

Chapter 20. Orbits of Light around the Spinning Black Hole

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- *What variety of paths does light follow around a spinning black hole?*
- *Can a spinning black hole reverse the direction of a light beam?*
- *Can a light beam go into orbit around a spinning black hole? If so, how many different orbits are available to it?*
- *What does a distant spinning black hole look like? How can I distinguish it visually from a non-spinning black hole?*

CHAPTER

20

Orbits of Light around the Spinning
Black Hole

Edmund Bertschinger & Edwin F. Taylor *

Light does strange things near a spinning black hole; it misleads you about the locations and shapes of things, so you must grope along as if you are in a haunted house. Seeing is definitely not believing!

—The authors

20.1 ■ INTRODUCTION: THE PURPOSES OF LOOKING

Who cares what we see?

New questions about
the spinning black hole

Chapter 11 described orbits of light around the non-spinning black hole. That earlier chapter focussed on the question, “What is the visual size of the black hole seen by a raindrop diver falling from a great distance?” The same question about the spinning black hole is of little interest today. Instead, we ask the questions:

- How can we identify a distant spinning black hole in the heavens?
- How can we measure its mass M and spin parameter a ?

To answer these questions, the present chapter does use a method similar to that of these earlier chapters:

Stone’s mass
goes to zero.

- Start with a stone of mass m .
- Let the stone’s mass go to zero.

We begin this program with a review of the stone’s equations of motion in global Doran coordinates.

*Draft of Second Edition of *Exploring Black Holes: Introduction to General Relativity*
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20-2 Chapter 20 Orbits of Light around the Spinning Black Hole**20.2.2 ■ DORAN GLOBAL EQUATIONS OF MOTION FOR THE STONE**

36 *Goal: Get rid of wristwatch time.*

Motion of a stone

37 This chapter analyzes the motion of light in the equatorial plane of the
 38 spinning black hole. We derive equations of motion for light by extending
 39 equations of motion for a free stone. Begin this process with equations (15),
 40 (21), and (20) and (16) for motion of a stone in Section 18.2: . Write them in
 41 the form:

$$\frac{dr}{d\tau} = \pm \frac{R}{mr} (E - V_L^+)^{1/2} (E - V_L^-)^{1/2} \quad (\text{stone}) \quad (1)$$

$$\frac{d\Phi}{d\tau} = \frac{L}{mR^2} + \frac{\sin^2 \alpha}{ma} \left[E - \omega L \pm \frac{1}{\beta} (E - V_L^+)^{1/2} (E - V_L^-)^{1/2} \right] \quad (\text{stone}) (2)$$

$$\frac{dT}{d\tau} = \left(\frac{R}{rH} \right)^2 \frac{1}{m} \left[E - \omega L \pm \beta (E - V_L^+)^{1/2} (E - V_L^-)^{1/2} \right] \quad (\text{stone}) (3)$$

$$V_L^\pm(r) \equiv \omega L \pm \frac{rH}{R} \left(m^2 + \frac{L^2}{R^2} \right)^{1/2} \quad (\text{stone}) \quad (4)$$

42 Box 1 in Section 18.2 defines α , β , and ω .

Photon
does not age.

43 To apply these equations to light, we must overcome a fundamental
 44 problem: The differential aging $d\tau$ of a light flash along its worldline is
 45 automatically zero (Section 1.4), so these equations of motion for the stone
 46 have no meaning for the light flash. To give them meaning, we eliminate $d\tau$
 47 from these equations of motion, recasting them without $d\tau$. Use the following
 48 equations:

$$\frac{dr}{dT} = \left(\frac{dr}{d\tau} \right) \left(\frac{d\tau}{dT} \right) \quad (\text{stone}) \quad (5)$$

$$\frac{d\Phi}{dT} = \left(\frac{d\Phi}{d\tau} \right) \left(\frac{d\tau}{dT} \right) \quad (\text{stone}) \quad (6)$$

49 Carry out this combination on equations (1) through (3). *Result:*
 50 equations for dr/dT and $d\Phi/dT$. Note that in this process, m disappears from
 51 the coefficients—but remains in the expression for $V_L^\pm(r)$ in equation (4).

52 To convert these equations to describe light, we take the limit of the
 53 resulting equation for dr/dT and $d\Phi/dT$ as $m \rightarrow 0$. Carry this out first on
 54 expressions that contain V_L^+ and V_L^- defined in equation :

$$[(E - V_L^+) (E - V_L^-)]^{1/2} \quad (7)$$

$$= \left[\left\{ E - \omega L - \frac{rH}{R} \left(m^2 + \frac{L^2}{R^2} \right)^{1/2} \right\} \left\{ E - \omega L + \frac{rH}{R} \left(m^2 + \frac{L^2}{R^2} \right)^{1/2} \right\} \right]^{1/2}$$

$$= \left[(E - \omega L)^2 - \left(\frac{rH}{R} \right)^2 \left(m^2 + \frac{L^2}{R^2} \right) \right]^{1/2} \quad (\text{stone}) \quad (8)$$

Section 20.2 Doran Global Equations of Motion for the Stone **20-3**Let $m \rightarrow 0$.55 To convert this expression to light, take the limit as $m \rightarrow 0$:

$$\lim_{m \rightarrow 0} [(E - V_L^+) (E - V_L^-)]^{1/2} = \lim_{m \rightarrow 0} \left[(E - \omega L)^2 - \left(\frac{rH}{R} \right)^2 \left(m^2 + \frac{L^2}{R^2} \right) \right]^{1/2} \quad (9)$$

$$= E \left[\left(1 - \omega \frac{L}{E} \right)^2 - \left(\frac{rH}{R} \right)^2 \left(\frac{L}{RE} \right)^2 \right]^{1/2} \quad (10)$$

$$= E \left[(1 - \omega b)^2 - \frac{b^2}{R^2} \left(\frac{rH}{R} \right)^2 \right]^{1/2} \quad (11)$$

$$\equiv E \times F_{\text{spin}}(a, b, r) \quad (\text{light}) \quad (12)$$

Impact parameter
 b for light56 Equation (11) defines a new constant b , while (12) defines a new function
57 $F_{\text{spin}}(a, b, r)$:

$$b \equiv \frac{L}{E} \quad (\text{impact parameter for light}) \quad (13)$$

$$F_{\text{spin}}(a, b, r) \equiv \left[(1 - \omega b)^2 - \frac{b^2}{R^2} \left(\frac{rH}{R} \right)^2 \right]^{1/2} \quad (\text{light}) \quad (14)$$

58

59

QUERY 1. Motion of light around the non-spinning black hole Show that when $a \rightarrow 0$, then $F_{\text{spin}}(a, b, r) \rightarrow F(b, r)$, defined in equation (16), Section 11.3.

62

Only L/E matters.

63 The spinning black hole shares an important simplification with the
64 non-spinning black hole (Section 11.2): the motion of light does not depend on
65 L or E separately, but only on their ratio, the impact parameter $b \equiv L/E$.

66 **Comment 1. Both L and b can be positive or negative.**

67 The angular momentum L of a stone can be positive (prograde orbit) or
68 negative (retrograde orbit). Equation (13) shows the same to be true of impact
69 parameter b . For the non-spinning black hole, orbits in the two directions are
70 simply mirror images of one another. In contrast, for the spinning black hole
71 counterclockwise and clockwise orbits of a stone are quite different, as Chapters
72 18 and 19 show. Prograde and retrograde orbits of light are also different from
73 one another, as Section 20.3 will show.

Equations of motion
for light

74 These results allow us to write down the equations of motion for light in
75 the equatorial plane of the spinning black hole:

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$$\frac{dr}{dT} = \pm \left(\frac{rH}{R} \right) \frac{F_{\text{spin}}}{1 - \omega b \pm \beta F_{\text{spin}}} \quad (\text{light}) \quad (15)$$

$$\frac{d\Phi}{dT} = \left(\frac{rH}{R} \right)^2 \frac{\frac{b}{R^2} + \frac{\sin^2 \alpha}{a} \left[1 - \omega b \pm \frac{1}{\beta} F_{\text{spin}} \right]}{1 - \omega b \pm \beta F_{\text{spin}}} \quad (\text{light}) \quad (16)$$

$$\frac{dr}{d\Phi} = \pm \frac{R}{r} \frac{F_{\text{spin}}}{\frac{b}{R^2} + \frac{\sin^2 \alpha}{a} \left[1 - \omega b \pm \frac{1}{\beta} F_{\text{spin}} \right]} \quad (\text{light}) \quad (17)$$

76

77

QUERY 2. And when $a \rightarrow 0$?

In Query 1 you showed that as $a \rightarrow 0$, $F_{\text{spin}} \rightarrow F$ for the non-spinning black hole. Apply the same limit to expressions in equations (15) through (17). Box 1 in Section 18.2 may be useful.

As $a \rightarrow 0$:

- A. $R \rightarrow r$
 B. $H^2 \rightarrow (1 - 2M/r)$
 C. $\omega \rightarrow 0$
 D. $\beta \rightarrow (2M/r)^{1/2}$
 E. $\sin^2 \alpha/a \rightarrow 0$

With these changes, show that equations (15) through (17) for the motion of light around the spinning black hole reduce to equations (17) through (19) in (Section 11.3) for the non-spinning black hole.

20.3. EFFECTIVE POTENTIAL FOR LIGHT

Orbits at a glance

Derive effective potential for light.

Now we derive the effective potential for light, which allows us to predict at a glance the r -motion of a light flash that moves in the equatorial plane of the spinning black hole. This derivation follows a procedure similar to that for the non-spinning black hole in Section 11.3. Run your finger down that earlier derivation to compare the following derivation for the spinning black hole.

To start, multiply both sides equation (15) by an expression that leaves only $\pm F_{\text{spin}}$ on the right side. Then multiply both sides by M/r and square both sides of the resulting equation. The result has the form:

$$A^2(a, b, r) \left(\frac{dr}{dT} \right)^2 = \left(\frac{M}{b} \right)^2 F_{\text{spin}}^2(a, b, r) \quad (\text{light}) \quad (18)$$

$$= \left(\frac{M}{b} \right)^2 - 2M\omega \left(\frac{M}{b} \right) + M^2\omega^2 - \frac{M^2}{R^2} \left(\frac{rH}{R} \right)^2 \quad (19)$$

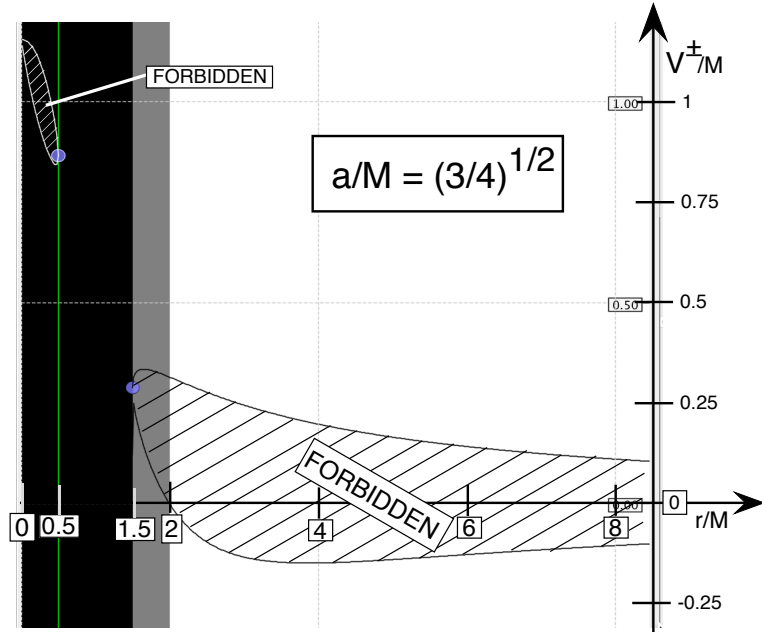


FIGURE 1 Effective potential for light with $a/M = (3/4)^{1/2}$. The gray region extends from the static limit at $r_S = 2M$ down to the event horizon at $r_{EH} = 1.5M$. The Cauchy horizon is at $r_{CH} = 0.5M$. There are two forbidden regions for light, one outside the event horizon and one inside the Cauchy horizon. Equation (22) shows that inside a forbidden region dr/dT is imaginary .

100 where

$$A(a, b, r) \equiv \left(\frac{M}{b}\right) \left(\frac{rH^2}{R}\right) [1 - \omega b \pm \beta F_{\text{spin}}(a, b, r)] \quad (20)$$

101 The right side of (19) is quadratic in M/b , so factor it using the quadratic
102 equation:

$$\frac{M}{b} = M\omega \pm \left[\frac{M^2}{R^2} \left(\frac{rH}{R}\right)^2 \right]^{1/2} \quad (21)$$

103 Substitute this expression for M/b into the second term on the right side of
104 (19) and collect terms, with the result:

$$A^2(a, b, r) \left(\frac{dr}{dT}\right)^2 = \left(\frac{M}{b}\right)^2 - \left[\frac{V^{\pm}(a, r)}{M}\right]^2 \quad (22)$$

105
106 where

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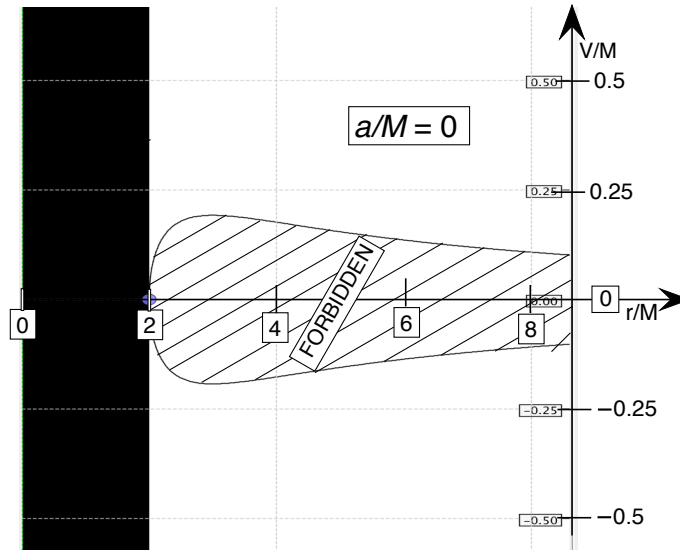


FIGURE 2 For comparison, effective potential for light for the non-spinning black hole ($a = 0$), to the same scale as Figure 1.

$$\left[\frac{V^\pm(a, r)}{M} \right]^2 \equiv \frac{M^2}{R^2} \left(\frac{rH}{R} \right)^2 \pm \frac{2M^2\omega}{R} \left(\frac{rH}{R} \right) - M^2\omega^2 \quad (23)$$

107

108 The superscript \pm on the left side of (23) is the same as the \pm on the right
109 side.

Track r -motion
of light.

110 Equation (22) tracks the r -motion of a light flash in the equatorial plane:
111 The first term on the right side is a function of b but not a function of a or r ,
112 while the second term—the square of the effective potential—is a function of a
113 and r , but not a function of b . Equation (22) then permits us to plot on the
114 same diagram the effective potential function (23)—Figure 1—and a
115 horizontal line that represents any given value of $(M/b)^2$ in (22).

Forbidden region
for light

116 Equation (22) tells us why the region between V^- and V^+ in Figure 1 is
117 forbidden: If $(M/b)^2$ is less than $(V^+/M)^2$ but greater than $(V^-/M)^2$, then
118 dr/dT is imaginary, which is indeed forbidden. Figure 2 reminds us of the
119 corresponding effective potential for the non-spinning black hole.

120

QUERY 3. Effective potential for the non-spinning black hole

Show that when $a \rightarrow 0$, equations (22) and (23) reduce to equations (25) and (26) in Section 11.3,
shown in Figure 2.

123

124

QUERY 4. Mass vs. massless

Compare Figure 1 for light with the corresponding Figure 1 in Section 18.3 for a stone, both for $a/M = (3/4)^{1/2}$. Is the following statement true or false: Outside the event horizon, the forbidden region for light is entirely enclosed in the forbidden region for the stone.

Where light goes

Global radial motion dr/dT must be real. Therefore the right side of (22) must be positive. *Result:* The light flash that moves along a horizontal line at $(M/b)^2$ in Figure 1, for example, cannot enter either forbidden region between the two effective potential curves. A light flash from far away that reaches one of these curves either reverses the sign of its r -motion or holds its r -value at the curve's maximum or minimum.

20.4 ■ EXERCISE**1. Does a spinning black hole appear smaller or larger than a non-spinning black hole?**

Think of two black holes. The first is a non-spinning black hole of a particular given mass M . The second is a spinning black hole of the *same* mass M . Can a distant observer tell them apart?

Specifically, (1) How large does a black hole look to an Earth observer? (2) Can the Earth observer determine whether or not the black hole is spinning or non-spinning? (3) Since most black holes in Nature spin, can the Earth observer determine *how fast* the spinning black hole rotates—that is, what is its value of a/M ? When you finish this exercise, you should be able to answer these three questions clearly and definitively.

Non-spinning black hole: Review Figures 3 and 12 in Chapter 11. They show that r -coordinate of the knife-edge circular orbit for light around a non-spinning black hole $r = 3M$ corresponds to the maximum of the effective potential curve $V(r)$, and that this knife-edge orbit determines the visual size of the non-spinning black hole through the critical impact parameter b_{critical} : the visual diameter of the non-spinning black hole is $2b_{\text{critical}}$.

Spinning black hole: The visual size of the spinning black hole in the equatorial plane is determined by two critical impact parameters b_{critical}^+ and b_{critical}^- that belong to the prograde and retrograde knife-edge orbits of light, respectively, as shown in Figure 3 for the special case $a = (3/4)^{1/2}M$.

A. Figure 1 in this chapter plots the effective potentials $V^\pm(r)$ for a light beam that moves near a black hole with $a/M = (3/4)^{1/2}$. From Figure 1, show that

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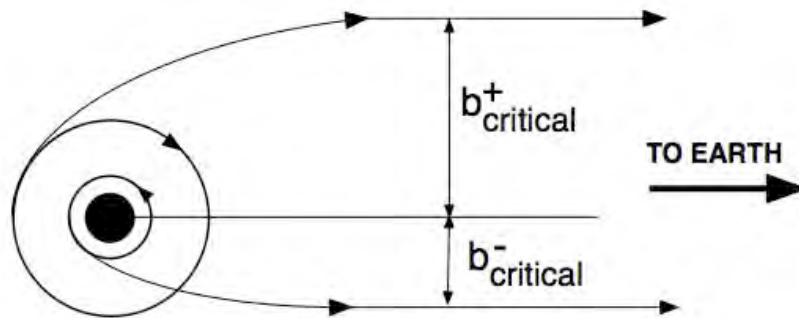


FIGURE 3 Figure for Exercise 1. Schematic diagram showing the visual size of a spinning black hole with $a/M = (3/4)^{1/2} = 0.866$.

$$b^+ = \frac{1}{V_{\max}^+} = \frac{1}{V^+(r_{\text{knife edge}}^+)} \quad \text{prograde knife-edge orbit} \quad (24)$$

$$b^- = \frac{1}{V_{\max}^-} = \frac{1}{V^-(r_{\text{knife edge}}^-)} \quad \text{retrograde knife-edge orbit} \quad (25)$$

162 B. Derive the following expressions for $b_{\text{critical}}^{\pm}$ and $r_{\text{knife edge}}^{\pm}$:

$$b_{\text{critical}}^{\pm} = \left(r_{\text{knife edge}}^{\pm} + 3M \right) \left(\frac{r_{\text{knife edge}}^{\pm}}{4M} \right)^{1/2} \quad (26)$$

$$r_{\text{knife edge}}^+ = 4M \cos^2 \Psi^+ \quad \text{where } \Psi = \frac{1}{3} \arccos \left(\mp \frac{a}{M} \right) \quad (27)$$

163 C. Evaluate equations (26) and (27) for $a = 0$ and show that both results
164 agree with equation (28) in Chapter 11.

165 D. Find the values of b^+ and b^- for a spinning black hole with (a)
166 $a/M = (3/4)^{1/2}$ and (b) $a/M = 1$ (maximum-spin black hole).

167 E. *Optional:* Plot the visual size ($b_{\text{critical}}^+ + b_{\text{critical}}^-$) of the spinning black
168 hole as a function of its spin parameter a/M .

169 F. Answer the question posed in the title of this exercise. Include a
170 sentence that starts, "This depends on".

Section 20.4 Exercise **20-9**

¹⁷¹ Download file name: Ch20OrbitsOfLightAroundSpinningBH170906v1.pdf