially? Look at the effective  
 106 potential in Newton's analysis of motion, the heavy line in Figure 3. This  
 107 heavy line has a minimum, the location at which the planet can ride around at  
 108 constant  $r$ -value, tracing out a circular orbit. But with a slightly higher  
 109 energy, it not only moves tangentially, it also oscillates radially in and out, as  
 110 shown by the two-headed arrow in Figure 3.

111 How long does it take for one in-and-out oscillation? That depends on the  
 112 shape of the effective potential curve near the minimum shown in Figure 3.  
 113 But if the amplitude of the oscillation is small, then the effective part of the  
 114 curve is very close to this minimum, and we can use a well-known  
 115 mathematical theorem: If a continuous, smooth curve has a local minimum,  
 116 then near that minimum a parabola approximates this curve. Figure 3 shows  
 117 such a parabola (thin curve) superimposed on the (heavy) effective potential  
 118 curve. From the diagram it is apparent that the parabola is a good  
 119 approximation of the potential, at least near that local minimum.

In-and-out motion  
 in parabolic potential . . .  
 . . . predicts simple  
 harmonic motion.

120 From introductory Newtonian mechanics, we know how a particle moves  
 121 in a parabolic potential. The motion is called **simple harmonic oscillation**,  
 122 described by the following expression:

$$x = A \sin \omega t \tag{1}$$

123 Here  $A$  is the amplitude of the oscillation and  $\omega$  (Greek lower case omega) tells  
 124 us how rapidly the oscillation occurs in radians per unit time. The potential  
 125 energy per unit mass,  $V/m$ , of a particle oscillating in a parabolic potential  
 126 follows the formula

$$\frac{V}{m} = \frac{1}{2} \omega^2 x^2 \tag{2}$$

127 To find the rate of oscillation  $\omega$  of the harmonic oscillator, take the second  
 128 derivative with respect to  $x$  of both sides of (2).

$$\frac{d^2 (V/m)}{dx^2} = \omega^2 \tag{3}$$

**10.3 ■ NEWTON'S ORBIT ANALYSIS**

130 *Round and round vs. in and out*

131 The in-and-out radial oscillation of Mercury does not take place around  $r = 0$   
 132 but around the  $r$ -value of the effective potential minimum. What is the  
 133  $r$ -coordinate of this minimum (call it  $r_0$ )? Start with Newton's equation (23)  
 134 in Section 8.4:

Newton's  
 equilibrium  $r_0$

$$\frac{1}{2} \left( \frac{dr}{dt} \right)^2 = \frac{E}{m} - \left( -\frac{M}{r} + \frac{L^2}{2m^2 r^2} \right) = \frac{E}{m} - \frac{V_L(r)}{m} \quad (\text{Newton}) \tag{4}$$

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135 This equation defines the effective potential,

$$\frac{V_L(r)}{m} \equiv -\frac{M}{r} + \frac{L^2}{2m^2r^2} \quad (\text{Newton}) \quad (5)$$

136 To locate the minimum of this effective potential, set its derivative equal to  
137 zero:

$$\frac{d(V_L/m)}{dr} = \frac{M}{r^2} - \frac{L^2}{m^2r^3} = 0 \quad (\text{Newton}) \quad (6)$$

138 Solve the right-hand equation to find  $r_0$ , the  $r$ -value of the minimum:

$$r_0 = \frac{L^2}{Mm^2} \quad (\text{Newton, equilibrium radius}) \quad (7)$$

Newton: In-and-out  
time equals round-  
and-round time.

139 We want to compare the rate  $\omega_r$  of in-and-out radial motion of Mercury with  
140 its rate  $\omega_\phi$  of round-and-round tangential motion. Use Newton's definition of  
141 angular momentum, with increment  $dt$  of Newton's universal time, similar to  
142 equation (10) of Section 8.2:

$$\frac{L}{m} \equiv r^2 \frac{d\phi}{dt} = r^2 \omega_\phi \quad (\text{Newton}) \quad (8)$$

143 where  $\omega_\phi \equiv d\phi/dt$ . Equation (8) gives us the angular velocity of Mercury along  
144 its almost-circular orbit.

145 Queries 1 and 2 show that for Newton the radial in-and-out angular  
146 velocity  $\omega_r$  is equal to the orbital angular velocity  $\omega_\phi$ .

---

**QUERY 1. Newton's angular velocity  $\omega_\phi$  of Mercury in orbit.**

Set  $r = r_0$  in (8) and substitute the result into (7). Show that at the equilibrium radius,  $\omega_\phi^2 = M/r_0^3$  for Newton.

---

**QUERY 2. Newton's radial oscillation rate  $\omega_r$  for Mercury's orbit**

We want to use (3) to find the angular rate of radial oscillation. Accordingly, take the second derivative of  $V_L$  in (5) with respect to  $r$ . Set  $r = r_0$  in the resulting expression and substitute your value for  $L^2$  in (7). Use (3) to show that at Mercury's orbital radius,  $\omega_r^2 = M/r_0^3$ , according to Newton.

---

158 **Important result:** *For Newton, Mercury's perihelion does not advance*  
159 *when one considers only the gravitational interaction between Mercury and the*  
160 *Sun.*

**10.4. EFFECTIVE POTENTIAL: EINSTEIN**

162 *Extra effective potential term advances perihelion.*

163 Now we repeat the analysis of radial and tangential orbital motion for the  
 164 general relativistic case. Chapter 9 predicts the radial motion of an orbiting  
 165 satellite. Multiply equations (4) and (5) of Section 9.1 through by 1/2 to  
 166 obtain an equation similar to (4) above for the Newton's case:

$$\begin{aligned} \frac{1}{2} \left( \frac{dr}{d\tau} \right)^2 &= \frac{1}{2} \left( \frac{E}{m} \right)^2 - \frac{1}{2} \left( 1 - \frac{2M}{r} \right) \left( 1 + \frac{L^2}{m^2 r^2} \right) \quad (9) \\ &= \frac{1}{2} \left( \frac{E}{m} \right)^2 - \frac{1}{2} \left( \frac{V_L(r)}{m} \right)^2 \quad (\text{Einstein}) \end{aligned}$$

Set up general  
relativity effective  
potential.

167 Equations (4) and (9) are of similar form, and we use this similarity to make a  
 168 general relativistic analysis of the harmonic radial motion of Mercury in orbit.  
 169 In this process we adopt the *algebraic manipulations* of Newton's analysis in  
 170 Sections 10.2 and 10.3 but apply them to the general relativistic expression (9).

Different time rates  
of different clocks  
do not matter.

171 Before we proceed, note three characteristics of equation (9). First,  $d\tau$  on  
 172 the left side of (9) is the differential wristwatch time  $d\tau$ , not the differential  $dt$   
 173 of Newton's universal time  $t$ . This different reference time is not necessarily  
 174 fatal, since we have not yet decided which relativistic measure of time should  
 175 replace Newton's universal time  $t$ . You will show in Section 10.11 that for  
 176 Mercury the choice of which time to use (wristwatch time, global map  
 177  $T$ -coordinate, or even shell time at the  $r$ -value of the orbit) makes a negligible  
 178 difference in our predictions about the rate of advance of the perihelion.

179 Note, second, that in equation (9) the relativistic expression  $(E/m)^2$   
 180 stands in the place of the Newtonian expression  $E/m$  in (4). However, both  
 181 are constant quantities, which is all that matters in the analysis.

182 Evidence that we are on the right track results when we multiply out the  
 183 second term of the first line of (9), which is the square of the effective  
 184 potential, equation (18) of Section 8.4, with the factor one-half. Note that we  
 185 have assigned the symbol  $(1/2)(V_L/m)^2$  to this second term.

$$\begin{aligned} \frac{1}{2} \left( \frac{V_L(r)}{m} \right)^2 &= \frac{1}{2} \left( 1 - \frac{2M}{r} \right) \left( 1 + \frac{L^2}{m^2 r^2} \right) \quad (\text{Einstein}) \quad (10) \\ &= \frac{1}{2} - \frac{M}{r} + \frac{L^2}{2m^2 r^2} - \frac{ML^2}{m^2 r^3} \end{aligned}$$

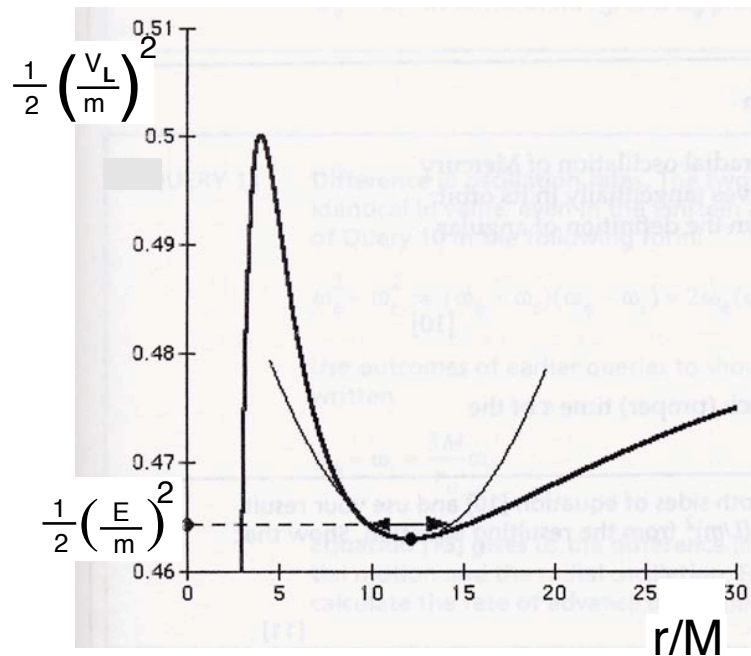
Details of relativistic  
effective potential

186 The heavy curve in Figure 4 plots this function. The second line in (10)  
 187 contains the two effective potential terms that made up the Newtonian  
 188 expression (5). The final term on the right of the second line of (10) describes  
 189 an added attractive potential from general relativity. For the Sun-Mercury  
 190 case at the  $r$ -value of Mercury's orbit, this term leads to the slight precession  
 191 of the elliptical orbit. As  $r$  becomes small, the  $r^3$  in the denominator causes  
 192 this term to overwhelm all other terms in (10), which results in the downward  
 193 plunge in the effective potential at the left side of Figure 4.



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**FIGURE 4** General-relativistic effective potential  $(V_L/m)^2/2$  (heavy curve) and its approximation at the local minimum by a parabola (light curve) in order to analyse the radial excursion (double-headed arrow) of Mercury as simple harmonic motion. The effective potential curve is for a black hole, not for the Sun, whose effective potential near the potential minimum would be indistinguishable from the Newton's effective potential on the scale of this diagram. However, this minute difference accounts for the tiny residual precession of Mercury's orbit.

194 Finally, note third that the last term  $(1/2)(V_L/m)^2$  in relativistic equation  
 195 (9) takes the place of the Newton's effective potential  $V_L/m$  in equation (4).

196 In summary, we can manipulate general relativistic expressions (9) and  
 197 (10) in nearly the same way that we manipulated Newton's expressions (4) and  
 198 (5) in order to analyze the radial component of Mercury's motion and small  
 199 perturbations of Mercury's elliptical orbit brought about by general relativity.

**10.5 ■ EINSTEIN'S ORBIT ANALYSIS**

201 *Einstein tweaks Newton's solution.*

202 Now analyze the radial oscillation of Mercury's orbit according to Einstein.

**QUERY 3. Local minimum of Einstein's effective potential**

Take the first derivative of the squared effective potential (10) with respect to  $r$ , that is find  $d[(1/2)(V_L/m)^2]/dr$ . Set this first derivative aside for use in Query 4. As a separate calculation, equate

this derivative to zero; set  $r = r_0$ , and solve the resulting equation for the unknown quantity  $(L/m)^2$  in terms of the known quantities  $M$  and  $r_0$ .

**QUERY 4. Einstein's radial oscillation rate  $\omega_r$  for Mercury in orbit.**

We want to use (3) to find the rate of oscillation  $\omega_r$  in the radial direction.

- A. Take the second derivative of  $(1/2)(V_L/m)^2$  from (10) with respect to  $r$ . Set the resulting  $r = r_0$  and substitute the expression for  $(L/m)^2$  from Query 3 to obtain

$$\left[ \frac{d^2}{dr^2} \left( \frac{1}{2} \frac{V_L^2}{m^2} \right) \right]_{r=r_0} = \omega_r^2 = \frac{M}{r_0^3} \frac{\left( 1 - \frac{6M}{r_0} \right)}{\left( 1 - \frac{3M}{r_0} \right)} \quad \text{(Einstein)} \quad (11)$$

$$\approx \frac{M}{r_0^3} \left( 1 - \frac{6M}{r_0} \right) \left( 1 + \frac{3M}{r_0} \right) \quad (12)$$

$$\approx \frac{M}{r_0^3} \left( 1 - \frac{3M}{r_0} \right) \quad (13)$$

where we have made repeated use of the approximation inside the front cover in order to find a result to first order in the fraction  $M/r$ .

- B. For our Sun,  $M \approx 1.5 \times 10^3$  meters, while for Mercury's orbit  $r_0 \approx 6 \times 10^{10}$  meters. Does the value of  $M/r_0$  justify the approximations in equations (12) and (13)?

Note that the coefficient  $M/r_0^3$  in these three equations equals Newton's expression for  $\omega_r^2$  derived in Query 1.

Now compare  $\omega_r$ , the in-and-out oscillation of Mercury's orbital  $r$ -coordinate with the angular rate  $\omega_\phi$  with which Mercury moves tangentially in its orbit. The rate of change of azimuth  $\phi$  springs from the definition of angular momentum in equation (10) in Section 8.2:

$$\frac{L}{m} = r^2 \frac{d\phi}{d\tau} \quad \text{(Einstein)} \quad (14)$$

Note the differential wristwatch time  $d\tau$  for the planet.

**QUERY 5. Einstein's angular velocity**

Square both sides of (14) and use your result from Query 3 to eliminate  $L^2$  from the resulting equation. Show that at the equilibrium  $r_0$  the result can be written

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$$\omega_\phi^2 \equiv \left(\frac{d\phi}{d\tau}\right)^2 = \frac{M}{r_0^3} \left(1 - \frac{3M}{r_0}\right)^{-1} \quad (\text{Einstein}) \quad (15)$$

$$\approx \frac{M}{r_0^3} \left(1 + \frac{3M}{r_0}\right) \quad (16)$$

where again we use our approximation inside the front cover. Compare this result with equation (13) and with Newton's result in Query 1.

10.6. ■ PREDICT MERCURY'S PERIHELION ADVANCE

235 *Simple outcome, profound consequences*

Einstein: in-out rate differs from circulation rate.

236 According to Einstein, the advance of Mercury's perihelion springs from the  
 237 difference between the frequency with which the planet sweeps around in its  
 238 orbit and the frequency with which it oscillates in and out in  $r$ . In Newton's  
 239 analysis these two frequencies are equal (for the interaction between Mercury  
 240 and the Sun). But Einstein's theory shows that these two frequencies are  
 241 *slightly* different; Mercury reaches its minimum  $r$  (its perihelion) at an  
 242 incrementally greater angular position in each successive orbit. *Result:* the  
 243 advance of Mercury's perihelion. In this section we compare Einstein's  
 244 prediction with observation. But first we need to define what we are  
 245 calculating.

246 What do we mean by the phrase "the period of a planet's orbit"? The  
 247 period with respect to what? Here we choose what is technically called the  
 248 **synodic period** of a planet, defined as follows:

Definition: synodic period

249 **DEFINITION 1. Synodic period of a planet**

250 The **synodic period** of a planet is the lapse in time (Newton) or lapse in  
 251 global  $T$ -value (Einstein) for the planet to revolve once around the Sun  
 252 with respect to the fixed stars.

"Fixed" stars?

253 **Comment 2. Fixed stars?**

254 What are the "fixed stars"? Chapter 14 The Expanding Universe shows that  
 255 stars are anything but fixed. With respect to our Sun, stars move! However, stars  
 256 that we now know to be very distant do not change angle rapidly from our point  
 257 of view. Over a few hundred years—the lifetime of the field of astronomy  
 258 itself—these stars may be called *fixed*.

259 The value  $T_r$  to make a complete in-and-out radial oscillation is

$$T_r \equiv \frac{2\pi}{\omega_r} \quad (\text{period of radial oscillation}) \quad (17)$$

260 In global coordinate lapse  $T_r$ , Mercury goes around the Sun, completing an  
 261 angle

Section 10.7 Compare Prediction with Observation **10-11**

$$\omega_\phi T_r = \frac{2\pi\omega_\phi}{\omega_r} = (\text{Mercury revolution angle in } T_r) \tag{18}$$

262 which exceeds one complete revolution in radians by:

$$\omega_\phi T_r - 2\pi = T_r (\omega_\phi - \omega_r) = (\text{excess angle per revolution}) \tag{19}$$

**QUERY 6. Difference in Einstein's oscillation rates**

The two angular rates  $\omega_\phi$  and  $\omega_r$  are *almost* identical in value, even in the Einstein analysis. Therefore we can write approximately:

$$\omega_\phi^2 - \omega_r^2 = (\omega_\phi + \omega_r)(\omega_\phi - \omega_r) \approx 2\omega_\phi(\omega_\phi - \omega_r) \tag{20}$$

A. Substitute equations (13) and (16) into the left side of (20):

$$\omega_\phi^2 - \omega_r^2 \approx \frac{M}{r_0^3} \left[ \left(1 + \frac{3M}{r_0}\right) - \left(1 - \frac{3M}{r_0}\right) \right] = \frac{M}{r_0^3} \frac{6M}{r_0} \tag{21}$$

B. Equation (20) becomes:

$$\omega_\phi^2 - \omega_r^2 \approx \frac{M}{r_0^3} \frac{6M}{r_0} \approx \omega_\phi^2 \frac{6M}{r_0} \approx 2\omega_\phi(\omega_\phi - \omega_r) \tag{22}$$

C. Simplify the right-hand equation in (22), write the result as:

$$\omega_\phi - \omega_r \approx \frac{3M}{r_0} \omega_\phi \quad (\text{angular rates, Einstein}) \tag{23}$$

Equation (23) shows the difference in angular velocity between the tangential motion and the radial oscillation. From this rate difference we will calculate the advance of the perihelion of Mercury in one Earth-century.

**Comment 3. What is X?**

Symbols  $\omega$  in (23) express rotation rates in radians per unit of—what? *Question:* What is  $X$  in the denominator of  $d\phi/dX \equiv \omega$ ? Does  $X$  equal global coordinate  $T$ ? planet wristwatch time  $\tau$ ? shell time  $t_{\text{shell}}$  at the average  $r$ -value of the orbit? *Answer:* It does not matter which of these quantities  $X$  represents, as long as this measure is the *same* on both sides of any resulting equation. Comment 1 told us to be relaxed about time. In the following Queries you use (23) to calculate the precession rate of Mercury in radians/second, then to convert this result to arcseconds/Earth-century.

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## Advance of Mercury's Perihelion

**10.7. ■ COMPARE PREDICTION WITH OBSERVATION**

284 *Check out Einstein!*

285 Now compare our approximate relativistic prediction with observation.

**QUERY 7. Mercury's angular velocity**

The synodic period of Mercury's orbit is  $7.602 \times 10^6$  seconds. To one significant digit,  $\omega_\phi \approx 8 \times 10^{-7}$  radian/second. What is its value to three significant digits?

**QUERY 8. Calculated coefficient**

The mass  $M$  of the Sun is  $1.477 \times 10^3$  meters and  $r_0$  of Mercury's orbit is  $5.80 \times 10^{10}$  meters. To one significant digit, the coefficient  $3M/r_0$  in (23) is  $1 \times 10^{-7}$ . Find this result to three significant digits.

**QUERY 9. Advance of Mercury's perihelion in radians/second**

From equation (23) and results of Queries 7 and 8, derive a numerical prediction of the advance of the perihelion of Mercury's orbit in radians/second. To one significant digit the result is  $6 \times 10^{-14}$  radians/second. Find the result to three significant digits.

**QUERY 10. Advance of Mercury's perihelion in arcseconds per Earth-century.**

Estimate the general relativity prediction of advance of Mercury's perihelion in arcseconds per century. Use results from preceding queries plus conversion factors inside the front cover plus the definition that 3600 arcseconds equals one degree. To one significant digit, the answer is 40 arcseconds/century. Find the result to three significant digits.

Observation and  
careful calculation  
agree.

310 A more accurate relativistic analysis predicts 42.980 arcseconds (0.011939  
311 degrees) per Earth-century (Table 10.1). The observed rate of advance of the  
312 perihelion is in perfect agreement with this value:  $42.98 \pm 0.1$  arcseconds per  
313 Earth-century. By what percentage did your prediction differ from  
314 observation?

All planet orbits  
precess.

**10.8. ■ ADVANCE OF THE PERIHELIA OF THE INNER PLANETS**

316 *Help from a supercomputer.*

317 Do the *perihelia* (plural of *perihelion*) of other planets in the solar system also  
318 advance as described by general relativity? Yes, but these planets are farther  
319 from the Sun, and their orbits are less eccentric, so the magnitude of the  
320 predicted advance is less than that for Mercury. In this section we compare our

Section 10.8 Advance of the Perihelia of the Inner Planets **10-13**

**TABLE 10.1** Advance of the perihelia of the inner planets

Planet	Advance of perihelion in seconds of arc per Earth-century (JPL calculation)	$r$ -value of orbit in AU*	Period of orbit in years
Mercury	$42.980 \pm 0.001$	0.38710	0.24085
Venus	$8.618 \pm 0.041$	0.72333	0.61521
Earth	$3.846 \pm 0.012$	1.00000	1.00000
Mars	$1.351 \pm 0.001$	1.52368	1.88089

\*Astronomical Unit (AU): average  $r$ -value of Earth's orbit; inside front cover.

Computer analysis of precessions.

JPL multi-program computation.

321 estimated advance of the perihelia of the inner planets Mercury, Venus, Earth,  
322 and Mars with results of an accurate calculation.

323 The Jet Propulsion Laboratory (JPL) in Pasadena, California, supports  
324 an active effort to improve our knowledge of the positions and velocities of the  
325 major bodies in the solar system. For the major planets and the moon, JPL  
326 maintains a database and set of computer programs known as the Solar System  
327 Data Processing System. The input database contains the observational data  
328 measurements for current locations of the planets. Working together, more  
329 than 100 interrelated computer programs use these data and the relativistic  
330 laws of motion to compute locations of planets at in the past and the future.  
331 The equations of motion take into account not only the gravitational  
332 interaction between each planet and the Sun but also interactions among all  
333 planets, Earth's moon, and 300 of the most massive asteroids, as well as  
334 interactions between Earth and Moon due to nonsphericity and tidal effects.

335 To help us with our project on perihelion advance, Myles Standish,  
336 Principal Member of the Technical Staff at JPL, kindly used the numerical  
337 integration program of the Solar System Data Processing System to calculate  
338 orbits of the four inner planets over four centuries, from A.D. 1800 to A.D.  
339 2200. In an overnight run he carried out this calculation twice, first with the  
340 full program including relativistic effects and second "with relativity turned  
341 off." Standish "turned off relativity" by setting the speed of light to  $10^{10}$  times  
342 its measured value, making light speed effectively infinite.

343 For each of the two runs, the perihelia of the four inner planets were  
344 computed for the four centuries. The results from the nonrelativistic run were  
345 subtracted from those of the relativistic run, revealing advances of the  
346 perihelia per Earth-century accounted for only by general relativity. The  
347 second column of Table 10.1 shows the results, together with the estimated  
348 computational error.

---

**QUERY 11. Approximate advances of the perihelia of the inner planets**

Compare the JPL-computed advances of the perihelia of Venus, Earth, and Mars in Table 10.1 with approximate results calculated using equation (23).

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## 10-14 Chapter 10

## Advance of Mercury's Perihelion

**10.9. CHECK THE STANDARD OF TIME**

355 *Whose clock?*

356 We have been casual about whose time tracks the advance of the perihelion of  
 357 Mercury and other planets; we even treated the global  $T$ -coordinate as a time,  
 358 which is against our usual rules. Does this invalidate our approximations?

359

**QUERY 12. Difference between shell time and Mercury's wristwatch time.**

Use special relativity to find the fractional difference between planet Mercury's wristwatch time increment  $\Delta\tau$  and the time increment  $\Delta t_{\text{shell}}$  read on shell clocks at the same average  $r_0$  at which Mercury moves in its orbit at the average velocity  $4.8 \times 10^4$  meters/second. By what fraction does a change of time from  $\Delta\tau$  to  $\Delta t_{\text{shell}}$  change the total angle covered in the orbital motion of Mercury in one century? Therefore by what fraction does it change the predicted angle of advance of the perihelion in that century?

366

367

368

**QUERY 13. Difference between shell time and global rain map  $T$ .**

Find the fractional difference between shell time increment  $\Delta t_{\text{shell}}$  at  $r_0$  and global map increment  $\Delta T$  for  $r_0$  equal to the average  $r$ -value of the orbit of Mercury. By what fraction does a change from  $\Delta t_{\text{shell}}$  to a lapse in global  $T$  alter the predicted angle of advance of the perihelion in that century?

372

374

**QUERY 14. Does the time standard matter?**

From your results in Queries 12 and 13, say whether or not the choice of a time standard—wristwatch time of Mercury, shell time, or map  $t$ —makes a detectable difference in the numerical prediction of the advance of the perihelion of Mercury in one Earth-century. Would your answer differ if the time were measured with clocks on Earth's surface?

380

**381 DEEP INSIGHTS FROM MORE THAN THREE CENTURIES AGO**

382 *Newton himself was better aware of the weaknesses inherent in his*  
 383 *intellectual edifice than the generations that followed him. This fact*  
 384 *has always roused my admiration.*

385 —Albert Einstein

386 We agree with Einstein. In the following quote from the end of his great work  
 387 *Principia*, Isaac Newton summarizes what he knows about gravity and what  
 388 he does not know. We find breathtaking the scope of what Newton says—and  
 389 the integrity with which he refuses to say what he does not know. In the  
 390 following, “feign” means “invent,” and since Newton's time “experimental  
 391 philosophy” has come to mean “physics.”

392 “I do not ‘feign’ hypotheses.”

393 *Thus far I have explained the phenomena of the heavens and of our*  
 394 *sea by the force of gravity, but I have not yet assigned a cause to*  
 395 *gravity. Indeed, this force arises from some cause that penetrates as*  
 396 *far as the centers of the sun and planets without any diminution of*  
 397 *its power to act, and that acts not in proportion to the quantity of*  
 398 *the surfaces of the particles on which it acts (as mechanical causes*  
 399 *are wont to do) but in proportion to the quantity of solid matter,*  
 400 *and whose action is extended everywhere to immense distances,*  
 401 *always decreasing as the squares of the distances. Gravity toward*  
 402 *the sun is compounded of the gravities toward the individual*  
 403 *particles of the sun, and at increasing distances from the sun*  
 404 *decreases exactly as the squares of the distances as far as the orbit*  
 405 *of Saturn, as is manifest from the fact that the aphelia of the*  
 406 *planets are at rest, and even as far as the farthest aphelia of the*  
 407 *comets, provided that those aphelia are at rest. I have not as yet*  
 408 *been able to deduce from phenomena the reason for these properties*  
 409 *of gravity, and I do not “feign” hypotheses. For whatever is not*  
 410 *deduced from the phenomena must be called a hypothesis; and*  
 411 *hypotheses, whether metaphysical or physical, or based on occult*  
 412 *qualities, or mechanical, have no place in experimental philosophy.*  
 413 *In this experimental philosophy, propositions are deduced from the*  
 414 *phenomena and are made general by induction. The*  
 415 *impenetrability, mobility, and impetus of bodies, and the laws of*  
 416 *motion and the law of gravity have been found by this method. And*  
 417 *it is enough that gravity really exists and acts according to the laws*  
 418 *that we have set forth and is sufficient to explain all the motions of*  
 419 *the heavenly bodies and of our sea.*

420

—Isaac Newton

## 10.10 ■ REFERENCES

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430 Myles Standish of the Jet Propulsion Laboratory ran the programs on the  
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**10-16** Chapter 10

Advance of Mercury's Perihelion

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