

## Chapter 7. Inside the Black Hole

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- 14 • *Why would anyone volunteer to dive to the center of a black hole?*
- 15 • *Why does everything inside the event horizon inevitably move to smaller*  
16  *$r$ ?*
- 17 • *How massive must a black hole be so that 20 years pass on my wristwatch*  
18 *between crossing the event horizon and arrival at the crunch point?*
- 19 • *How can I construct a local inertial frame that is valid inside the event*  
20 *horizon?*
- 21 • *What do I see ahead of me and behind me as I approach the crunch*  
22 *point?*
- 23 • *Is my death quick and painless?*

## CHAPTER

## 7

25

## Inside the Black Hole

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26 *Alice had not a moment to think about stopping herself before*  
 27 *she found herself falling down what seemed to be a very deep*  
 28 *well. Either the well was very deep, or she fell very slowly, for*  
 29 *she had plenty of time as she went down to look about her,*  
 30 *and to wonder what was going to happen next. First she tried*  
 31 *to look down and make out what she was coming to, but it was*  
 32 *too dark to see anything . . . So many out-of-the-way things*  
 33 *had happened lately that Alice had begun to think that very few*  
 34 *things indeed were really impossible.*

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—Lewis Carroll, *Alice in Wonderland*

## 7.1 ■ INTERVIEW OF A DIVING CANDIDATE

37 *Few things are really impossible.*

38 So you are applying to be a member of the black hole diving research group.

39 *Yes.*

40 Have you personally had experience diving into black holes?

41 *This question is a joke, right?*42 Why do you want to be part of this diving group, since your research results  
43 cannot be reported back to us outside the event horizon?44 *We want to see for ourselves whether or not our carefully-studied*  
45 *predictions are correct. You know very well that 27 percent of*  
46 *qualified Galaxy Fleet personnel volunteered for this mission.*47 Tell me, why doesn't the black-hole diving group use local shell coordinates to  
48 make measurements inside the event horizon?

\*Draft of Second Edition of *Exploring Black Holes: Introduction to General Relativity*  
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**7-2 Chapter 7 Inside the Black Hole**

49 *Inside the event horizon no one can build a spherical shell that*  
50 *stays at constant  $r$  in Schwarzschild coordinates. Instead we make*  
51 *measurements in a series of local inertial frames that can exist*  
52 *anywhere except on the singularity.*

53 Then how will you measure your  $r$ -coordinate without a spherical shell?

54 *We track the decreasing value of  $r$  by measuring the decreasing*  
55 *separation between us and a test particle beside us that is also*  
56 *diving radially inward.*

57 What clocks will you use in your experiments?

58 *Our wristwatches.*

59 When does your diving group cross the event horizon?

60 *As measured on whose clock?*

61 You *are* savvy. When does your diving group cross the event horizon as read  
62 on your wristwatches?

63 *Zeroing different clocks in different locations is arbitrary. The*  
64 *Astronautics Commission has a fancy scheme for coordinating the*  
65 *various clock readings, mostly for convenience in scheduling. Want*  
66 *more details?*

67 Not now. Is there any service that we on the outside can provide for your  
68 diving group once you are inside the event horizon?

69 *Sure. We will welcome radio and video bulletins of the latest news*  
70 *plus reports of scientific developments outside the event horizon.*

71 And will your outgoing radio transmissions from inside the event horizon  
72 change frequency during their upward transit to us?

73 *Another joke, I see.*

74 Yes. Does your personal—ah—end seem mercifully quick to you?

75 *The terminal “spaghettification” will take place in a fraction of a*  
76 *second as recorded on my wristwatch. Many of you outside the*  
77 *event horizon would welcome assurance of such a quick end.*

78 What will you personally do for relaxation during the trip?

79 *I am a zero-g football champion and grandmaster chess player.*  
80 *Also, my fiancé has already been selected as part of the team.*  
81 *We will be married before launch.*

82 [We suppress the transcript of further discussion about the ethical and moral  
83 status of bringing children into the diving world.]

### Box 1. Eggbeater Spacetime?

Being “spaghettified” as you approach the center of a black hole is bad enough. But according to some calculations, your atoms will be scrambled by violent, chaotic tidal forces before you reach the center—especially if you fall into a young black hole.

The first theory of the creation of a black hole by J. Robert Oppenheimer and Hartland Snyder (1939) assumed that the collapsing structure is spherically symmetric. Their result is a black hole that settles quickly into a placid final state. A diver who approaches the singularity at the center of the Oppenheimer-Snyder black hole is stretched with steadily increasing force along the  $r$ -direction and compressed steadily and increasingly from all sides perpendicular to the  $r$ -direction.

In Nature an astronomical collapse is rarely spherically symmetric. Theory shows that when a black hole forms, the asymmetries exterior to the event horizon are quickly radiated away in the form of gravitational waves—in a few seconds measured on a distant clock! Gravitational radiation captured inside the event horizon, however, evolves and influences spacetime inside the black hole.

So what happens? There is no way to verify any predictions about events inside the event horizon (Objection 1), but

that does not stop us from making them! Vladimir Belinsky, Isaac Markovich Khalatnikov, Evgeny Mikhailovich Lifshitz, and independently Charles Misner discovered that Einstein’s equations predict more than one kind of singularity.

Their theory says that as a diving observer approaches the center point, spacetime can oscillate chaotically, squeezing and stretching the poor traveler in random directions like an electric mixer (eggbeater). These oscillations increase in both amplitude and frequency as the astronaut approaches the singularity of the black hole. Any physical object, no matter what stresses it can endure, is necessarily utterly destroyed at an eggbeater singularity.

However, there is some theoretical evidence that eggbeater oscillations will die away, so an astronaut who waits a while to dive after the black hole has formed may not encounter them. Before these eggbeater oscillations die away—if they do—spacetime in the chaotic regions is definitely NOT described by the Schwarzschild metric!

In the present chapter we assume the non-spinning black hole under exploration is an ancient one and that we can ignore possible eggbeater oscillations of spacetime. We predict (and hope!) that as our astronaut colony approaches the center, the “spacetime weather” is clear and calm.

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**Objection 1.** *What kind of science are these people talking about? Obviously nothing more than science fiction! No one who crosses the event horizon of a black hole can report observations to the scientific community outside the event horizon. Therefore all observations carried out inside the event horizon—and conclusions drawn from them—remain private communications. Private communication is not science!*

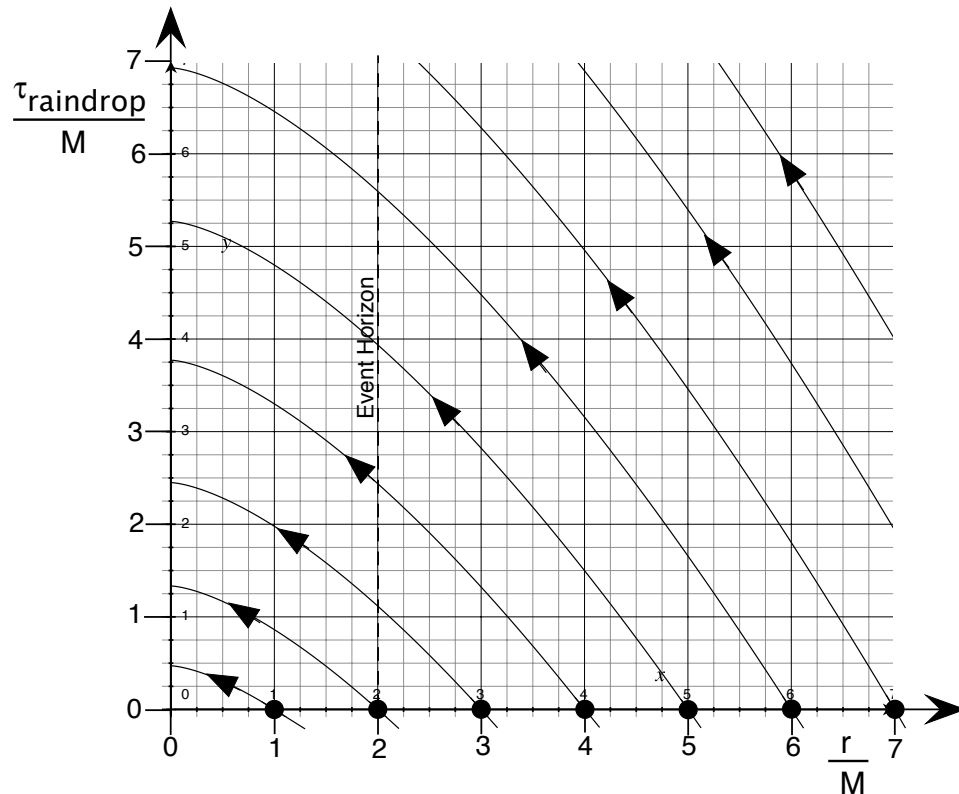
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Yours is one sensible view of science, but if the “spacetime weather is clear and calm” inside the event horizon (Box 1), then the diving research group may have decades of life ahead of them, as recorded on their wristwatches. They can receive news and science updates from friends outside, view the ever-changing pattern of stars in the heavens (Chapter 12), carry out investigations, discuss observations among themselves, and publish their own exciting research journal.

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We recognize that the event horizon separates two communities of investigators with a one-way surface or “membrane.” Outsiders cannot receive reports of experiments that test their predictions about life inside the event horizon. They must leave it to insiders to verify or disprove these predictions with all the rigor of a lively in-falling research community. Later chapters on the *spinning* black hole raise the possibility that an explorer might navigate in such a way as to reemerge from the event horizon, possibly into a different spacetime region.

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**FIGURE 1** Raindrop wristwatch time vs. global  $r$ -coordinate from (2) for a series of raindrops that pass  $r = M, 2M, \dots, 9M$  at  $\tau_{\text{raindrop}} = 0$  (little filled circles along the horizontal axis). The shapes of these curves are identical, just displaced vertically with respect to one another. Every raindrop moves smoothly across the event horizon when clocked on its wristwatch, but not when tracked with global Schwarzschild  $t$ -coordinate (Figure 2).

John A. Wheeler:  
radical conservatism

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**Comment 1. Wheeler’s “radical conservatism”**

John Archibald Wheeler (1911-2008), who co-authored the first edition of *Exploring Black Holes*, rescued general relativity from obscurity in the 1950s and helped to jump-start the present golden age of gravitational physics. He was immensely inventive in research and teaching; for example he adopted and publicized the name *black hole* (initial quote, Chapter 3). Wheeler’s professional philosophy was **radical conservatism**, which we express as: *Follow well-established physical principles while pushing each to its extreme limits. Then develop a new intuition!* The black hole—both outside and inside its event horizon—is a perfect structure on which to apply Wheeler’s radical conservatism, as we do throughout this book.

**Comment 2. Non-spinning vs. spinning black hole**

Chapters 2 through 13 describe spacetime around a black hole: a black hole that does not rotate. We call this a **non-spinning black hole**. The Universe is full of black holes that spin; many of them spin very fast, with deep

consequences for their structure and for spacetime around them. We call each of these a **spinning black hole**, the subject of Chapters 17 through 21.

### 7.2 ■ RAINDROP WORLDLINE

“*Raindrops keep fallin’ on my head . . .*” song by Hal David and Burt Bacharach

We start with the raindrop of Chapter 6. The *raindrop* is a stone (wearing a wristwatch) that drops from initial rest very far from the black hole (Definition 2, Section 6.4). From equation (23) of that section:

$$d\tau_{\text{raindrop}} = - \left( \frac{r}{2M} \right)^{1/2} dr \quad (1)$$

In the following Queries you integrate (1) and apply the result to a raindrop that first falls past an Above  $r$ -coordinate  $r_A$  then falls past a sequence of lower  $r$ -coordinates (Figure 1).

#### QUERY 1. Raindrop wristwatch time lapse between the above $r_A$ and lower $r$

- A. Integrate (1) to determine the elapsed raindrop wristwatch time from the instant the raindrop falls past the Above coordinate  $r_A$  until it passes a sequence of smaller  $r$ -coordinates. Express this elapsed raindrop wristwatch time with the notation  $[r_A \rightarrow r]$  for  $r$ -limits.

$$\tau_{\text{raindrop}} [r_A \rightarrow r] = \frac{4M}{3} \left[ \left( \frac{r_A}{2M} \right)^{3/2} - \left( \frac{r}{2M} \right)^{3/2} \right] \quad (2)$$

Figure 1 plots this equation for a series of raindrops after each passes through a different given  $r_A$  at  $\tau_{\text{raindrop}} = 0$ .

- B. What happens to the value of the raindrop wristwatch time lapse in (2) when the initial  $r_A$  becomes very large? Explain why you are not disturbed by this result.

#### QUERY 2. Raindrop wristwatch time lapse from event horizon to crunch.

Suppose you ride the raindrop, and assume (incorrectly) that you survive to reach the center. This Query examines how long (on your wristwatch) it takes you to drop from the event horizon to the singularity—the crunch point.

- A. Adapt your result from Query 1 to show that:

$$\tau_{\text{raindrop}} [2M \rightarrow 0] = \frac{4M}{3} \quad (\text{event horizon to crunch, in meters}) \quad (3)$$

- B. *Predict:* Does every curve in Figure 1 satisfy (3)?

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C. Use constants inside the front cover to find the event horizon-to-crunch raindrop wristwatch time *in seconds* for a black hole of mass  $M/M_{\text{Sun}}$  times the mass of our Sun:

$$\tau_{\text{raindrop}}[2M \rightarrow 0] = 6.57 \times 10^{-6} \frac{M}{M_{\text{Sun}}} \quad (\text{event horizon to crunch, in seconds}) \quad (4)$$

D The monster black hole at the center of our galaxy has mass  $M \approx 4 \times 10^6 M_{\text{Sun}}$ : its mass is about four million times the mass of our Sun. Assume (incorrectly) that this black hole does not spin. How long, in seconds on your wristwatch, will it take you—riding on the raindrop—to fall from the event horizon of this monster black hole to its singularity?

E. **Discussion question:** How can the  $r$ -value of the event horizon  $r = 2M$  possibly be greater than the wristwatch time  $4M/3$  that it takes the raindrop to fall from the event horizon to the singularity? Does the raindrop move faster than light inside the event horizon? (*Hint:* Do global coordinate separations dependably predict results of our measurements?)

**QUERY 3. Mass of the “20-year black hole.”**

The black hole we feature in this chapter has a mass such that it takes 20 years—recorded on the wristwatch of the raindrop—to fall from event horizon to singularity.

- A. Find the mass of the “20-year black hole” (a) in meters, (b) as a multiple of the mass of our Sun, and (c) in light-years.
- B. An average galaxy holds something like  $10^{11}$  stars of mass approximately equal to that of our Sun. The “20-year black hole” has the mass of approximately how many average galaxies?
- C. What is the value of the  $r$ -coordinate at the event horizon of the “20-year black hole” in light-years?

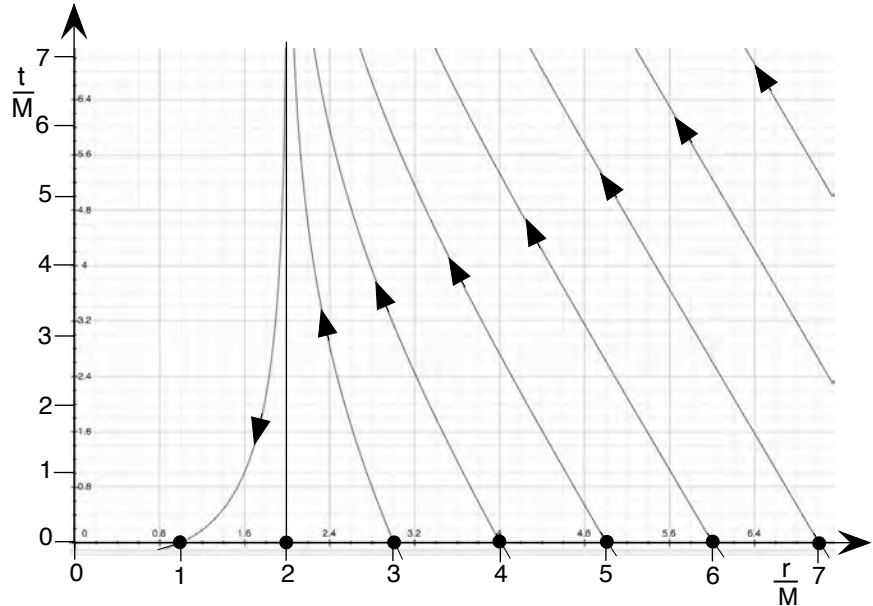
Now turn attention to the motion of the raindrop in global Schwarzschild coordinates. Equation (22) in Section 6.4 gives the Schwarzschild map velocity of the raindrop:

$$\begin{aligned} \frac{dr}{dt} &= - \left(1 - \frac{2M}{r}\right) \left(\frac{2M}{r}\right)^{1/2} && (\text{raindrop map velocity}) && (5) \\ &= \left(\frac{r}{2M}\right)^{-3/2} \left(1 - \frac{r}{2M}\right) \end{aligned}$$

We want to find  $r(t)$ , the  $r$ -coordinate of the raindrop as a function of the  $t$ -coordinate. This function defines a worldline (Section 3.10).

To find the worldline of the raindrop, manipulate (5) to read:

$$dt = \frac{\left(\frac{r}{2M}\right)^{3/2} dr}{1 - \frac{r}{2M}} = \frac{4Mu^4 du}{1 - u^2} \quad (\text{raindrop}) \quad (6)$$



**FIGURE 2** Schwarzschild worldlines of raindrops from (10), plotted on the  $[r, t]$  slice. These particular raindrops pass  $t_A/M = 0$  at different values of  $r_A/M$  (filled dots along the horizontal axis). The curves for  $r_A/M > 2$  are identical in shape, simply displaced vertically with respect to one another. These worldlines are not continuous across the event horizon (compare Figure 1).

176 where the expression on the right side of (6) results from substitutions:

$$u \equiv \left(\frac{r}{2M}\right)^{1/2} \quad \text{so} \quad du = \frac{1}{4M} \left(\frac{r}{2M}\right)^{-1/2} dr \quad \text{and} \quad dr = 4Mudu \quad (7)$$

177 From a table of integrals:

$$\int \frac{u^4 du}{1-u^2} = -\frac{u^3}{3} - u + \frac{1}{2} \ln \left| \frac{1+u}{1-u} \right| \quad (8)$$

178 Integrate (6) from  $u_A$  to  $u$ , where A stands for Above. The integral of (6)  
179 becomes:

$$t - t_A = 4M \left[ \frac{u_A^3}{3} - \frac{u^3}{3} + u_A - u + \frac{1}{2} \ln \left| \frac{1+u}{1-u} \right| - \frac{1}{2} \ln \left| \frac{1+u_A}{1-u_A} \right| \right] \quad (\text{raindrop9})$$

180 Substitute the expression for  $u$  from (7) into (9):

$$t - t_A = \frac{4M}{3} \left[ \left(\frac{r_A}{2M}\right)^{3/2} - \left(\frac{r}{2M}\right)^{3/2} + 3 \left(\frac{r_A}{2M}\right)^{1/2} - 3 \left(\frac{r}{2M}\right)^{1/2} \right. \\ \left. + \frac{3}{2} \ln \left| \frac{1 + \left(\frac{r}{2M}\right)^{1/2}}{1 - \left(\frac{r}{2M}\right)^{1/2}} \right| - \frac{3}{2} \ln \left| \frac{1 + \left(\frac{r_A}{2M}\right)^{1/2}}{1 - \left(\frac{r_A}{2M}\right)^{1/2}} \right| \right] \quad (\text{raindrop}) \quad (10)$$



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181 Equation (10) is messy, but the computer doesn't care and plots the curves in  
182 Figure 2 for  $t_A = 0$  and  $r_A/M = 1$  through 9.



184 **Objection 2.** *Wait! The  $t/M$  versus  $r/M$  worldline in Figure 2 tells us that*  
185 *the raindrop takes an unlimited  $t$  to reach the event horizon. Do you mean*  
*to tell me that our raindrop does not cross the event horizon?*



187 Recall our “strong advice” in Section 5.6: “To be safe, it is best to assume  
188 that global coordinate separations do not have any measured meaning.”  
189 The worldlines in Figure 2 that rise without limit in  $t$ -coordinate as  
190  $r/M \rightarrow 2^+$  (from above) do not tell us directly what any observer  
191 measures. In contrast, the observer riding on the raindrop reads and  
192 records her wristwatch time  $\tau$  as she passes each shell. At each such  
193 instant on her wristwatch, she also her direct reading of the  $r$ -coordinate  
194 stamped on the shell she is passing. When timed on her wristwatch, the  
raindrop passes smoothly inward across the event horizon (Figure 1).



196 **Objection 3.** *There is still a terrible problem with Figure 2. Why does the*  
197 *worldline of the raindrop that somehow makes it inward across  $r/M = 2$*   
*run **backward** in the Schwarzschild  $t$ -coordinate?*



199 We saw this earlier in the light-cone diagram of Figure 8 in Section 3.7, in  
200 which the  $t$ -coordinate can run backward along a worldline. However, that  
201 has no measurable consequence. You cannot grow younger by falling into  
202 a black hole. Sorry! The global  $t$ -coordinate is *not* time. Does the idea of  
203 backward-running global  $t$ -coordinate along a worldline make you  
204 uncomfortable? Get used to it! In contrast, the time you read on your  
205 wristwatch *always* runs forward along your worldline, in particular along the  
206 raindrop worldline (Figure 1). In Section 7.4 we develop a set of **global**  
207 **rain coordinates** that not only labels each event—which we require of  
208 every set of global coordinates—but also yields predictions more  
209 comfortable to our intuition about the “respectable sequence” of global  
210 coordinates along a worldline. This (arbitrary) choice of global coordinates  
is purely for our own convenience: Nature doesn't care!

### 7.3 ■ THE LOCAL RAIN FRAME IN SCHWARZSCHILD COORDINATES

212 *Carry out experiments as you pass smoothly through the event horizon*

213 Does passing through the event horizon disturb local experiments that we may  
214 be conducting during this passage? To answer this questions go step by step  
215 from the raindrop to the local inertial rain frame and (in Section 7.4) from the  
216 local frame to a new global description. Start with local shell coordinates  
217 outside the horizon and use special relativity to derive local rain frame  
218 coordinates.

219 Equations (9) through (11) in Section 5.7 give us local shell coordinates  
220 expressed in Schwarzschild global coordinates:

Section 7.3 The Local Rain Frame in Schwarzschild Coordinates **7-9**

$$\Delta t_{\text{shell}} \equiv \left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \Delta t \quad (\text{from Schwarzschild}) \quad (11)$$

$$\Delta y_{\text{shell}} \equiv \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \Delta r \quad (12)$$

$$\Delta x_{\text{shell}} \equiv \bar{r} \Delta \phi \quad (13)$$

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Now use the Lorentz transformation equations of Section 1.10 to find to the local time lapse  $\Delta t_{\text{rain}}$  in an inertial frame in which the raindrop is at rest. With respect to the local shell frame, the raindrop moves with velocity  $v_{\text{rel}}$  in the  $-\Delta y_{\text{shell}}$  direction. From equation (18) in Section 6.4:

$$v_{\text{rel}} = -\left(\frac{2M}{r}\right)^{1/2} \quad \text{so} \quad \gamma_{\text{rel}} \equiv \frac{1}{(1 - v_{\text{rel}}^2)^{1/2}} = \left(1 - \frac{2M}{r}\right)^{-1/2} \quad (14)$$

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Then from the first of equations (41) in Section 1.10:

$$\begin{aligned} \Delta t_{\text{rain}} &= \gamma_{\text{rel}} (\Delta t_{\text{shell}} - v_{\text{rel}} \Delta y_{\text{shell}}) \quad (15) \\ &= \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \left[ \left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \Delta t + \left(\frac{2M}{\bar{r}}\right)^{1/2} \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \Delta r \right] \end{aligned}$$

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so that

$$\Delta t_{\text{rain}} = \Delta t + \left(\frac{2M}{\bar{r}}\right)^{1/2} \left(1 - \frac{2M}{\bar{r}}\right)^{-1} \Delta r \quad (16)$$

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And from the second of equations (41) in Section 1.10:

$$\begin{aligned} \Delta y_{\text{rain}} &= \gamma_{\text{rel}} (\Delta y_{\text{shell}} - v_{\text{rel}} \Delta t_{\text{shell}}) \quad (17) \\ &= \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \left[ \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \Delta r + \left(\frac{2M}{\bar{r}}\right)^{1/2} \left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \Delta t \right] \end{aligned}$$

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so that

$$\Delta y_{\text{rain}} = \left(1 - \frac{2M}{\bar{r}}\right)^{-1} \Delta r + \left(\frac{2M}{\bar{r}}\right)^{1/2} \Delta t \quad (18)$$

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Finally, from the third of equations (41) in Section 1.10, the shell and rain coordinates transverse to the direction of relative motion have equal values:

$$\Delta x_{\text{rain}} = \Delta x_{\text{shell}} = \bar{r} \Delta \phi \quad (19)$$

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**7-10** Chapter 7 Inside the Black Hole

235 The right sides of local rain frame equations (16) and (18) suffer from the  
 236 same disease as their parent global Schwarzschild coordinates: They blow up at  
 237 the event horizon. Nevertheless, we can use these *local* coordinate equations to  
 238 derive a new set of *global* rain coordinates that, at long last, cures this disease  
 239 and allows us to predict observations made in the local rain frame as we ride  
 240 smoothly inward across the event horizon and all the way to the singularity.

**7.4 ■ GLOBAL RAIN COORDINATES**

242 *Convert from Schwarzschild- $t$  to global rain  $T$ .*

243 Schwarzschild coordinates are completely legal, but they make us  
 244 uncomfortable because they do not describe the motion of a stone or light  
 245 flash inward across the event horizon in a finite lapse of the  $t$ -coordinate. So in  
 246 this section we find a new global coordinate—that we label the  
 247  $T$ -coordinate—which advances smoothly along the global worldline of a  
 248 descending stone, even when the stone crosses the event horizon.

249 The result is a new set of global coordinates with the old global  
 250 coordinates  $r$  and  $\phi$  but a new  $T$ -coordinate. We call this new set of  
 251 coordinates **global rain coordinates**. They are often called  
 252 **Painlevé-Gullstrand coordinates** after Paul Painlevé and Alvar Gullstrand  
 253 who independently developed them in 1921 and 1922, respectively. The present  
 254 section develops global rain (Painlevé-Gullstrand) coordinates. Section 7.5 uses  
 255 these new global coordinates to derive the **global rain metric**

Global rain coordinates or "Painlevé-Gullstrand coordinates."



256 **Objection 4.** *Hold on! How can we have two different global coordinate*  
 257 *systems for the same spacetime?*



258 For the same reason that a flat Euclidean plane can be described by either  
 259 Cartesian coordinates or polar coordinates. More than one global  
 260 coordinate system can describe the same spacetime. Indeed, an unlimited  
 261 number of global coordinate systems exist for any configuration of  
 262 mass-energy-pressure (Box 3).



263 **Objection 5.** *I am awash in arbitrary global coordinates here. What makes*  
 264 *practical sense of all this formalism? What can I hang onto and depend*  
 265 *upon?*



266 The answer comes from invariant wristwatch time and invariant ruler  
 267 distance. These direct observables are outputs of the global metric. In  
 268 John Wheeler's words, "*No phenomenon is a real phenomenon until it is*  
 269 *an observed phenomenon.*" Spacetime is effectively flat on a local patch;  
 270 on that flat patch we use the approximate global metric to derive local  
 271 coordinates that we choose to be inertial (Section 5.7). The observer  
 272 makes a measurement and expresses the result in those local

Primary goal:  
 Predict result of  
 local measurements

273 coordinates. **Every set of global coordinates must lead to the same**  
 274 **predicted result of a given measurement.** That is what makes sense of  
 275 the formalism; that is what you can hang onto and depend on.

276 The troublemaker in Schwarzschild coordinates is the  $t$ -coordinate, which  
 277 Figure 2 shows to be diseased at the event horizon. The cure is a new global  
 278 rain coordinate which we call capital  $T$ . The other two global rain coordinates,  
 279  $r$  and  $\phi$  remain the same as the corresponding Schwarzschild coordinates.

280 **Comment 3. Why conversion from Schwarzschild to global rain?**  
 281 Most often in this book we simply display a global metric with its global  
 282 coordinate system without derivation. In what follows, we make a “conversion” of  
 283 Schwarzschild coordinates to global rain coordinates that leads to the global rain  
 284 metric. Why this conversion? Why don’t we simply display the global rain metric  
 285 and its coordinate system? We do this to show that there are two ways to derive  
 286 a valid global metric. The first way is to submit the (almost) arbitrary global  
 287 coordinates to Einstein’s equations, which return the correct global metric. The  
 288 second way is simply to transform the already-validated global metric directly.  
 289 That conversion does not require Einstein’s equations.

You cannot derive  
 global coordinates from  
 local coordinates. . . .

290 To find the new global  $T$  coordinate we do something apparently illegal:  
 291 We derive it from *local* rain coordinate  $\Delta t_{\text{rain}}$  in equation (16). From the  
 292 beginning of this book we have emphatically declared that you cannot derive  
 293 global coordinates from local coordinates. Why not? Because in considering  
 294 any flat local inertial frame, we lose details of the global curvature of the  
 295 spacetime region. The local  $\Delta t_{\text{rainA}}$  from equation (16) is unique to Frame A  
 296 which depends on the average value  $\bar{r}_A$ ; adjacent Frame B has a different  
 297  $\Delta t_{\text{rainB}}$  which depends on a different average value  $\bar{r}_B$ . This leads to a  
 298 discontinuity at the boundary between these two local frames. The local frame  
 299 with its local coordinates is useful for us because it allows us to apply special  
 300 relativity to an experiment or observation made in a limited spacetime region  
 301 in globally curved spacetime. But the local frame does have this major  
 302 drawback: We cannot connect adjacent flat frames smoothly to one another in  
 303 curved spacetime. That keeps us from generalizing from local frame  
 304 coordinates to global coordinates.

. . . except when  
 local coordinates  
 lead to an exact  
 differential.

305 But there is an exception. To understand this exception, pause for a quick  
 306 tutorial in the mathematical theory of calculus (invented in the late 1600s by  
 307 both Isaac Newton and Gottfried Wilhelm Leibniz). In calculus we use  
 308 differentials, for example  $dt$  and  $dr$ . The technical name for the kind of  
 309 differential we use in this book is **exact differential**, sometimes called a  
 310 **perfect differential**. Formally, an exact differential (as contrasted with an  
 311 inexact differential or a partial differential) has the form  $dQ$  or  $dT$ , where  $Q$   
 312 and  $T$  are **differentiable functions**. What is a differentiable function? It is  
 313 simply a function whose derivative exists at every point in its domain.

Global coordinates  
 are differentiable.

314 *Question:* Are global coordinates differentiable? *Answer:* We choose global  
 315 coordinates ourselves, then submit them to Einstein’s equations which are  
 316 differential equations that return to us the global metric. So if one of our  
 317 chosen global coordinates is not differentiable, it is our own fault. *Conclusion:*

## 7-12 Chapter 7 Inside the Black Hole

318 For a global coordinate to be useful in general relativity, it must be  
 319 differentiable and thus have an exact differential (except at a physical  
 320 singularity).

321 So we purposely choose a set of global coordinates that are differentiable.  
 322 If we then make a transformation between sets of global coordinates, the  
 323 transformed global coordinate is also a differentiable function and therefore  
 324 has an exact differential.

325 A close look at equation (16) shows that we can turn it into an exact  
 326 differential, as follows:

$$\lim_{\Delta t \rightarrow 0} \Delta t_{\text{rain}} = dt + \frac{df(r)}{dr} dr = d[t + f(r)] \equiv dT \quad (20)$$

327 where, from (16),  $T = t + f(r)$  is a differentiable function with

$$\frac{df(r)}{dr} = \left(\frac{2M}{r}\right)^{1/2} \left(1 - \frac{2M}{r}\right)^{-1} \quad (21)$$

328 Equation (20) is immensely significant. It tells us that there is an exact  
 329 differential of what we call a new global  $T$ -coordinate. This means that  $T$  is a  
 330 differentiable function of global rain coordinates. This coordinate  $T = t + f(r)$   
 331 is global because (a) Schwarzschild's  $t$  and  $r$  (along with  $\phi$ ) already label every  
 332 event for  $r > 0$ , and (b) Schwarzschild coordinates and metric satisfy  
 333 Einstein's equations. From (20) and (21):

$$dT = dt + d[f(r)] = dt + \left(\frac{2M}{r}\right)^{1/2} \left(1 - \frac{2M}{r}\right)^{-1} dr \quad (22)$$

334 To validate the new global rain coordinate  $T$ , we need to express it in the  
 335 already-validated Schwarzschild coordinates. To start this process integrate  
 336 (21):

$$f(r) \equiv \int_0^r \left(\frac{2M}{r}\right)^{1/2} \left(1 - \frac{2M}{r}\right)^{-1} dr = -(2M)^{1/2} \int_0^r \frac{r^{1/2} dr}{2M - r} \quad (23)$$

337 A table of integrals helps us to integrate the right side of (23). The result is:

$$f(r) = 4M \left(\frac{r}{2M}\right)^{1/2} - 2M \ln \left| \frac{1 + (2M/r)^{1/2}}{1 - (2M/r)^{1/2}} \right| \quad (24)$$

338 Finally, from (22) and (24):

$$T = t + f(r) = t + 4M \left(\frac{r}{2M}\right)^{1/2} - 2M \ln \left| \frac{1 + (2M/r)^{1/2}}{1 - (2M/r)^{1/2}} \right| \quad (25)$$

339 where we have arbitrarily set some constants of integration equal to zero.  
 340 Equation (25) is the final proof that the new global rain  $T$  is valid, since it  
 341 transforms directly from valid Schwarzschild  $t$  and  $r$  global coordinates.  
 342

343 **?** **Objection 6.** *I still think that your definition of  $T$  derived from a local*  
 344 *coordinate increment  $\Delta t_{\text{rain}}$  in equations (22) through (25) violates your*  
 345 *own prohibition in Section 5.7 of a local-to-global transformation.*

346 **!** No, we do *not* transform from local rain coordinates to global rain  
 347 coordinates, which would be illegal. Instead, we start with the global  
 348 Schwarzschild expression for  $\Delta t_{\text{rain}}$  in (16), take its differential limit in  
 349 (20), which converts it to the global rain differential  $dT$ . The key step in  
 350 that conversion is to recognize that equation (20) contains an exact  
 351 differential in global Schwarzschild coordinates. Note that this does not  
 352 happen for shell time in Schwarzschild coordinates:

$$\lim_{\Delta t \rightarrow 0} \Delta t_{\text{shell}} \rightarrow \left(1 - \frac{2M}{r}\right)^{1/2} dt \quad (26)$$

353 The right side of (26) cannot be expressed as  $d[T(r, t)]$ , the exact  
 354 differential of a coordinate  $T(r, t)$ .

355 The exact differential in (20) allows us to complete the derivation and  
 356 validation of the global rain  $T$ -coordinate in equation (25). Is this magic?  
 357 No, but it does require sophisticated use of calculus.

358 **QUERY 4. Differentiate  $T$  (Optional)**

Take the differential of (25) to show that the result is (22).

Coordinate transformation to global rain coordinates

Payoff: "Intuitive" global rain metric

362 We worked in Schwarzschild coordinates to find a function  
 363  $T(t, r) = t + f(r)$  whose differential matches local rain frame time. We can free  
 364 this  $T$  of its origin and regard it as a new label for each spacetime event. But  
 365 after that addition there is no need to use both  $T$  and  $t$  to label events. We  
 366 can now eliminate  $t$  from the list, because we can always find it if we know the  
 367 value of  $T$  (along with  $r$ ) through  $t = T - f(r)$ . So we perform the **coordinate**  
 368 **transformation** (25), in which we replace one set of global coordinates  
 369  $(t, r, \phi)$  with a new set of global coordinates  $(T, r, \phi)$ . We call the result **global**  
 370 **rain coordinates**—or historically, **Painlevé-Gullstrand coordinates**.

371 Why go to all this bother? In order to derive (in the next section) a global  
 372 rain metric that describes the motion of a stone or light flash across the event  
 373 horizon in a manner more comfortable to our intuition. The global rain metric  
 374 encourages our unlimited, free exploration of *all* spacetime outside, at, and  
 375 inside the event horizon.

376 An immediate payoff of global rain coordinates is local shell coordinates  
 377 expressed in global rain coordinates (Box 2).

378 **Comment 4. Global  $T$  is defined everywhere.**

379 Although we started from local shell coordinates which exist only for  $r > 2M$ ,  
 380 our new  $T$  coordinate is defined everywhere, even at and inside the event

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**Box 2. Local shell coordinates expressed in global rain coordinates**

We now derive local shell coordinates as functions of global rain coordinates. Equation (11) gives  $\Delta t_{\text{shell}}$  as a function of the Schwarzschild  $t$ -coordinate increment:

$$\Delta t_{\text{shell}} \equiv \left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \Delta t \quad (27)$$

The Schwarzschild  $t$ -coordinate does increase without limit along the worldline of a stone that approaches  $r = 2M$ , but shells exist only outside this event horizon, where (27) is well-behaved. Write equation (22) in approximate form:

$$\Delta t \approx \Delta T - \left(\frac{2M}{\bar{r}}\right)^{1/2} \left(1 - \frac{2M}{\bar{r}}\right)^{-1} \Delta r \quad (28)$$

Substitute (28) into (27) to yield:

$$\Delta t_{\text{shell}} \equiv \left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \Delta T - \left(\frac{2M}{\bar{r}}\right)^{1/2} \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \Delta r \quad (29)$$

Equations for  $\Delta y_{\text{shell}}$  and  $\Delta x_{\text{shell}}$  do not depend on  $\Delta t$ , so we copy them directly from (12) and (13).

$$\Delta y_{\text{shell}} \equiv \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \Delta r \quad (30)$$

$$\Delta x_{\text{shell}} \equiv \bar{r} \Delta \phi \quad (31)$$

*Note:* Expressions for  $\Delta t_{\text{shell}}$  and  $\Delta y_{\text{shell}}$  are real only for  $r > 2M$ , consistent with the conclusion that no shell can exist inside the event horizon.

381 horizon, because its defining equation (25) contains only global Schwarzschild  
 382 coordinates, which span all of spacetime.  
 383 Should we worry that the last term on the right side of (25) blows up as  
 384  $r \rightarrow 2M$ ? No, because Schwarzschild  $t$  in (25) blows up in the opposite  
 385 direction in such a way that  $T$  is continuous across  $r = 2M$ . Our only legitimate  
 386 worry is continuity of the resulting metric, which (32). Section 7.5 shows this  
 387 metric to be continuous across  $r = 2M$ .

**7.5 THE GLOBAL RAIN METRIC**

389 *Move inward across the event horizon.*

390 Section 7.4 created a global  $T$ -coordinate, validated it by direct transformation  
 391 from already-approved global Schwarzschild coordinates, and installed it in a  
 392 new set of global rain coordinates. To derive the **global rain metric**, solve  
 393 the  $dT$ -transformation (22) for  $dt$ , substitute the result into the Schwarzschild  
 394 metric, and collect terms to yield:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dT^2 - 2 \left(\frac{2M}{r}\right)^{1/2} dT dr - dr^2 - r^2 d\phi^2 \quad (32)$$

$-\infty < T < +\infty, \quad 0 < r < \infty, \quad 0 \leq \phi < 2\pi$  (global rain metric)

395  
 396 Metric (32), with its connectedness (topology), provides a complete  
 397 description of spacetime around a non-spinning black hole, just as the

398 Schwarzschild metric does. In addition, all Schwarzschild-based difficulties  
 399 with worldlines that cross the event horizon disappear.

**QUERY 5. Global rain metric**

Substitute  $dt$  from (22) into the Schwarzschild metric to verify the global rain metric (32).

**QUERY 6. Flat spacetime far from the black hole**

Show that as  $r \rightarrow \infty$  metric (32) becomes the metric for flat spacetime in global coordinates  $T, r, \phi$ .

**Use global  
rain metric  
from now on.**

408 **Comment 5. USE THE GLOBAL RAIN METRIC FROM NOW ON.**  
 409 **From now on in this book we use the global rain metric—and expressions**  
 410 **derived from it—to analyze events in the vicinity of the non-spinning black**  
 411 **hole.**

**Principle of  
General Covariance**

412 *Question:* When all is said and done, which set of coordinates is the  
 413 “correct” one for the non-spinning black hole: Schwarzschild coordinates or  
 414 rain coordinates or some other set of global coordinates? *Answer: Every global*  
 415 *coordinate system is valid provided it is either (a) submitted to Einstein’s*  
 416 *equations, which return a global metric or (b) transformed from an*  
 417 *already-validated global coordinate system.* This reflects a fundamental  
 418 principle of general relativity with the awkward technical name **Principle of**  
 419 **General Covariance**. In this book we repeat over and over again that no  
 420 single observer measures map coordinates directly (back cover). To overlook  
 421 the Principle of General Covariance by attaching physical meaning to global  
 422 coordinates is wrong and sets oneself up to make fundamental errors in the  
 423 predictions of general relativity.

**EVERY GENERATION MUST LEARN ANEW**

424 *One of the fundamental principles of general relativity (the principle of*  
 425 *general covariance) states that all [global] spacetime coordinate systems*  
 426 *are equally valid for the description of nature and that metrics that are*  
 427 *related by a coordinate transformation are physically equivalent. This*  
 428 *principle sounds simple enough, but one repeatedly finds in the literature*  
 429 *arguments that amount to advocacy for the interpretation based on one*  
 430 *set of [global] coordinates to the exclusion of the interpretation that is*  
 431 *natural when using another set of [global] coordinates for the same set of*  
 432 *events. It has been said that each generation of physicists must learn*  
 433 *anew (usually the hard way) the meaning of Einstein’s postulate of*  
 434 *general covariance.*

436

—Richard C. Cook and M. Shane Burns



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**Box 3. An Unlimited Number of Global Coordinate Systems**

General relativity uses two methods to derive a global metric: *Method 1*: Choose an arbitrary set of global coordinates and submit them to Einstein’s equations, which return the global metric expressed in those coordinates. Karl Schwarzschild did this to derive the Schwarzschild metric. *Method 2*: Transform an already-validated set of global coordinates to another set, then substitute the new coordinates into the already-verified metric of Method 1 to yield a metric in the new global coordinates. In the present chapter we use Method 2 to go from the Schwarzschild metric to the global rain metric.

How many different global coordinate systems are there for the non-spinning black hole? An unlimited number! Any one-to-one transformation from one valid set of global coordinates to another set of global coordinates makes the second set valid as well, provided it meets the usual criteria of uniqueness and smoothness (Section 5.8).

As a simple case, make the transformation:

$$T^\dagger \equiv KT \tag{33}$$

where  $K$  is any real number—or even a simple minus sign! With this substitution, global rain metric (32) becomes

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) \left(\frac{dT^\dagger}{K}\right)^2 - 2\left(\frac{2M}{r}\right)^{1/2} \frac{dT^\dagger}{K} dr - dr^2 - r^2 d\phi^2 \tag{34}$$

If  $K$  is negative, then global  $T^\dagger$  runs backward along the worldline of every stone. This does not bother us; Schwarzschild  $t$  does the same along some worldlines (Figure 2). Equations for local shell and local rain coordinates are similarly modified, which does not change any prediction or the result of any measurement in these local frames.

In a similar manner we can let  $r^\dagger \equiv Qr$  and/or  $\phi^\dagger \equiv -\phi$ . You can write down new metrics with any one or any pair of these new coordinates. These changes may seem trivial, but they are not. For example, choose  $K = Q = M^{-1}$ . The result is the variables  $T/M$  and  $r/M$ , so-called **unitless coordinates**, in which curves are plotted in many figures of this book. Plots with unitless coordinates are correct for every non-spinning black hole, independent of its mass  $M$ .

*Result*: Global rain metric (32) is only one of an unlimited set of alternative, equally-valid global metrics for the non-spinning black hole, each one expressed in a different global coordinate system. The examples in this box may be no more useful than the original global rain metric, but there are other global metrics that highlight some special property of the non-spinning black hole. Look up **Eddington-Finkelstein coordinates** and **Kruskal-Szekeres coordinates** and their metrics.

437 *In particular*: Fixation on an interpretation based on one set of arbitrary  
 438 coordinates can lead to the mistaken belief that global coordinate differences  
 439 correspond to measurable quantities (Section 2.7).

Reminder of  
Einstein’s error

440 **Comment 6. You know some relativity that Einstein missed!**  
 441 Chapter 5 says that Einstein took seven years to appreciate that global  
 442 coordinate separations have no measurable meaning. But even then he did not  
 443 fully understand this fundamental idea. At a Paris conference in 1922—seven  
 444 years after he completed general relativity—Einstein worried about what would  
 445 happen at a location where the denominator of the  $dr^2$  term in the  
 446 Schwarzschild metric, namely  $(1 - 2M/r)$ , goes to zero. He said it would be “an  
 447 unimaginable disaster for the theory; and it is very difficult to say *a priori* what  
 448 would occur physically, because the theory would cease to apply.” (In 1933 the  
 449 Belgian priest Georges Lemaître recognized that the apparent singularity in  
 450 Schwarzschild coordinates at  $r = 2M$  is “fictional.”) Einstein was also baffled by  
 451 the  $dTdr$  cross term in the global rain metric (32) presented to him by Paul  
 452 Painlevé and rejected Painlevé’s solution out of hand. This led to the eclipse of  
 453 this metric for decades.

454 *Congratulations*: You know some relativity that Einstein did not understand!

**QUERY 7. Map energy in global rain coordinates**

- A. Use the global rain metric and the Principle of Maximal Aging to derive the map energy of a stone in global rain coordinates. You can model the procedure on the derivation of the Schwarzschild coordinate expression for  $E/m$  in Section 6.2, but will need to alter some of the notation. Demonstrate this result:

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right) \frac{dT}{d\tau} - \left(\frac{2M}{r}\right)^{1/2} \frac{dr}{d\tau} \quad (\text{global rain coordinates}) \quad (35)$$

- B. Substitute for  $dT$  from (22) and show that the result yields the Schwarzschild expression for map energy, equation (8) in Section 6.2.
- C. Show that the map energy equation (35) applies to a stone in orbit around the black hole, not just to one that moves along the  $r$ -coordinate line.
- D. Find an expression for  $dr/dT$  of the raindrop in rain map coordinates? Start with the first line of (5) and multiply both sides by  $dt/dT$  from (22). Show that the result is:

$$\frac{dr}{dT} = \frac{dr}{dt} \frac{dt}{dT} = -\left(\frac{2M}{r}\right)^{1/2} \quad (\text{raindrop}) \quad (36)$$

- E. Show that the map worldline of a raindrop is given by the equation:

$$T - T_A = \frac{4M}{3} \left[ \left(\frac{r_A}{2M}\right)^{3/2} - \left(\frac{r}{2M}\right)^{3/2} \right] \quad (\text{raindrop}) \quad (37)$$

where  $(r_A, T_A)$  locates the initial event on the plotted curve (Figure 3).



471  
472

**Objection 7.** Wait! The right side of (37) is identical to the right side of (2). The two equations are both correct if and only if the left sides are equal:

$$T - T_A = \tau_{\text{raindrop}}[r_A \rightarrow r] \quad (38)$$

473  
474  
475

But these two quantities are completely different—apples and oranges! The left side is the difference in a global coordinate, while the right side is the lapse of wristwatch time of a particular falling stone.



476  
477  
478  
479  
480  
481

For our own convenience, we chose global rain coordinates so that equation (38) is valid. This is playing with fire, of course, because it is dangerous to assume that any given global coordinate corresponds to a measurable quantity (Section 5.8). But we are adults now, able to see the pitfalls of mature life. The goal, as always, is correct prediction of measurements and observations.

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482 Raindrop equation (36) for  $dr/dT$  looks quite different from raindrop  
 483 equation (5) for  $dr/dt$ , but the two must predict the same raindrop wristwatch  
 484 time from  $r_A$  to a smaller  $r$ -coordinate given by (2). Let's check this: For a  
 485 raindrop, set  $E/m = 1$  in (35) and multiply through by  $d\tau_{\text{raindrop}}$ :

$$d\tau_{\text{raindrop}} = \left(1 - \frac{2M}{r}\right) dT - \left(\frac{2M}{r}\right)^{1/2} dr \quad (39)$$

486 Solve (36) for  $dT$  and substitute into (39).

$$\begin{aligned} d\tau_{\text{raindrop}} &= -\left(1 - \frac{2M}{r}\right) \left(\frac{2M}{r}\right)^{-1/2} dr - \left(\frac{2M}{r}\right)^{1/2} dr \quad (40) \\ &= -\left(\frac{r}{2M}\right)^{1/2} dr \end{aligned}$$

487 which is the same as equation (1), so its integral from  $r_A$  to  $r$  must be (2), as  
 488 required. This is an example of an important property of different global  
 489 metrics: *Every global metric must predict the same result of a given*  
 490 *measurement or observation* (Objection 5).

491 Box 4 carries out a Lorentz transformation from local shell coordinates in  
 492 Box 2 to local rain coordinates, then verifies that local rain coordinates are  
 493 valid everywhere, not just outside the event horizon.

494 **Comment 7. An observer passes through a sequence of local frames.**

495 The rain observer rides on a raindrop (Definition 4, Section 7.7). In curved  
 496 spacetime, local inertial frames are limited in both space and time. During her  
 497 fall, the rain observer passes through a series of local rain frames as shown in  
 498 Figure 3. Equations (42) through (44) contain an  $\bar{r}$ , assumed to have the same  
 499 value everywhere in that local frame. Although absent from the equations, similar  
 500 average  $\bar{T}$  and  $\bar{\phi}$  are implied by all three local rain coordinate equations. *Result:*  
 501 Each rain observer passes through a sequence of local inertial frames. Similar  
 502 statements also apply to, and may seem more natural for, local shell frames with  
 503 local coordinates (29) through (31).

504 Box 4 derives *local* rain coordinates expressed in *global* rain coordinates.  
 505 This simplifies local rain coordinates compared with those expressed in  
 506 Schwarzschild coordinates in equations (16) through (19).

507 Figure 3 displays several rain observer worldlines on the  $[r, T]$  slice. We  
 508 surround one worldline with a worldtube—shown in cross section—that  
 509 contains local rain frames through which this rain observer passes in sequence.  
 510 With  $\Delta t_{\text{rain}} = \Delta T = 0$ , equation (43) tells us that  $\Delta y_{\text{rain}}$  coordinate lines are  
 511 horizontal in this figure. Finally,  $\Delta x_{\text{rain}}$  coordinate lines, which are  
 512 perpendicular to both  $\Delta t_{\text{rain}}$  and  $\Delta y_{\text{rain}}$  coordinate lines, project outward,  
 513 perpendicular to the page in Figure 3.

514 We could cover the worldtube in Figure 3 with adjacent or overlapping  
 515 local rain frames. The resulting figure would be analogous to Figure 5 in  
 516 Section 2.2, which places overlapping local flat maps along the spatial path  
 517 from Amsterdam to Vladivostok along Earth's curved surface.

### Box 4. Local rain coordinates expressed in global rain coordinates

Apply the Lorentz transformation to local shell coordinates in Box 2 to derive local rain coordinates. Relative velocity in the Lorentz transformation lies along the common  $\Delta y_{\text{shell}}$  and  $\Delta y_{\text{rain}}$  line. With this change, Lorentz transformation equations of Section 1.10 become:

$$\Delta t_{\text{rain}} = \gamma_{\text{rel}} (\Delta t_{\text{shell}} - v_{\text{rel}} \Delta y_{\text{shell}}) \quad (41)$$

$$\Delta y_{\text{rain}} = \gamma_{\text{rel}} (\Delta y_{\text{shell}} - v_{\text{rel}} \Delta t_{\text{shell}})$$

$$\Delta x_{\text{rain}} = \Delta x_{\text{shell}}$$

Substitute  $v_{\text{rel}}$  and  $\gamma_{\text{rel}}$  from (14), along with local shell coordinates from Box 2, into equations (41) to obtain expressions for local rain coordinates as functions of global rain coordinates:

$$\Delta t_{\text{rain}} \equiv \Delta T \quad (42)$$

$$\Delta y_{\text{rain}} \equiv \Delta r + \left( \frac{2M}{\bar{r}} \right)^{1/2} \Delta T \quad (43)$$

$$\Delta x_{\text{rain}} \equiv \bar{r} \Delta \phi \quad (44)$$

Coefficients on the right sides of these equations remain real inside the event horizon, so local rain coordinates are valid there (Figure 3).

Wait! How can we justify our derivation of rain coordinates from local shell coordinates, which are valid only outside the event horizon? To do so, we need to show that local rain coordinates lead back to the global rain metric, which is valid everywhere outside the singularity.

$$\Delta \tau^2 \approx \Delta t_{\text{rain}}^2 - \Delta y_{\text{rain}}^2 - \Delta x_{\text{rain}}^2 \quad (45)$$

Substitute into (45) from (42) through (44):

$$\Delta \tau^2 \approx \Delta T^2 - \left[ \Delta r + \left( \frac{2M}{\bar{r}} \right)^{1/2} \Delta T \right]^2 - \bar{r}^2 \Delta \phi^2 \quad (46)$$

Multiply out:

$$\Delta \tau^2 \approx \left( 1 - \frac{2M}{\bar{r}} \right) \Delta T^2 - 2 \left( \frac{2M}{\bar{r}} \right)^{1/2} \Delta T \Delta r - \Delta r^2 - \bar{r}^2 \Delta \phi^2 \quad (47)$$

In the calculus limit, equation (47) becomes the global rain metric (32). The global rain metric is valid everywhere down to the singularity; therefore local rain coordinates can be constructed down to the singularity.

518 Box 5 derives the global rain embedding diagram, Figure 4, from the  
519 global rain metric (32) and compares it with the embedding diagram for  
520 Schwarzschild coordinates.

## 7.6 ■ TETRAD FORMS OF THE GLOBAL RAIN METRIC

522 *A difference of squares hides the cross term.*

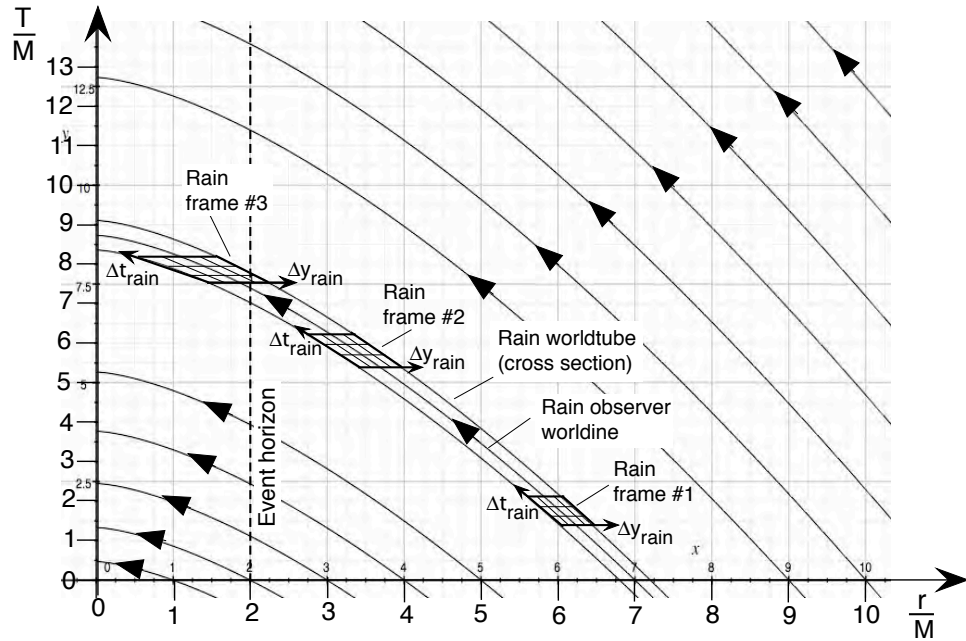
523 Global rain metric (32) has a cross term. The metric for any local inertial  
524 frame derived from this global metric does not have a cross term. For example:

$$\Delta \tau^2 \approx \Delta t_{\text{shell}}^2 - \Delta y_{\text{shell}}^2 - \Delta x_{\text{shell}}^2 \quad (49)$$

$$\Delta \tau^2 \approx \Delta t_{\text{rain}}^2 - \Delta y_{\text{rain}}^2 - \Delta x_{\text{rain}}^2 \quad (50)$$

525 Why this difference between global and local metrics? Can we use a set of  
526 local inertial coordinates to create a form of the global metric that consists of  
527 the sum and difference of squares? Try it! From expressions for local shell and

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**FIGURE 3** Raindrop worldlines plotted on an  $[r, T]$  slice. Note that these worldlines are continuous through the event horizon (compare Figure 2). All these worldlines have the same shape and are simply displaced vertically with respect to one another. Around one of these worldlines we construct a worldtube (shown in cross section on this slice) that bounds local rain frames through which that rain observer passes.

528 rain coordinates in Boxes 2 and 4, respectively, simply write down two  
 529 differential forms of the global metric. From local shell coordinates (Box 2):

$$d\tau^2 = \left[ \left(1 - \frac{2M}{r}\right)^{1/2} dT - \left(\frac{2M}{r}\right)^{1/2} \left(1 - \frac{2M}{r}\right)^{-1/2} dr \right]^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\phi^2 \quad (\text{global rain metric})$$

$$-\infty < T < +\infty, \quad 0 < r < \infty, \quad 0 \leq \phi \leq 2\pi$$

530

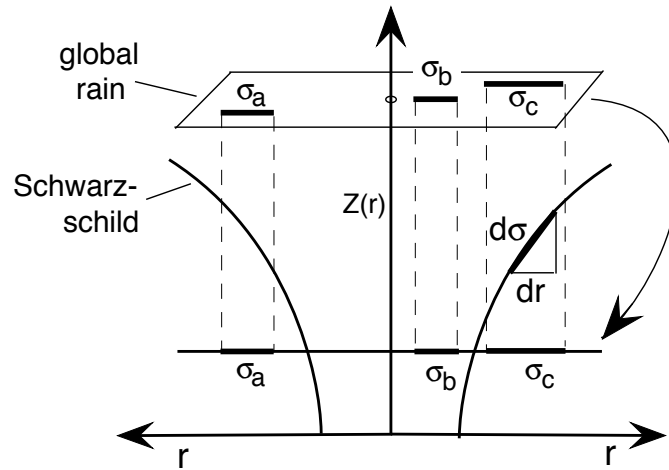
531 And from local rain coordinates (Box 4):

$$d\tau^2 = dT^2 - \left[ dr + \left(\frac{2M}{r}\right)^{1/2} dT \right]^2 - r^2 d\phi^2 \quad (\text{global rain metric})$$

$$-\infty < T < +\infty, \quad 0 < r < \infty, \quad 0 \leq \phi \leq 2\pi$$

532

Section 7.6 Tetrad Forms of the Global Rain Metric 7-21



**FIGURE 4** Figure for Box 5. Compare the embedding diagram outside the event horizon for Schwarzschild coordinates (the funnel) in Figures 11 through 13 in Section 3.9 with the flat embedding diagram of global rain coordinates (the flat surface shown in perspective across the top). The function  $Z(r)$  is the fictional dimension we add in order to visualize these surfaces.

**Box 5. Embedding diagrams for Schwarzschild and global rain coordinates.**

Set  $dT = 0$  in global rain metric (32), which then retains terms that include only  $r$  and  $\phi$ :

$$d\sigma^2 = ds^2 = dr^2 + r^2 d\phi^2 \quad (dT = 0) \quad (48)$$

Surprise! The differential ruler distance  $d\sigma$  obeys Euclidean flat-space geometry, which leads to the flat embedding diagram at the top of Figure 4 (point at  $r = 0$  excluded). Because the global rain embedding diagram is flat, we can simply sum increments  $d\sigma$  to draw arbitrary lines or curves with measured lengths  $\sigma_a$ ,  $\sigma_b$ , and  $\sigma_c$  (for simplicity, drawn as parallel straight lines in Figure 4).

Figure 4 also repeats the embedding diagram outside the event horizon in Schwarzschild global coordinates from Figures 11 through 13 in Section 3.9.

Outside the event horizon both embedding diagrams are valid for what they describe. And the flat global rain embedding diagram is valid inside the event horizon as well.

What's going on here? Is space flat or funnel-shaped around this black hole? *That depends on our choice of global coordinates!* Spacetime as a unity is curved; but Nature does not care how we share the description of spacetime curvature among the terms of the global metric. In global rain coordinates the  $dT^2$  and  $dTdr$  terms describe spacetime curvature, which leaves the  $[r, \phi]$  embedding diagram to show flat space. In contrast, for the Schwarzschild metric the  $dt^2$  term and the  $dr^2$  term share the description of spacetime curvature, which yields a funnel outside the event horizon on the  $[r, \phi]$  embedding diagram. Each embedding diagram and global light cone diagram is a child of our (arbitrary!) choice of global coordinates in which they are expressed.

Recall Herman Minkowski's declaration (Section 2.7): "Henceforth space by itself and time by itself are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality." Each global metric displays that union in a different way.

533 Are (51) and (52) valid global metrics? Yes! In Query 8 you multiply out  
 534 these global metrics to show that they are algebraically equivalent to the  
 535 original global rain metric (32).

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**QUERY 8. The same global metric**

Expand the right side of (51) and the right side of (52). Show that in both cases the result is equal to the right side of the original global rain metric (32).

541 *Conclusion:* All three forms of the global rain metric, (32), (51), and (52)  
 542 are simply algebraic rearrangements of one another. So what? Why bother?  
 543 Here's why: Suppose we are first given either metric (51) or metric (52). In  
 544 that case we can immediately write down the expressions  $\Delta t_{\text{frame}}$ ,  $\Delta y_{\text{frame}}$ ,  
 545 and  $\Delta x_{\text{frame}}$  for *some* local inertial frame. (We may not know right away *which*  
 546 local inertial frame it is.)

547 A metric of the form (51) or (52) is called the **tetrad form** of the global  
 548 metric. "Tetra" means "four," in this case the four dimensions of spacetime. A  
 549 tetrad form rearranges the global metric in a form more useful to us.

550 **DEFINITION 1. A tetrad form of a global metric**

551 A **tetrad form of a global metric** consists of a sum and difference of  
 552 squares, with no additional terms.

553 **DEFINITION 2. Tetrad**

554 A **tetrad** is a set of four differential expressions, each of which is  
 555 squared in a "tetrad form" of a global metric.

556 *Example:* Equation (52) is the global rain metric in the tetrad form that leads  
 557 to the local rain frame coordinates in Box 4.

558 ?

559 **Objection 8.** *You said the tetrad consists of four differential expressions. I see only three.*

560 !

561 You're right. In most of this book we use only two global space dimensions,  
 562 those on a slice through the center of the black hole. A third global space  
 563 dimension would add a fourth  $\Delta z_{\text{rain}}$  component to the tetrad. We retain  
 the professional terminology *tetrad* in spite of our simplification.

564 **Comment 8. Tetrad as link**

565 A tetrad is the link between a global metric and a local inertial frame. To specify  
 566 a particular tetrad, give both the local inertial frame and the global metric. The  
 567 local inertial frame stretches differentials  $d$  in the global metric to deltas  $\Delta$  in the  
 568 local inertial frame. This stretch creates elbow room to make measurements in  
 569 the local inertial frame.

570 **The global metric in tetrad form immediately translates into**  
 571 **expressions for local inertial coordinates (Box 6). The remainder of**  
 572 **this book will make primary use of such global tetrad metrics.**

**Box 6. A Brief History of Tetrads**

The concept of a tetrad originates from the “repère mobile” (*moving frame*), introduced by Élie Cartan in the 1930s. Cartan showed that a sequence of local inertial coordinate systems, grouped along a set of curves such as worldlines, can provide a complete global description of a curved spacetime. He did so by introducing new calculus concepts on curved spaces, which extended the foundations of differential geometry laid by Bernhard Riemann in 1854.

Cartan’s moving-frame theory was incomprehensible to Einstein, but later physicists found it useful and even necessary to study elementary particle physics in curved spacetime.

In this book we simplify the moving frame, or tetrad, to its most basic element: a set of local inertial frames in which motion is described using local inertial coordinates, with each frame and each local coordinate system related to the global coordinates provided by the metric. The tetrad is the bridge between the global metric and the local inertial metric in which we carry out all measurements and observations.

Because this metric is the sum and difference of the squares of the tetrad—such as in equations (51) and (52)—general relativists sometimes call the tetrad the **square root of the metric**.

**7.7 ■ RAIN WORLDLINES OF LIGHT**

574 *The light flash is ingoing or outgoing—or “outgoing.”*

575 In the present chapter we consider only *r*-motions, motions that can be either  
 576 ingoing or outgoing along *r*-coordinate lines. (Chapter 11 analyzes the general  
 577 motion of light in global coordinates.) The *r*-motion of light is easily derived  
 578 from global rain metric (53), which you show in Query 8 to be equivalent to  
 579 global rain metric (32).

580

**QUERY 9. Light <sup>7.7.1</sup> motion in global rain coordinates**

A. Multiply out the right side of (53) to show that it is equivalent to the global rain metric (32):

$$d\tau^2 = - \left[ dr + \left\{ 1 + \left( \frac{2M}{r} \right)^{1/2} \right\} dT \right] \left[ dr - \left\{ 1 - \left( \frac{2M}{r} \right)^{1/2} \right\} dT \right] - r^2 d\phi^2 \tag{53}$$

(global rain metric)

583

B. For light ( $d\tau = 0$ ) that moves along the *r*-coordinate line ( $d\phi = 0$ ), show that (53) has two solutions for the rain map velocity of light, which are summarized by equation (54):

$$\frac{dr}{dT} = - \left( \frac{2M}{r} \right)^{1/2} \pm 1 \quad (\text{light flash that moves along the } r\text{-coordinate line}) \tag{54}$$

( − = incoming light; + = outgoing or “outgoing” light)

C. Look separately at each element of the plus-or-minus sign in (54). Show that the solution with the lower sign (−) describes an *incoming* flash (light moving inward along the *r*-coordinate line). Then show that the solution with the upper sign (+) describes an *outgoing* flash (light moving outward along the *r*-coordinate line) outside the event horizon, but an “*outgoing*” flash inside the event horizon (Definition 3).



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**DEFINITION 3. “Outgoing” light flash.**

The “outgoing” light flash—with quotes—is a flash inside the event horizon whose  $r$ -motion is described by (54) with the plus sign. Inside the event horizon  $dr$  is negative as  $T$  advances (positive  $dT$ ), so that even the “outgoing” light flash moves to smaller  $r$ -coordinate. The light cone diagram Figure 5 shows this, and Figure 6 displays longer worldlines of light flashes emitted sequentially by a plunging raindrop. Compare the global rain worldline in Figure 6 with the Schwarzschild worldlines in Figure 2.

We want to plot worldlines of incoming and outgoing (and “outgoing”) light flashes. Rewrite (54) to read

$$dT = \frac{dr}{-\left(\frac{2M}{r}\right)^{1/2} \pm 1} = \frac{r^{1/2} dr}{-(2M)^{1/2} \pm r^{1/2}} \quad (r\text{-moving flash}) \quad (55)$$

We carry the plus-or-minus sign along as we integrate to find expressions for light that moves in either  $r$ -direction. Make the substitution:

$$u \equiv -(2M)^{1/2} \pm r^{1/2} \quad (56)$$

From (56),

$$r^{1/2} = \pm[u + (2M)^{1/2}] \quad \text{and} \quad dr = +2[u + (2M)^{1/2}] du \quad (57)$$

With these substitutions, equation (55) becomes

$$\begin{aligned} dT &= \pm 2 \frac{[u + (2M)^{1/2}]^2 du}{u} \quad (r\text{-moving flash}) \quad (58) \\ &= \pm 2 \left[ u + 2(2M)^{1/2} + \frac{2M}{u} \right] du \end{aligned}$$

Integrate the second line of (58) from an initial  $u_0$  to a final  $u$ . The result is:

$$\pm(T - T_0) = u^2 - u_0^2 + 4(2M)^{1/2}(u - u_0) + 4M \ln \left| \frac{u}{u_0} \right| \quad (r\text{-moving flash}) \quad (59)$$

To restore global rain coordinates in (59), reverse the substitution in (56). (*Hint:* To save time—and your sanity—replace  $\pm$  in (59) with a symbol such as  $Q$ .) There are two cases: *First case:* an incoming flash from a larger  $r$ -coordinate  $r_0 = r_A$  at coordinate  $T_0 = T_A$  to a lower  $r$  at coordinate  $T$ . For

Section 7.7 Rain Worldlines of Light **7-25**

Incoming flash 613 this incoming flash, take the lower minus signs in (56) and (59). Multiply both  
614 sides of the result by minus one to obtain:

$$T - T_A = (r_A - r) - 4M \left[ \left( \frac{r_A}{2M} \right)^{1/2} - \left( \frac{r}{2M} \right)^{1/2} \right] \tag{60}$$

$$+ 4M \ln \left[ \frac{1 + (r_A/2M)^{1/2}}{1 + (r/2M)^{1/2}} \right] \quad (\text{incoming flash})$$

**QUERY 10. Horizon-to-crunch global  $T$ -coordinate lapse for light**

- A. Verify that for  $r = r_A$  in (60), the elapsed global  $T$ -coordinate  $T - T_A$  is zero, as it must be for light. 618
- B. Show that from the event horizon  $r_A = 2M$  to the crunch point  $r = 0$ , the elapsed  $T$ -coordinate  $T - T_A = 0.773M$ . 620
- C. Compare the result of Item B with the event-horizon-to-crunch wristwatch time  $\tau_{\text{raindrop}} = (4/3)M$  in equation (3). Why is the result for a light flash in (60) less than the result for the raindrop? Are the plots in Figure 6 consistent with this inequality? 624

Outgoing flash 625 *Second case:* the outgoing and “outgoing” light flashes from an initial  
626  $r$ -coordinate  $r_0 = r_L$  at coordinate  $T_L$  to a final  $r$  at coordinate  $T$ . In this case  
627 we must take the upper, plus signs in (56) and (59). The result is:

$$T - T_L = (r - r_L) + 4M \left[ \left( \frac{r}{2M} \right)^{1/2} - \left( \frac{r_L}{2M} \right)^{1/2} \right] \tag{61}$$

$$+ 4M \ln \left[ \frac{1 - (r/2M)^{1/2}}{1 - (r_L/2M)^{1/2}} \right] \quad (\text{outgoing flash})$$

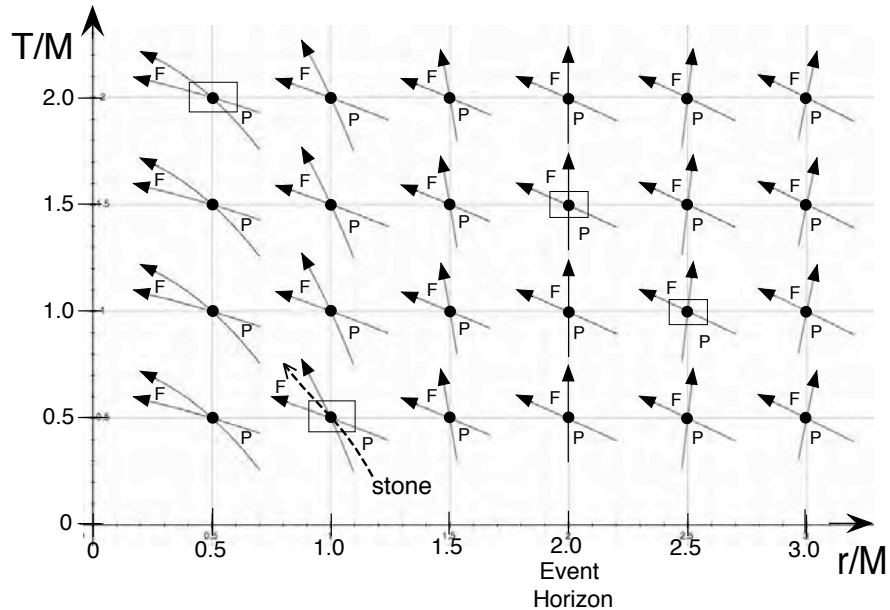
Inside event horizon, 628 What does equation (61) predict when  $r_L < 2M$ ? Worldlines of light that  
 “outward” means 629 sprout upward from the raindrop worldline in Figure 6 show that after the  
 worldline with  $dr < 0$ . 630 diver falls through the event horizon, even the “outward” flash moves to  
631 smaller  $r$  in global rain coordinates.

632 Figure 5 uses equations (60) and (61) to plot light cones for a selection of  
633 events inside and outside of the event horizon.

**QUERY 11. Motion to smaller  $r$  only**

Use the dashed worldline of a stone in Figure 5 to explain, in one or two sentences, why “everything moves to smaller  $r$ -coordinate” inside the event horizon. *Hint:* Think of the connection between worldlines of stones and future light cones. 639

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**FIGURE 5** Light cone diagram on the  $[r, T]$  slice, plotted from equations (60) and (61) for events both outside and inside the event horizon. Past and future events of each filled-dot-event are corralled inside the past (P) and future (F) light cone of that event. At each  $r$ -coordinate, the light cone can be moved up or down vertically without change of shape, as shown. Inside the event horizon, light and stones can move only to smaller  $r$ -coordinate. A few sample boxes show locally flat patches around a single event. The global rain  $T$ -coordinate conveniently runs forward along every worldline (in contrast to the Schwarzschild  $t$ -coordinate along some worldlines in the light cone diagram of Figure 8, Section 3.7).

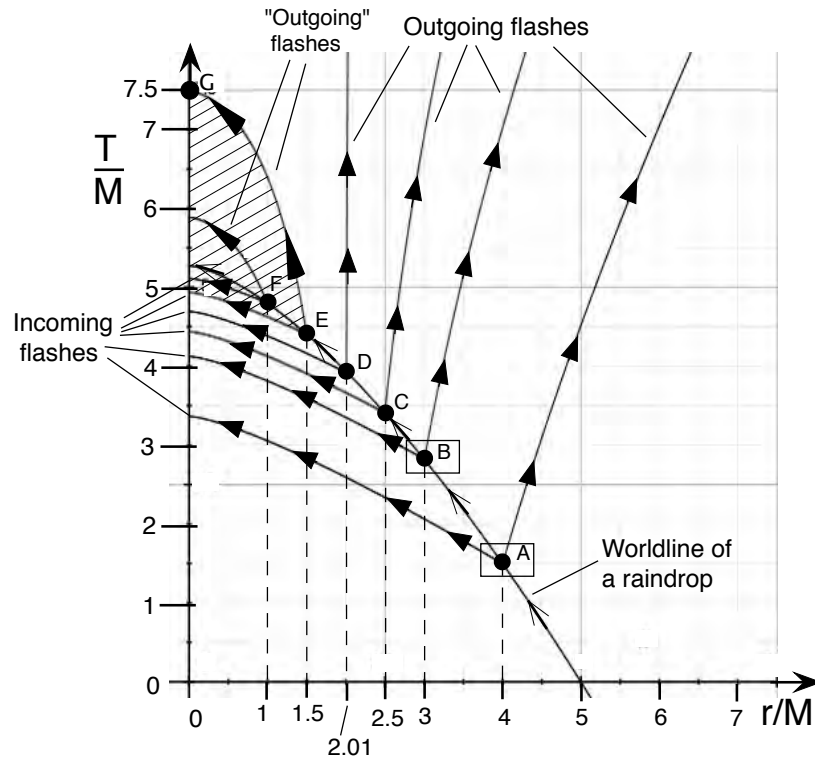
**QUERY 12. Detailed derivation *Optional***

Show details of the derivation of equations (60) and (61) from equation (59). Recall the hint that follows equation (59)

644

645 Time to celebrate! The raindrop worldline in Figure 6 is continuous and  
 646 smooth as it moves inward across the horizon. Global rain coordinates yield  
 647 predictions that are natural and intuitive for us. With global rain coordinates,  
 648 we no longer need to reconcile the awkward contrast between the  
 649 discontinuous Schwarzschild worldlines of Figure 2 and the smooth advance of  
 650 raindrop wristwatch time in Figure 1 (even though both of these plots are  
 651 valid and technically correct). The simplicity of results for global rain  
 652 coordinates leads us to use them from now on to describe the non-spinning  
 653 black hole. Farewell, Schwarzschild metric!

Section 7.8 The rain observer looks—and acts 7-27



**FIGURE 6** A raindrop passes  $r/M = 5$  at  $T/M = 0$  and thereafter emits both incoming and outgoing flashes at events A through F. “Outgoing” flashes—with quotes—from events E and F move to smaller global  $r$ -coordinates, along with everything else inside the event horizon. Little boxes at A and B represent two of the many locally flat patches through which the rain observer passes as she descends. When the rain diver reaches event E, her “range of possible influence” consists of events in the shaded region, for example event F.

**7.8 THE RAIN OBSERVER LOOKS—AND ACTS**

655 Which distant events can the rain observer see? Which can she influence?

656 You ride a raindrop; in other words, you fall from initial rest far from the  
 657 black hole. What do you see radially ahead of you? behind you? Of all events  
 658 that occur along this  $r$ -line, which ones can you influence from where you are?  
 659 Which of these events can influence you? When can you no longer influence  
 660 any events? To answer these questions we give the raindrop some elbow room,  
 661 turn her into a **rain observer** who makes measurements and observations in a  
 662 series of local rain frames through which she falls. This definition specializes  
 663 the earlier general definition of an observer (Definition 4, Section 5.7).

**DEFINITION 4. Rain observer**

Definition:  
**Rain observer**

664 A **rain observer** is a person or a data-collecting machine that rides a  
 665 raindrop. As she descends, the rain observer makes a sequence of  
 666

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667 measurements, each measurement limited to a local inertial rain frame  
 668 (Box 4).



669 **Objection 9.** *Wait: Go back! You have a fundamental problem that ruins*  
 670 *everything. The global rain metric (32) contains the  $r$ -coordinate, but you*  
 671 *have not defined the  $r$ -coordinate inside the event horizon. Section 3.3*  
 672 *defined the  $r$ -coordinate as “reduced circumference,” that is, the*  
 673 *circumference of a shell divided by  $2\pi$ . But you cannot build a shell inside*  
 674 *the event horizon, so you cannot define global coordinate  $r$  there.*  
 675 *Therefore you have no way even to describe the worldline of the rain*  
 676 *observer once she crosses the event horizon.*



677 **!** Guilty as charged! Box 4 defined local rain coordinates and justified their  
 678 validity inside the event horizon, but we have not formally defined the  
 679  $r$ -coordinate inside the event horizon, or how an observer might determine  
 680 its value there. Here is one way (Box 7): As the rain observer drops from  
 681 rest far from the black hole, she simultaneously releases a stone test  
 682 particle from rest beside her and perpendicular to her direction of motion.  
 683 Thereafter she uses radar or a meter stick to measure the distance to the  
 684 stone. In this way she monitors her  $r$ -coordinate as she descends inside  
 685 the event horizon.

686 Box 4 introduced local rain frames in which we can carry out and record  
 687 measurements using special relativity. Small boxes in Figures 5 and 6 represent  
 688 effectively flat patches on which we can construct local inertial frames. In this  
 689 chapter we allow the rain observer to look only at events that lie before and  
 690 behind her along her worldline. She can also send light flashes and stones to  
 691 influence (as much as possible) this limited set of events. (Chapter 11 allows  
 692 the raindrop observer to look all around her.)

**QUERY 13. Observe ingoing and outgoing light flashes in a local rain frame.**

How do light flashes that we describe as ingoing, outgoing, and “outgoing” in global rain coordinates (Definition 3) move when observed entirely within a local rain frame? Answer this question with the following procedure or some other method.

A. From (42) and (43) show that:

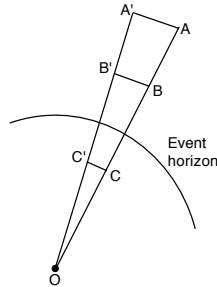
$$\Delta y_{\text{rain}} = \left[ \frac{\Delta r}{\Delta T} + \left( \frac{2M}{\bar{r}} \right)^{1/2} \right] \Delta t_{\text{rain}} \quad (\text{light flash that moves along the } r\text{-coordinate line}) \quad (66)$$

B. Use an approximate version of (54) to replace the square bracket expression in (66):

$$\frac{\Delta y_{\text{rain}}}{\Delta t_{\text{rain}}} = \pm 1 \quad (\text{light flash that moves along the } r\text{-coordinate line}) \quad (67)$$

C. Is equation (67) a surprise—or obvious? What does each sign mean for measurement of light velocity inside a rain frame?

**Box 7. Define the Value of  $r$  Inside the Event Horizon**



**FIGURE 7** The rain observer measures her  $r$ -coordinate inside the event horizon.

*Question:* How can a rain observer inside the event horizon determine her current  $r$ -coordinate? *Answer:* To adapt the Chapter 3 definition of the  $r$ -coordinate—reduced circumference of the shell outside the event horizon—she measures only a tiny arc inside the event horizon.

The rain observer takes the path ABCO in Figure 7; the stone that accompanies her takes the converging path A'B'C'O. Draw a circular arc AA' and similar circular arcs BB' and CC'. The angle AOA' is the same for every arc, so the length of each arc represents the same fraction of the circumference of its corresponding circle. In equation form,

$$\frac{\left( \begin{array}{c} \text{length of} \\ \text{arc AA'} \end{array} \right)}{\left( \begin{array}{c} \text{circumference} \\ \text{of shell thru A} \end{array} \right)} = \frac{\left( \begin{array}{c} \text{length of} \\ \text{arc BB'} \end{array} \right)}{\left( \begin{array}{c} \text{circumference} \\ \text{of shell thru B} \end{array} \right)} \quad (62)$$

The values of  $r$ -coordinates  $r_A$  and  $r_B$  are stamped on the shells outside the event horizon, so the denominators of the two sides of the equation become  $2\pi r_A$  and  $2\pi r_B$ , respectively, and we cancel the common factor  $2\pi$ .

If the angle at the center is small enough, we can replace the length of each circular arc with the straight-line distance measured between, say, A and A' shown in Figure 7. Call this measured distance AA'. And call BB' the corresponding straight-line distance measured between B and B'. Then (62) becomes,

$$\frac{AA'}{r_A} \approx \frac{BB'}{r_B} \quad (63)$$

The rain observer monitors the distance to her accompanying stone as she descends, with radar or—if the stone lies near enough—directly with a meter stick. While she is outside the event horizon, the rain observer reads the value of the  $r$ -coordinate  $r_A$  stamped on that spherical shell as she passes it and the measured distance AA' between the two rain frames, and later BB' as she passes and reads off  $r_B$ . She verifies that this direct reading with the value of  $r_B$  is the same as that calculated with the equation:

$$r_B \approx \frac{BB'}{AA'} r_A \quad (64)$$

At any point C inside the event horizon, the observer measures distance CC' and defines her instantaneous  $r$ -coordinate  $r_C$  as:

$$r_C \equiv \frac{CC'}{AA'} r_A \quad (\text{definition}) \quad (65)$$

This definition of the  $r$ -coordinate inside the event horizon is a direct extension of its definition outside the event horizon and is valid for any observer falling along an  $r$ -coordinate line.

- D. From observations inside a rain frame, is there any difference between a light flash we describe as *outgoing* and one we describe as “*outgoing*”? More generally, can observations carried out entirely inside a rain frame tell us whether that rain frame is outside of, at, or inside the event horizon?

“Range of possible influence”

As our rain observer arrives at any of the emission points A through F in Figure 6, she can try—by firing an ingoing or outgoing stone or light flash—to influence a later event located within the region embraced by the worldlines of the incoming and outgoing (or “outgoing”) flashes from that event. The farther toward the singularity the rain observer falls, the smaller is this “range

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712 of possible influence.” When she arrives at event E, for example, she can  
 713 influence only events in the shaded region in Figure 6, including event F.

714

**QUERY 14. Future events that the rain observer can still influence.**

Make four photocopies of Figure 6. On each copy, choose one emission event A through D or F.

- A. Shade the spacetime region in which a rain observer can influence future events once she has arrived at that emission event.
- B. Which of these emission points is the last one from which the rain observer can influence events that occur at  $r_0 > 2M$ ?

721

722 As she crosses the event horizon, how long will it be on her wristwatch  
 723 before she reaches the singularity? Equation (3) tells us this wristwatch time is  
 724  $4M/3$  meters.

?

725 **Objection 10.** *Ha! I can live a lot longer inside the event horizon than your*  
 726 *measly  $4M/3$  meters of time. All I have to do, once I get inside the event*  
 727 *horizon, is to turn on my rockets and boost myself radially outward. For*  
 728 *example, I can fire super-powerful rockets at event E and follow the*  
 729 *“outgoing” photon flash from E that reaches the singularity at Event G (top*  
 730 *left corner of that figure). That final  $T$ -value is much greater than the*  
 731  *$T$ -value where the raindrop worldline reaches the singularity.*

!

732 **Be careful!** You want to maximize wristwatch time, not the span of global  $T$   
 733 which, remember, is usually *not* measureable time. The wristwatch time is  
 734 zero along the worldline of a light flash, so the closer you come to that  
 735 worldline the smaller will be your wristwatch time during descent from  
 736 Event E to an event just below G in Figure 6.

?

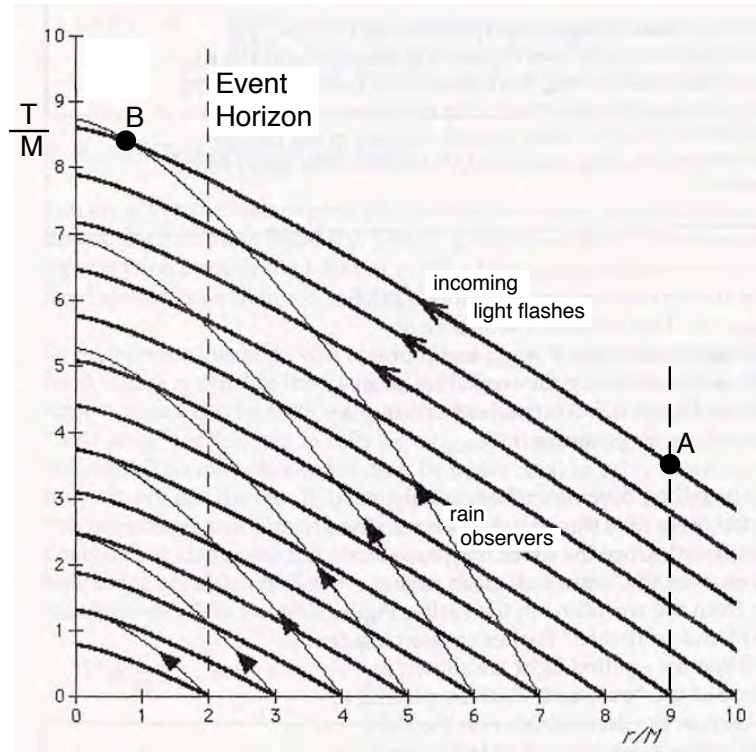
737 **Objection 11.** *Okay, then! I'll give up the rocket blast, but I still want to*  
 738 *know what is the longest possible wristwatch time for me to live after I*  
 739 *cross the event horizon.*

!

740 **Part B of Exercise 3** at the end of this chapter shows how to extend your  
 741 lifetime to  $\pi M$  meters after you cross the event horizon, which is a bit  
 742 longer than the raindrop  $4M/3$  meters. You will show that the way to  
 743 achieve this is to drip from the shell just outside the event horizon; that is,  
 744 you release yourself from rest in global coordinates at  $r = 2M^+$ .

?

745 **Objection 12.** *Can I increase my lifetime inside the event horizon by*  
 746 *blasting rockets in either  $\phi$  direction to add a tangential component to my*  
 747 *global velocity?*



**FIGURE 8** Worldlines of rain observers (thin curves) and incoming light flashes (thick curves) plotted on the  $[r, T]$  slice. All rain observer worldlines have the same form and can be moved up and down without change in shape. Light-flash worldlines also have the same form. A shell observer at  $r/M = 9$  emits a signal from Event A that a rain observer receives inside the event horizon at Event B.



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The present chapter analyzes only  $r$ -motion. In the exercises of Chapter 8 you will show that the answer to your question is no; a tangential rocket blast *decreases* your lifetime inside the event horizon. A wristwatch time lapse of  $\pi M$  is the best you can do. Sorry.

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Figure 8 displays both global worldlines of light flashes—thick curves derived from (60)—and worldlines of raindrops—thin curves derived from (2). It is evident from Figure 8 that news bulletins—incoming or outgoing electromagnetic radio bursts fired outside the event horizon—can be scheduled to catch up with the diver community at any predetermined  $r$ -coordinate.

**QUERY 15. Can you see a rain diver ahead of you?**

Compare Figures 6 and 8. Label as #1 the rain diver whose worldline is plotted in Figure 6. Label as #2 a second rain diver who falls along the *same*  $r$ -coordinate line in space as rain diver #1, but at a  $T$ -coordinate greater by  $\Delta T$ . Use worldlines of Figure 8 to answer the following questions. (*Optional:*



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Derive analytic solutions to these questions and compare the results with your answers derived from the figure.)

763

- A. Over what range of delays  $\Delta T$  will rain diver #2 be able to see the flash from Event E emitted by diver #1 but not the flash from Event F?
- B. Over what range of delays  $\Delta T$  will rain diver #2 be able to see the flash from Event D emitted by diver #1 but not the flash from Event E?
- C. Answer this question decisively: Can any later diver #2 see a flash emitted by diver #1 at say,  $r/M = 0.1$ , just before diver #1 reaches the singularity?

770

771

**QUERY 16. Can you see a rain diver behind you?**

Extend the results of Query 15 to analyze a third rain diver labeled #3 who falls along the same  $r$ -coordinate line in space as the earlier two rain divers, but at a  $T$ -coordinate that is smaller by  $-\Delta T$ . Diver #3 looks outward at light pulses emitted by diver #1. Use the worldlines of Figure 6 to answer the following questions. (*Optional:* Derive analytic solutions to these questions and compare the results with your answers derived from the figure.)

- A. Over what range of earlier launches  $-\Delta T$  will rain diver #3 be able to see the flash from Event D emitted by diver #1 but not the flash from Event F?
- B. Over what range of earlier launches  $-\Delta T$  will rain diver #3 be able to see the flash from Event A emitted by diver #1 but not the flash from Event D?
- C. Answer this question decisively: Can any earlier diver #3 see a flash emitted by diver #1 at say,  $r/M = 0.1$ , just before diver #1 reaches the singularity?

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785

**QUERY 17. Can you see the crunch point ahead of you?**

You are the rain diver whose worldline is plotted in Figure 6. By some miracle, you survive to reach the center of the black hole. Show that you *cannot* see the singularity ahead of you before you arrive there. (What a disappointment after all the training, preparation, and sacrifice!)

790

791

**QUERY 18. Rain frame energy**

The rain frame is inertial. Therefore the expression for energy of a stone in rain frame coordinates is that of special relativity, namely  $E_{\text{rain}}/m = dt_{\text{rain}}/d\tau$  (Section 1.7). Recall also from the differential version of (42) that  $dt_{\text{rain}} = dT$ .

- A. Use (35) together with the special relativity expression for the rain frame energy of the stone and the above identification of global rain  $T$  with rain frame time from (42) to show that:

**Box 8. The River Model**



**FIGURE 9** In the river model of a black hole, fish that swim at different rates encounter a waterfall. The fastest fish represents a photon. “The fish upstream can make way against the current, but the fish downstream is swept to the bottom of the waterfall.” The event horizon corresponds to that point on the waterfall at which the upward-swimming photon-fish stands still. [From Hamilton and Lisle, see references]

Andrew Hamilton and Jason Lisle created a **river model** of the black hole. In their model, water “looks like ordinary flat space, with the distinctive feature that space itself is flowing inward at the Newtonian escape velocity. The place where the infall velocity hits the speed of light . . . marks the event horizon. . . . Inside the event horizon, the infall velocity exceeds the speed of light, carrying everything with it.” At every  $r$ -coordinate near a black hole the river of space flows past at the speed of a raindrop, namely a stone that falls from initial rest far from the waterfall.

Envision flat spacetime distant from a black hole as still water in a large lake with clocks that read raindrop wristwatch time  $\tau_{\text{raindrop}}$  floating at rest with respect to the water. At one side

of the lake the water drifts gently into a river and that carries the raindrop clocks with it. River water moves faster and faster as it approaches and flows over the brink of the waterfall. Each jet of falling water narrows as it accelerates downward. Fish represent objects that move in the river/space; the fastest fish represents a photon. At some point below the lip of the waterfall, not even the photon-fish can keep up with the downward flow and is swept to the bottom of the falls (Figure 9) The black hole event horizon corresponds to the point at which the upward-swimming “photon-fish” stands still.

The river model helps us to visualize many effects observed near the black hole. Hamilton and Lisle write, “It explains why light cannot escape from inside the event horizon, and why no star can come to rest within the event horizon. It explains how an extended object will be stretched radially by the inward acceleration of the river, and compressed transversely by the spherical convergence of the flow. It explains why an object that falls through the event horizon appears to an outsider redshifted and frozen at the event horizon: as the object approaches the event horizon, light [a photon-fish] emitted by it takes an ever-longer global time to forge against the onrushing current of space and eventually to reach the outside observer.”

Hamilton and Lisle show that the river model is consistent with the results of general relativity. In that sense the river model is *correct and complete*.

The river model is a helpful visualization, but that visualization comes at a price. It carries two misleading messages: First, that space itself—represented by the river—is observable. We easily observe various flows of different rivers on Earth, but no one—and no instrument—registers or observes any “flow of space” into a black hole. Second, the river model embodies global rain coordinates, but we have seen that there are an unlimited number of global coordinates for the black hole, many of which cannot be envisioned by the river model.

$$\frac{E_{\text{rain}}}{m} \equiv \lim_{\Delta\tau \rightarrow 0} \frac{\Delta t_{\text{rain}}}{\Delta\tau} \equiv \frac{dT}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1} \left[ \frac{E}{m} + \left(\frac{2M}{r}\right)^{1/2} \frac{dr}{d\tau} \right] \quad (68)$$

- B. Is it possible for  $E_{\text{rain}}/m$  to become negative inside the event horizon? Would any observer complain if it did?
- C. Same questions as Item B for  $E/m$ , the global map energy per unit mass.
- D. Perform a Lorentz transformation (Section 1.10) with  $v_{\text{rel}}$  and  $\gamma_{\text{rel}}$  from (14) to obtain  $E_{\text{shell}}$  in terms of  $E$ . Compare with  $E$  and  $E_{\text{shell}}$  in Schwarzschild coordinates from Sections 6.2 and 6.3.

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**Box 9. The Planck length**

General relativity is a classical—non-quantum—theory (Box 7, Section 6.7). One of its beauties is that, when applied to the black hole, general relativity points to its own limits. The Schwarzschild metric plus the Principle of Maximal Aging predict that everything which moves inward across the horizon will end up on the singularity, a point. We know that this does not satisfy quantum mechanics: The Heisenberg uncertainty principle of quantum mechanics tells us that a single electron confined to a point has unlimited momentum. So not even a single electron—much less an entire star gobbled up by the black hole—can be confined to the singularity. In this book we assume that classical general relativity is valid until very close to the singularity. How close? One estimate is the so-called **Planck length**, derived from three fundamental constants:

$$\text{Planck length} = \left( \frac{hG}{2\pi c^3} \right)^{1/2} = 1.616\,199 \times 10^{-35} \tag{69}$$

in meters, with an uncertainty of  $\pm 97$  in the last two digits. The presence in this equation of Planck’s constant  $h = 1.054\,571\,726 \times 10^{-34}$  Joule-second ( $\pm 15$  in the last two digits) tells us that we have entered the realm of quantum mechanics, where classical general relativity is no longer valid. Cheer up! Before any part of you arrives at the Planck distance from the singularity, you will no longer feel any discomfort.

What happens when a single electron arrives at a Planck length away from the singularity? Nobody knows!

**7.9 ■ A MERCIFUL ENDING?**

806 *How long does the “terminal spaghettification” process last?*

Is death near  
the singularity  
painful?

807 To dive into a black hole is to commit suicide, which may go against religious,  
808 moral, or ethical principles—or against our survival instinct. Aside from such  
809 considerations, no one will volunteer for your black-hole diver research team if  
810 she predicts that as she approaches the crunch point her death will be painful.  
811 Your task is to estimate the **ouch time**  $\tau_{\text{ouch}}$ , defined as the lapse of time on  
812 the wristwatch of the diver between her first discomfort and her arrival at the  
813 singularity,  $r = 0$ .

**QUERY 19. Preliminary: Acceleration  $g$  in units of inverse meters.**

Newton’s expression for gravitational force in conventional units:

$$F_{\text{conv}} \equiv m_{\text{conv}} g_{\text{conv}} = - \frac{GM_{\text{conv}} m_{\text{conv}}}{r^2} \quad (\text{Newton}) \tag{70}$$

A. Verify the resulting gravitational acceleration in units of inverse meters:

$$g \equiv \frac{g_{\text{conv}}}{c^2} = - \frac{M}{r^2} \quad (\text{Newton}) \tag{71}$$

B. Show that at Earth’s surface the Newtonian acceleration of gravity has the value given inside the front cover, namely

$$|g_{\text{Earth}}| \equiv |g_{\text{E}}| = \left| - \frac{M_{\text{Earth}}}{r_{\text{Earth}}^2} \right| = 1.09 \times 10^{-16} \text{ meter}^{-1} \quad (\text{Newton}) \tag{72}$$

Rain frame  
observer  
feels tides.

821 The rain observer is in free fall and does not feel any net force as a result  
822 of local acceleration. However, she does feel radially stretched due to a  
823 *difference* in acceleration between her head and her feet, along with a  
824 compression from side to side. We call these differences **tidal accelerations**.

**QUERY 20. Tidal acceleration along the  $r$ -coordinate line**

We want to know how much this acceleration *differs* between the head and the feet of an in-falling rain observer. Take the differential of  $g$  in (71). Convert the result to increments over a patch of average  $\bar{r}$ . Show that

$$\Delta g \approx \frac{2M}{\bar{r}^3} \Delta r \quad (\text{Newton}) \quad (73)$$

831 What does Einstein say about tidal acceleration? Section 9.7 displays the  
832 correct general relativistic expressions for the variation of local gravity with  
833 spatial separation. *Surprise:* The expression for tidal acceleration in *any*  
834 inertial frame falling along the  $r$ -coordinate line has a form identical to the  
835 Newtonian result (73), and thus for the local rain frame becomes:

$$\Delta g_{\text{rain}} \approx \frac{2M}{\bar{r}^3} \Delta y_{\text{rain}} \quad (74)$$

Define "discomfort."

836 What are the criteria for discomfort? Individual rain observers will have  
837 different tolerance to tidal forces. To get a rough idea, let the rain observer's  
838 body be oriented along the  $r$ -coordinate line as she falls, and assume that her  
839 stomach is in free fall, feeling no stress whatever. Assume that the rain  
840 observer first becomes uncomfortable when the *difference* in local acceleration  
841 between her free-fall stomach and her head (or her feet), that stretches her, is  
842 equal to the acceleration at Earth's surface. By this definition, the rain  
843 observer becomes uncomfortable when her feet are pulled downward with a  
844 force equal to their weight on Earth and her head is pulled upward with a  
845 force of similar magnitude. Let her height be  $h$  in her frame, and the ruler  
846 distance between stomach and either her head or her feet be half of this, that  
847 is,  $\Delta y_{\text{rain}} = h/2$ .

**QUERY 21. The  $r$ -value for the start of "ouch."**

From our criteria above for discomfort, we have:

$$\Delta g_{\text{rain ouch}} \equiv g_E \quad (\text{stomach-to-foot distance}) \quad (75)$$

Show that the  $r$ -value for the start of "ouch," namely  $r_{\text{ouch}}$ , is:

$$r_{\text{ouch}} = \left( \frac{Mh}{g_E} \right)^{1/3} \quad (h = \text{head-to-foot height}) \quad (76)$$

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**QUERY 22. Three cases for the start of “ouch.”**

Approximate  $h$ , the head-to-foot distance, as 2 meters. Find the value of the ratio  $r_{\text{ouch}}/(2M)$  for these cases:

- A black hole with ten times the mass of our Sun.
- The “20-year black hole” in Query 3.
- Suppose that the ouch  $r$ -coordinate is at the event horizon. What is the mass of the black hole as a multiple of the Sun’s mass?

**QUERY 23. The wristwatch ouch time  $\tau_{\text{ouch}}$** 

- Use equation (72) to show that the raindrop ouch time  $\tau_{\text{ouch}}$  (the wristwatch time between initial ouch and arrival at the singularity) is *independent of the mass of the black hole*:

$$\tau_{\text{ouch}} = \frac{1}{3} \left( \frac{2h}{g_E} \right)^{1/2} \quad (\text{raindrop wristwatch ouch time in meters}) \quad (77)$$

Here, recall,  $h$  is the height of the astronaut, about 2 meters, and the value of  $g_E$  is given in (72). Show that the wristwatch ouch time in seconds, the same for *all* non-spinning black holes, is:

$$\tau_{\text{ouch}} = \frac{1}{3c} \left( \frac{2h}{g_E} \right)^{1/2} \quad (\text{raindrop wristwatch ouch time in seconds}) \quad (78)$$

- Substitute numbers into equation (78). Show that the duration of raindrop wristwatch ouch time is about  $2/9$  of a second for *every* non-spinning black hole, independent of its mass  $M$ .  
*Guess: Will pain signals travel from your extremities to your brain during this brief wristwatch ouch time?*

**MUTABILITY OF PHYSICAL LAWS**

*By 1970, I had become convinced not only that black holes are an inevitable consequence of general relativity theory and that they are likely to exist in profusion in the universe, but also that their existence implies the mutability of physical law. If time can end in a black hole, if space can be crumpled to nothingness at its center, if the number of particles within a black hole has no meaning, then why should we believe that there is anything special, anything unique, about the laws of physics that we discover and apply? These laws must have come into existence with the Big Bang as surely as space and time did.*

—John Archibald Wheeler

**7.10 ■ EXERCISES****886 1. Crossing the Event Horizon**

887 Pete Brown disagrees with the statement, “No special event occurs as we fall  
 888 through the event horizon.” He says, “Suppose you go feet first through the  
 889 event horizon. Since your feet hit the event horizon before your eyes, then your  
 890 feet should disappear for a short time on your wristwatch. When your eyes  
 891 pass across the event horizon, you can see again what’s inside, including your  
 892 feet. So tie your sneakers tightly or you will lose them in the dark!” Is Pete  
 893 correct? Analyze his argument without criticizing him.

**894 2. Equations of Motion of the Raindrop**

895 From equations in Chapter 6 we can derive the equations of motion for a  
 896 raindrop in Schwarzschild coordinates. From the definition of the raindrop,

$$897 \quad \frac{E}{m} = 1 \quad \text{and} \quad \frac{d\phi}{d\tau} = 0 \quad \begin{array}{l} \text{(raindrop in Schwarzschild coordinates (79)} \\ \text{and in global rain coordinates)} \end{array}$$

898 In addition, equation (23) in Section 6.4 tells us that

$$899 \quad \frac{dr}{d\tau} = - \left( \frac{2M}{r} \right)^{1/2} \quad \begin{array}{l} \text{(raindrop in Schwarzschild coordinates (80)} \\ \text{and in global rain coordinates)} \end{array}$$

900 From equation (13) in Section 6.4, you can easily show that

$$901 \quad \frac{dt}{d\tau} = \left( 1 - \frac{2M}{r} \right)^{-1} \quad \begin{array}{l} \text{(raindrop in Schwarzschild coordinates (81)} \\ \text{and in global rain coordinates)} \end{array}$$

902 Now derive the raindrop equations of motion in global rain coordinates.  
 903 First, show that both  $dr/d\tau$  and  $d\phi/d\tau$  have the same form in global rain  
 904 coordinates as in Schwarzschild coordinates, as stated in the labels of  
 905 equations (80) and (81). Second, use equations (80) and (81) plus equation  
 906 (35) to show that

$$\frac{dT}{d\tau} = 1 \quad \begin{array}{l} \text{(raindrop in global rain coordinates)} \\ \end{array} \quad (82)$$

**907 Comment 9. Simple definition of global rain  $T$** 

908 Equation (82) can be used as the definition of global rain coordinate differential  
 909  $dT$ . In other words, we *choose*  $dT$  equal to the differential lapse of wristwatch  
 910 time on a falling raindrop.

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911 **3. Different masses for the “20-year black hole.”**

912 This chapter describes a “20-year black hole,” defined as one for which the  
 913 wristwatch on a raindrop registers a 20-year lapse between its crossing of the  
 914 event horizon and its arrival at the singularity. But the wristwatch may be on  
 915 a hailstone, flung radially inward from far away; or on a drip, dropped from  
 916 rest from a shell outside the event horizon. What is the required mass of the  
 917 “20-year black hole” in these two cases?

918 A. We fling an incoming **hailstone** inward along the  $r$ -coordinate line  
 919 with initial shell speed  $|v_{\text{far}}|$  from far away from the black hole. A  
 920 lengthy derivation of the wristwatch time from event horizon to the  
 921 singularity yields the result:

$$\tau_{\text{Aail}}[2M \rightarrow 0] = M \left[ \frac{2}{v_{\text{far}}^2 \gamma_{\text{far}}} - \frac{1}{v_{\text{far}}^3 \gamma_{\text{far}}^3} \ln \left( \frac{1 + v_{\text{far}}}{1 - v_{\text{far}}} \right) \right] \quad (83)$$

922 where, remember,  $\gamma \equiv (1 - v^2)^{-1/2}$  and we treat  $v_{\text{far}}$  as a (positive)  
 923 speed. Answer questions in the following items:

- 924 a. *Guess:* In the case of the hailstone, will the mass of “20-year black  
 925 hole” be greater or less than that for the raindrop?  
 926 b. Consider  $\gamma_{\text{far}} = 2$ . What is the value of  $v_{\text{far}}$ ?  
 927 c. Show that for this particular value  $\gamma_{\text{far}} = 2$ , the first term inside  
 928 the square bracket in (83) alone gives the same result as the  
 929 raindrop in equation (3).  
 930 d. Was your guess in Item **a** correct or incorrect?  
 931 e. What is the mass of the “20-year black hole” for that hailstone?  
 932 How does it compare to the mass of the “20-year black hole” for  
 933 the raindrop?

934 B. A **drip** drops from rest on a shell of global coordinate  $r_0 > 2M$ .  
 935 Another lengthy derivation of the wristwatch time from event horizon  
 936 to the singularity yields the result:

$$\tau_{\text{drip}}[2M \rightarrow 0] = \quad (84)$$

$$2M \left( \frac{2M}{r_0} \right)^{-3/2} \left[ - \left( \frac{2M}{r_0} \right)^{1/2} \left( 1 - \frac{2M}{r_0} \right)^{1/2} + \arctan \left( \frac{2M/r_0}{1 - 2M/r_0} \right)^{1/2} \right]$$

- 937 a. *Guess:* In the case of a drip, will the mass of “20-year black hole”  
 938 be greater or less than that for the raindrop?  
 939 b. Next, take the limiting case  $r_0 \rightarrow 2M$ . Show that in this limit  
 940 arctan takes the value  $\pi/2$ .  
 941 c. Show that in this case  $\tau_{\text{drip}}[2M \rightarrow 0] \rightarrow \pi M$ .  
 942 d. Was your guess in Item **a** correct or incorrect?

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943 e. What is the mass of the “20-year black hole” for that drip? How  
 944 does it compare to the mass of the “20-year black hole” for the  
 945 raindrop?

946 C. *Fascinating but optional:* The next to last paragraph in Box 8 states  
 947 that every stone that passes inward across the event horizon at  $r = 2M$   
 948 moves at that  $r$ -value with shell velocity  $v_{\text{shell}} = -1$ , the speed of light  
 949 (as a limiting case). Since this holds for *all*  $r$ -diving stones, how can the  
 950 masses of “20-year black holes” possibly differ for raindrops, hailstones,  
 951 and drips?

952 **4. Map energy of a drip released from  $r_0$** 

953 A. Derive the following expression for  $E/m$  in global rain coordinates for a  
 954 drip released from rest with respect to the local shell frame at  $r_0 > 2M$ :

$$\frac{E}{m} = \left(1 - \frac{2M}{r_0}\right)^{1/2} \quad (\text{drip released from rest at } r_0) \quad (85)$$

955 Compare with equation (33) of Chapter 6. Are you surprised by what  
 956 you find? Should you be?

957 B. What are the maximum and minimum values of  $E/m$  in (85) as a  
 958 function of  $r_0$ ? How can the minimum value possibly be less than the  
 959 rest energy  $m$  of the stone measured in an inertial frame?

960 C. Is expression (85) consistent with the value  $E/m = 1$  for a raindrop?

961 D. Is expression (85) valid for  $r_0 < 2M$ ? What is the physical reason for  
 962 your answer?

963 E. For  $r_0 > 2M$ , is expression (85) still valid when that stone arrives  
 964 inside the event horizon?

965 **5. Map energy of a hailstone**

966 A. Derive the following expression for  $E/m$  in global rain coordinates for a  
 967 hailstone hurled radially inward with speed  $v_{\text{far}}$  from a shell very far  
 968 from the black hole.

$$\frac{E}{m} = \gamma_{\text{far}} \equiv (1 - v_{\text{far}}^2)^{-1/2} \quad (\text{hailstone}) \quad (86)$$

969 Compare this expression with results of Exercise 7 in Chapter 6. Are  
 970 you surprised by what you find? Should you be?

971 B. Is expression (86) for the hailstone consistent with expression (85) for  
 972 the drip? consistent with  $E/m = 1$  for a raindrop?



**7-40** Chapter 7 Inside the Black Hole973 **6. Motion of outgoing light flash outside and at the event horizon**

974 Find the maximum value of  $r/M$  at which the “outgoing” flash moves to  
 975 larger  $r$ , that is  $dr > 0$ , at each of these global map velocities:

976 A.  $dr/dT = 0.99$

977 B.  $dr/dT = 0.9$

978 C.  $dr/dT = 0.5$

979 D.  $dr/dT = 0$

980 **7. Motion of the “outgoing” flash inside the event horizon.**

981 Find the value of  $r/M$  at which the “outgoing” flash moves to smaller  $r$ , that is  
 982  $dr < 0$ , at each of these global map velocities:

983 A.  $dr/dT = -0.1$

984 B.  $dr/dT = -0.5$

985 C.  $dr/dT = -1$

986 D.  $dr/dT = -9$

987 **8. Motion of the incoming flash**

988 At each value of  $r/M$  found in Exercises 6 and 7, find the value of  $dr/dT$  for  
 989 the *incoming* flash.

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