

Chapter 16. Gravitational Waves

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- 17 • *What are gravitational waves?*
- 18 • *How do gravitational waves differ from ocean waves?*
- 19 • *How do gravitational waves differ from light waves?*
- 20 • *What is the source (or sources) of gravitational waves?*
- 21 • *Why has it taken us so long to detect gravitational radiation?*
- 22 • *Why is the Laser Interferometer Gravitational-Wave Observatory*
23 *(LIGO) so big?*
- 24 • *Why are LIGOs located all over the Earth?*
- 25 • *What will the next generation of gravitational wave detectors look like?*

CHAPTER

16

Gravitational Waves

Edmund Bertschinger & Edwin F. Taylor *

28 *If you ask me whether there are gravitational waves or not, I*
 29 *must answer that I do not know. But it is a highly interesting*
 30 *problem.*

—Albert Einstein

16.1 ■ THE PREDICTION AND DISCOVERY OF GRAVITATIONAL WAVES

33 *Gravitational wave: a tidal acceleration that propagates through spacetime.*

34 General relativity predicts black holes with properties utterly foreign to
 35 Newtonian and quantum physics. And general relativity predicts gravitational
 36 waves, also foreign to Newtonian and quantum physics.

Newton: Gravity
 propagates
 instantaneously.

37 Without quite saying so, Newton assumed that gravitational interaction
 38 propagates instantaneously: When the Earth moves around the Sun, the
 39 Earth’s gravitational field changes all at once everywhere. When Einstein
 40 formulated special relativity and recognized its requirement that no
 41 information can travel faster than the speed of light in a vacuum, he realized
 42 that Newtonian gravity would have to be modified. Not only would static
 43 gravitational effects differ from the Newtonian prediction in the vicinity of
 44 compact masses, but also gravitational effects would propagate as waves that
 45 move with the speed of light.

Einstein: No signal
 propagates faster
 than light.

46 Einstein’s conceptual prototype for gravitational waves was
 47 electromagnetic radiation. In 1873 James Clerk Maxwell demonstrated that
 48 the laws of electricity and magnetism predicted electromagnetic radiation.
 49 Einstein was born in 1879. Heinrich Hertz demonstrated electromagnetic waves
 50 experimentally in 1888. The adult Einstein realized that a general relativity
 51 theory would not look like Maxwell’s electromagnetic theory, but he and
 52 others were able to formulate the corresponding gravitational wave equations.

Compare gravitational
 waves to
 electromagnetic waves.

53 What do we mean by a “gravitational wave”? The gravitational wave is a
 54 tidal acceleration that propagates; that is all it is. As a gravitational wave
 55 passes over you, you are alternately stretched and compressed in ways that

Gravitational waves
 propagate tidal
 accelerations.

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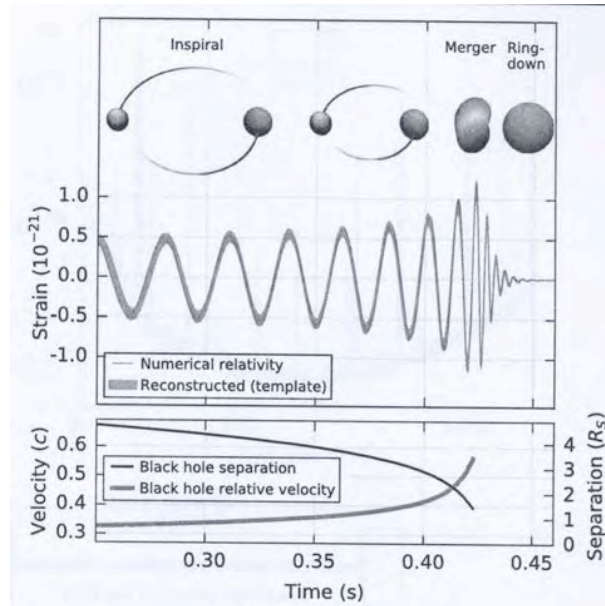


FIGURE 1 Predicted “chirp” of the gravitational wave as two black holes in a binary system merge. Frequency and amplitude increase, followed by a “ring down” due to oscillation of the merged black hole. The present chapter explains details of this figure.

56 depend on the particular form of the wave. In principle there is no limit to the
 57 amplitude of a gravitational wave. In the vicinity of the coalescence,
 58 gravity-wave-induced tidal forces would be lethal. Far from such a source,
 59 gravitational waves are tiny, which makes them difficult to detect.

60 In 2015, the most sensitive gravitational wave detector is the **Laser**
 61 **Interferometer Gravitational Wave Observatory**, or **LIGO** for short.
 62 Gravitational waves were first detected on 14 September 2015 with two LIGO
 63 detectors, one at Hanford, Washington state USA, the other at Livingston,
 64 Louisiana state. These detections give us confidence that gravitational waves
 65 from various sources continually sweep over us on Earth. Sections 16.3 and
 66 16.7 describe some of these sources.

67 Basically we observe gravitational waves by detecting changes in
 68 separation between two test masses suspended near to one another—changes
 69 in gravitational-wave tidal effects. Changes in this separation are *extremely*
 70 small for gravitational waves detected on Earth.

71 Current gravitational wave detectors on Earth are interferometers in which
 72 light reflects back and forth between “free” test masses (mirrors) positioned at
 73 the ends of two perpendicular vacuum chambers. A passing gravitational wave
 74 changes the relative number of wavelengths along each leg, with a resulting
 75 change in interference between the two returning waves. The “free” test masses
 76 are hung from wires that are in turn supported on elaborate shock-absorbers
 77 to minimize the vibrations from passing trucks and even ocean waves crashing

Gravitational wave
 on Earth:
 An extremely small
 traveling tidal effect

Gravitational wave
 detectors are
 interferometers.

78 on a distant shore. The pendulum-like motions of these test masses are free
 79 enough to permit measurement of their change in separation due to tidal
 80 effects of a passing gravitational wave, caused by some remote gigantic distant
 81 event such as the coalescence of two black holes modeled in Figure ??.



82 **Objection 1.** *Does the change in separation induced by gravitational*
 83 *waves affect everything, for example a meter stick or the concrete slab on*
 84 *which a gravitational wave detector rests?*



85 The structure of a meter stick and a concrete slab are determined by
 86 electromagnetic forces mediated by quantum mechanics. The two ends of
 87 a meter stick are not freely-floating test masses. The tidal force of a
 88 passing gravitational wave is much weaker than the internal forces that
 89 maintain the shape of a meter stick—or the concrete slab supporting the
 90 vacuum chamber of a gravitational-wave observatory; these are stiff
 91 enough to be negligibly affected by a passing gravitational wave.

92 **Comment 1. Why not “gravity wave”?**

93 Why do we use the five-syllable *gravitational* to describe this waves, and not the
 94 three-syllable *gravity*? Because the term *gravity wave* is already taken. *Gravity*
 95 *wave* describes the disturbance at an interface—for example between the sea
 96 and the atmosphere—where gravity provides the restoring force.

16.2 ■ GRAVITATIONAL WAVE METRIC

98 *Tiny but significant departure from the inertial metric*

99 Our analysis examines effects of a particular gravitational wave: a plane wave
 100 from a distant source that moves in the z -direction. Every gravitational wave
 101 we discuss in this chapter (except those shown in Figure ??) represents a very
 102 small deviation from flat spacetime. Here is the metric for a gravitational
 103 plane wave that propagates along the z -axis.

Gravitational wave
metric

$$d\tau^2 = dt^2 - (1 + h)dx^2 - (1 - h)dy^2 - dz^2 \quad (h \ll 1) \quad (1)$$

104 First, for light $d\tau = 0$. Then, as usual, no experiment or observation is global;
 105 every one is local. At the LIGO detector the local metric has the form:

$$\begin{aligned} 0 &\approx \Delta t_{\text{LIGO}}^2 - [(1 + h)^{1/2} \Delta x_{\text{LIGO}}]^2 - [(1 - h)^{1/2} \Delta y_{\text{LIGO}}]^2 - \Delta z_{\text{LIGO}}^2 & (2) \\ &\approx \Delta t_{\text{LIGO}}^2 - [(1 + h/2) \Delta x_{\text{LIGO}}]^2 - [(1 - h/2) \Delta y_{\text{LIGO}}]^2 - \Delta z_{\text{LIGO}}^2 & (h \ll 1) \end{aligned}$$

$h/2 =$ gravitational
wave strain

106 In this metric $h/2$ is the tiny fractional deviation from the flat-spacetime
 107 coefficients of dx^2 and dy^2 . The technical name for fractional deviation of
 108 length is **strain**, so $h/2$ is also called the **gravitational wave strain**. Metric
 109 (1) describes a transverse wave, since h is a perturbation in the x and y
 110 directions transverse to the z -direction of propagation. The metric guarantees
 111 that t will vary, along with x and y .

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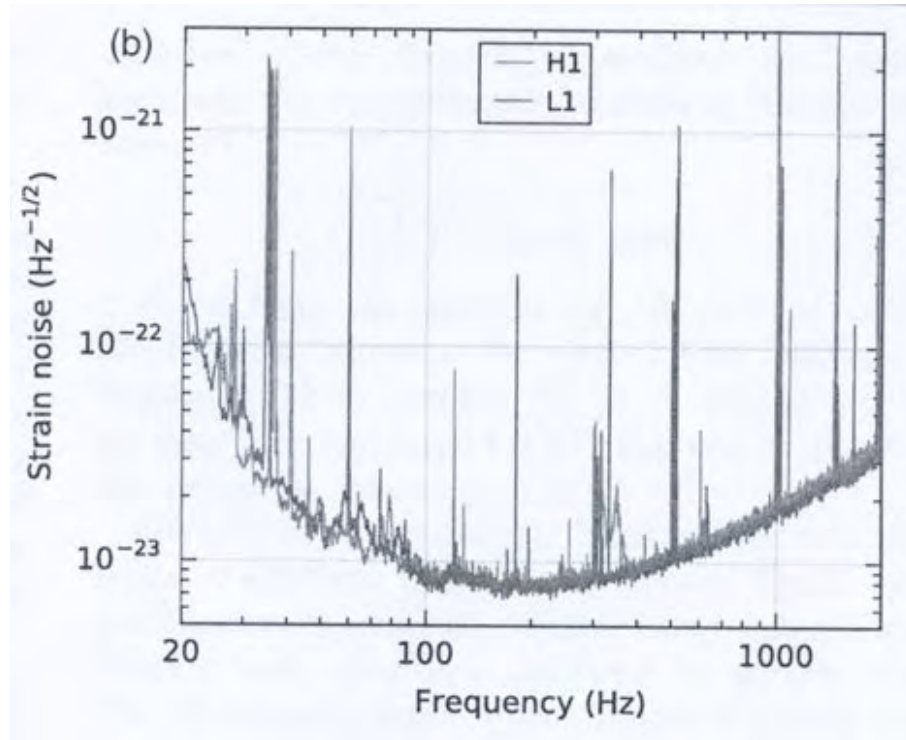


FIGURE 2 Strain noise of LIGO detectors at Hanford, Washington state (curve H1) and at Livingston, Louisiana state (curve L1) at the first detection of a gravitational wave on 14 September, 2015. On the vertical axis $h = 10^{-23}$, for example, means a fractional change in separation of 10^{-23} between test masses. Spikes occur at frequencies of electrical or acoustical noise. To be detectable, gravitational wave signals must cause fractional change above these noise curves.

112 Let two free test masses be at rest D apart in the x or y direction. When a
 113 z -directed gravitational wave passes over them, the change in their separation,
 114 called the **displacement**, equals $h/2 \times D$, which follows from the definition of
 115 $h/2$ as a “fractional deviation.”

116 **?** **Objection 2.** *Awkward! Why define the strain as $h/2$ instead of simply h ?*

117 **!** *Response:* This results from squared values of separation in both global
 118 and local metrics. We could use $(1 - 2h)$ instead of $(1 - h)$ in global
 119 metric (1), but that would be awkward in another way. As usual, we get to
 120 choose the awkwardness, but cannot eliminate or ignore it!

121 Einstein's field equations yield predictions about the magnitude of the
 122 function h in equation (1) for various kinds of astronomical phenomena.
 LIGO gravity wave detector 123 Current gravity wave detectors use laser interferometry and go by the full
 124 name **Laser Interferometer Gravitational Wave Observatory**, or **LIGO**
 125 for short.

126 Figure 2 shows the noise spectrum of the two LIGO instruments that were
 127 the first to detect a gravitational wave. The displacement sensitivity is
 128 expressed in the units of meter/(hertz)^{1/2} because the amount of noise limiting
 129 the measurement grows with the frequency range being sampled. Note that
 Various kinds of noise 130 the instruments are designed to be most sensitive near 150 hertz. This
 131 frequency is determined by the different kinds of noise faced by experimenters:
 132 Quantum noise ("shot noise") limits the sensitivity at high frequencies, while
 133 seismic noise (shaking of the Earth) is the largest problem at low frequencies.

LIGO sensitivity 134 If the range of sampled frequencies—*bandwidth*—is 100 hertz, then LIGO's
 135 best sensitivity is about $10^{-21} \times 100^{1/2} = 10^{-23}$. This means that along a
 136 length of 4 kilometers = 4×10^3 meters, the change in length is approximately
 137 $10^{-21} \times 4 \times 10^3 = 4 \times 10^{-18}$ meters, which is one thousandth the size of a
 138 proton, or a hundred million times smaller than a single atom!

139 **?** **Objection 3.** *Your gravitational wave detector sits on Earth's surface, but
 140 equation (1) says nothing about curved spacetime described, for example,
 141 by the Schwarzschild metric. The expression $2M/r$ measures departure
 142 from flatness in the Schwarzschild metric. At Earth's surface,
 143 $2M/r \approx 1.4 \times 10^{-9}$, which is 10^{13} —ten million million!—times greater
 144 than the corresponding gravitational wave factor $h \sim 10^{-22}$. Why doesn't
 145 the quantity $2M/r$ —which is much larger than h —appear in (1)?*

146 **!** The factor $2M/r$ is essentially constant across the structure of LIGO, so
 147 we can ignore its change as the gravitational wave sweeps over it. LIGO is
 148 totally insensitive to the *static* curvature introduced by the factor $2M/r$ at
 149 Earth's surface. Indeed, the LIGO detector is "tuned" to detect gravitational
 150 wave frequencies near 150 hertz. For this reason, we simply omit static
 151 curvature factors from equation (1), effectively describing gravitational
 152 waves "in free space" for the predicted $h \ll 1$.

Einstein's equations become a wave equation. 153 In flat spacetime and for small values of h , Einstein's field equations
 154 reduce to a wave equation for h . For the most general case, this wave has the

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155 form $h = h(t, x, y, z)$. When t, x, y, z are all expressed in meters, this wave
 156 equation takes the form:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \frac{\partial^2 h}{\partial t^2} \quad (\text{flat spacetime and } h \ll 1) \quad (3)$$

157 For simplicity, think of a plane wave moving along the z -axis. The most
 158 general solution to the wave equation under these circumstances is

$$h = h_{+z}(z - t) + h_{-z}(z + t) \quad (4)$$

Assume gravity
 wave moves
 in $+z$ direction.

159 The expression $h_{+z}(z - t)$ means a function h of the single variable $z - t$.
 160 The function $h_{+z}(z - t)$ describes a wave moving in the positive z -direction
 161 and the function $h_{-z}(z + t)$ describes a wave moving in the negative
 162 z -direction. In this chapter we deal only with a gravitational wave propagating
 163 in the positive z -direction (Figure 5) and hereafter set

$$h \equiv h(z - t) \equiv h_{+z}(z - t) \quad (\text{wave moves in } +z \text{ direction}) \quad (5)$$

164 The argument $z - t$ means that h is a function of *only* the combined variable
 165 $z - t$. Indeed, h can be *any function whatsoever* of the variable $(z - t)$. The
 166 form of this variable tells us that, whatever the profile of the gravitational
 167 wave, that profile displaces itself in the positive z -direction with the speed of
 168 light (local light speed = one in our units).

LIGO sensitive
 75 to 500 hertz

169 Figure 2 shows that the LIGO gravitational wave detector has maximum
 170 sensitivity for frequencies between 75 and 500 hertz, with a peak sensitivity at
 171 around 150 hertz. Even at 500 hertz, the wavelength of the gravitational wave
 172 is very much longer than the overall 4-kilometer dimensions of the LIGO
 173 detector. Therefore *we can assume in the following that the value of h is*
 174 *spatially uniform over the entire LIGO detector.*

QUERY 1. Uniform h ?

Using numerical values, verify the claim in the preceding paragraph that h is effectively uniform over
 the LIGO detector. 178

Analogy: draw global
 map coordinates
 on rubber sheet.

180 It is important to understand that coordinates in metric (1) are global and
 181 to recall that global coordinates are arbitrary; we choose them to help us
 182 visualize important aspects of spacetime. For $h \neq 0$, these global coordinates
 183 are invariably distorted. Think of the three mutually perpendicular planes
 184 formed by (x, y) , (y, z) , and (z, x) pairs. Draw a grid of lines on a rubber sheet
 185 lying in each corresponding plane. By analogy, the passing gravitational wave
 186 distorts these rubber sheets.

Gravitational wave
 distorts rubber
 sheet.

187 Glue map clocks to intersections of these grid lines on a rubber sheet so
 188 that they move as the rubber sheet distorts. A gravitational wave moving in
 189 the $+z$ direction (Figure 3) passes through a rubber sheet and acts in different

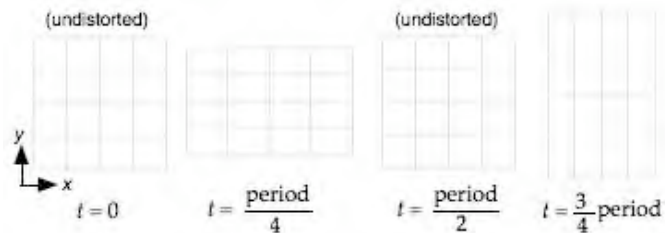


FIGURE 3 Change in shape (greatly exaggerated!) of the map coordinate grid at the same x, y location at four sequential t -values as a periodic gravitational wave passes through in the z -direction (perpendicular to the page). NOTE carefully: The x -axis is stretched while the y -axis is compressed and vice versa. The areas of the panels remain the same.

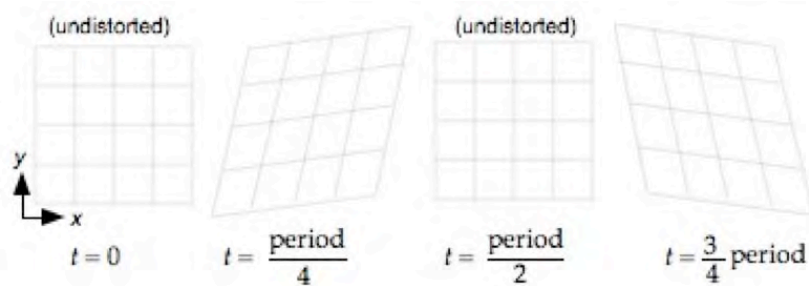


FIGURE 4 Effects of a periodic gravitational wave with polarization “orthogonal” to that of Figure 3 on the map grid in the xy plane. Note that the axes of compression and expansion are at 45 degrees from the x and y axes. All grids stay in the xy plane as they distort. As in Figure 3, the areas of the panels are all the same.

Map t read on
clocks glued to
the rubber sheet.

190 directions within the plane of the sheet (Figures 3 and 4). The map clocks
191 glued at intersections of map coordinate grid lines ride along with the grid as
192 the sheet distorts, so the map coordinates of any clock do not change.

193 Think of two ticks on a single map clock. Between ticks the map
194 coordinates of the clock do not change: $dx = dy = dz = 0$. Therefore metric (1)
195 tells us that the wristwatch time $d\tau$ between two ticks is also map dt between
196 ticks. Map t corresponds to the time measured on the clocks glued to the
197 rubber sheet, even when the strain $h/2$ varies at their locations.

198 Figure 3 represents the map distortion of the rubber sheet with t at a
199 given location due to a particular polarization of the gravitational wave.
200 Although gravitational waves are transverse like electromagnetic waves, the
201 polarization forms of gravitational waves are different from those of
202 electromagnetic waves. Figure 4 shows the distortion caused by a polarization
203 “orthogonal” to that shown in Figure 3.

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16.3 ■ SOURCES OF GRAVITATIONAL WAVES

205 *Many sources; only one type leads to a clear prediction*

No linear “antenna”
for gravitational waves

206 Sources of gravitational waves include collapsing stars, exploding stars, stars in
207 orbit around one another, and the Big Bang itself. Neither an electromagnetic
208 wave nor a gravitational wave results from a spherically symmetric
209 distribution of charge (for electromagnetic waves) or matter (for gravitational
210 waves), even when that spherical distribution pulses symmetrically in and out
211 (Birkhoff’s Theorem, Section 6.5). Therefore, a *symmetric* collapse or
212 explosion emits no waves, either electromagnetic or gravitational. The most
213 efficient source of electromagnetic radiation, for example along an antenna, is
214 oscillating pairs of electric charges of opposite sign moving back and forth
215 along the antenna, the resulting waves technically called **dipole radiation**.
216 But mass has only one “polarity” (there is no negative mass), so there is no
217 gravity dipole radiation from masses that oscillate back and forth along a line.
218 Emission of gravitational waves requires *asymmetric* movement or oscillation;
219 the technical name for the simplest result is **quadrupole radiation**. Happily,
220 most collapses and explosions are asymmetric; even the motion in a binary
221 system is sufficiently asymmetric to emit gravitational waves.

Binary system
emits gravity
waves . . .

222 We study here gravitational waves emitted by a binary system consisting
223 of two black holes orbiting about one another (Section 16.7). The pair whose
224 gravitational waves were detected are a billion light-years distant, so are not
225 visible to us. As the two objects orbit, they emit gravitational waves, so the
226 orbiting objects gradually spiral in toward one another. These orbits are well
227 described by Newtonian mechanics until about one millisecond before the two
228 objects coalesce.

. . . whose
amplitude is
predictable.

229 Emitted gravitational waves are nearly periodic during the Newtonian
230 phase of orbital motion. As a result, these particular gravitational waves are
231 easy to predict and hence to search for. When the two objects coalesce, they
232 emit a burst of gravitational waves (Figures ?? and 1). After coalescence the
233 resulting black hole vibrates (“rings down”), emitting additional gravitational
234 waves as it settles into its final state.

235 **Comment 2. Amplitude, not intensity of gravitational waves**

236 The gravitational wave detector measures the *amplitude* of the wave. The wave
237 amplitude received from a small source decreases as the inverse *r*-separation.

238 In contrast, our eyes and other detectors of light respond to its *intensity*, which is
239 proportional to the square of its amplitude, so the received intensity of light
240 decreases as the inverse *r*-separation.

241 **QUERY 2. Increased volume containing detectable sources**

242 If LIGO sensitivity is increased by a factor of two, what is the increased volume ratio from which it can
243 detect sources?

From other sources:
hard to predict.

244 Binary coalescence is the only source for which we can currently make a
245 clear prediction of the signal. Other possible sources include supernovae and

248 the collapse of a massive star to form a black hole—the event that triggers a
 249 so-called **gamma-ray burst**. We can only speculate about how far away any
 250 of these can be and still be detectable by LIGO.

251 **Comment 3. Detectors do not affect gravitational waves**
 252 We know well that metal structures can distort or reduce the amplitude of
 253 electromagnetic waves passing across them. Even the presence of a receiving
 254 antenna can distort an electromagnetic wave in its vicinity. The same is not true
 255 of gravitational waves, whose generation requires massive moving structures.
 256 Gravitational wave detectors have negligible effect on the waves they detect.

QUERY 3. Electromagnetic waves vs. gravitational waves. Discussion.

What property of electromagnetic waves makes their interaction with conductors so huge compared with the interaction of gravitational waves with matter of any kind?

16.4 ■ MOTION OF LIGHT IN MAP COORDINATES

263 *Light reflected back and forth between mirrored test masses*

264 Currently the LIGO detector system consists of two *interferometers* that
 265 employ mirrors mounted on “test masses” suspended at rest at the ends of an
 266 L-shaped vacuum cavity. The length of each leg $L = 4$ kilometers for
 267 interferometers located in the United States. Gravitational wave detection
 268 measures the changing interference of light waves round-trip *time delays* sent
 269 down the two legs of the detector.

LIGO is an
interferometer.

270 Suppose that a gravitational wave of the polarization illustrated in Figure
 271 3 moves in the z -direction as shown in Figure 5 and that one leg of the
 272 detector along the x -direction and the other leg along the y -direction. In order
 273 to analyze the operation of LIGO, we need to know (a) how light propagates
 274 along the x and y legs of the interferometer and (b) how the test masses at the
 275 ends of the legs move when the z -directed gravitational wave passes over them.

Motion of light in
map coordinates.

276 With what map speed does light move in the x -direction in the presence of
 277 a gravitational wave implied by metric (1)? To answer this question, set
 278 $dy = dz = 0$ in that equation, yielding

$$d\tau^2 = dt^2 - (1 + h)dx^2 \tag{6}$$

279 As always, the wristwatch time is zero between two adjacent events on the
 280 worldline of a light pulse. Set $d\tau = 0$ to find the map speed of light in the
 281 x -direction.

$$\frac{dx}{dt} = \pm(1 + h)^{-1/2} \quad (\text{light moving in } x \text{ direction}) \tag{7}$$

282 The plus and minus signs correspond to a pulse traveling in the positive or
 283 negative x -direction, respectively—that is, in the plane of LIGO in Figure 5.

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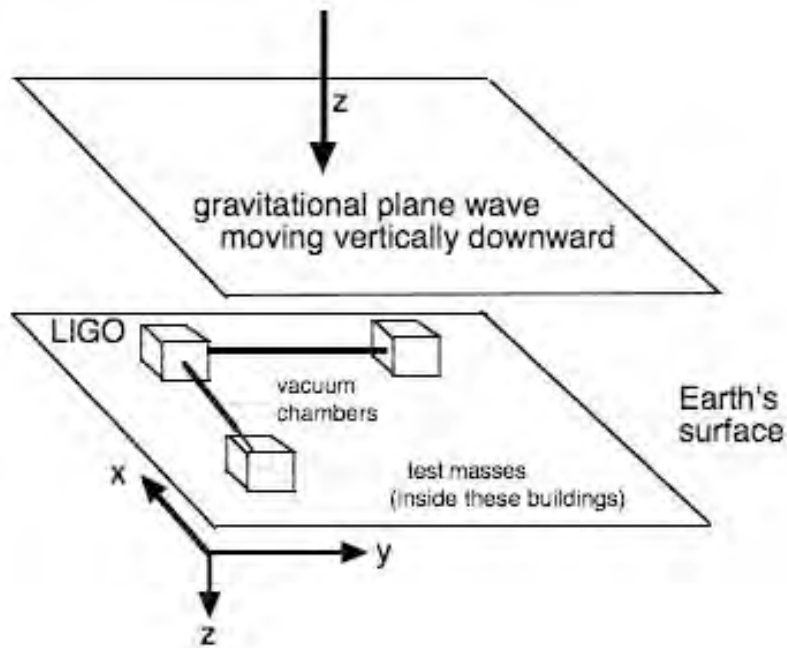


FIGURE 5 Perspective drawing of the relative orientation of legs of the LIGO interferometer lying in the x and y directions on the surface of Earth and the z -direction of the incident gravitational wave descending vertically. [Illustrator: Rotate lower plate and contents CCW 90 degrees, so corner box is above the origin of the coordinate system. Same for Figure 10.]

284 Remember that the magnitude of h is very much smaller than one, so we use
 285 the approximation inside the front cover. To first order:

$$(1 + \epsilon)^n \approx 1 + n\epsilon \quad |\epsilon| \ll 1 \text{ and } |n\epsilon| \ll 1 \quad (8)$$

286 Apply this approximation to (7) to obtain

$$\frac{dx}{dt} \approx \pm(1 - \frac{h}{2}) \quad (\text{light moving in } x \text{ direction}) \quad (9)$$

Gravitational wave
 modifies map
 speed of light.

287 In words, the map speed of light changes (slightly!) in the presence of our
 288 gravitational wave. Since h is a function of t as well as x and y , the map speed
 289 of light in the x -direction is not constant, but varies as the wave passes
 290 through. (Should we worry that the speed in (9) does not have the standard
 291 value one? No! This is a *map speed*—a mythical beast—measured directly by
 292 no one.)

293 By similar arguments, the map speeds of light in the y and z directions for
 294 the wave described by the metric (1) are:

$$\frac{dy}{dt} \approx \pm(1 + \frac{h}{2}) \quad (\text{light moving in } y \text{ direction}) \quad (10)$$

Section 16.5 Zero motion of Ligo Test Masses in Map Coordinates 16-11

$$\frac{dz}{dt} = \pm 1 \quad (\text{light moving in } z \text{ direction}) \quad (11)$$

16.5 ■ ZERO MOTION OF LIGO TEST MASSES IN MAP COORDINATES

296 *“Obey the Principle of Maximal Aging!”*

297 Consider two test masses with mirrors suspended at opposite ends of the x -leg
 298 of the detector. The signal of the interferometer due to the motion of light
 299 along this leg will be influenced only by the x -motion of the test masses due to
 300 the gravitational wave. In this case the metric is the same as (6).

How does the
test mass move?

301 How does a test mass move as the gravitational wave passes over it? As
 302 always, to answer this question we use the Principle of Maximal Aging to
 303 maximize the wristwatch time of the test mass across two adjoining segments
 304 of its worldline between fixed end-events. In what follows we verify the
 305 surprising result, anticipated in Section 16.2, that a test mass initially at rest
 306 in map coordinates rides with the expanding and contracting map coordinates
 307 drawn on the rubber sheet, so this test mass does not move with respect to
 308 map coordinates as a gravitational wave passes over it. This result comes from
 309 showing that an out-and-back jog in the vertical worldline in map coordinates
 310 leads to smaller aging and therefore does not occur for a free test mass.

Idealized case:
Linear jogs
out and back.

311 Figure 6 pictures the simplest possible round-trip excursion: an
 312 incremental linear deviation from a vertical worldline from origin 0 to the
 313 event at $t = 2t_0$. Along Segment A the displacement x increases linearly with
 314 t : $x = v_0 t$, where v_0 is a constant. Along segment B the displacement returns
 315 to zero at the same constant rate. Twice the strain h has average values \bar{h}_A
 316 and \bar{h}_B along segments A and B respectively. We use the Principle of Maximal
 317 Aging to find the value of the speed v_0 that maximizes the wristwatch time
 318 along this worldline. We will find that $v_0 = 0$. In other words, the free test
 319 mass initially at rest in map coordinates stays at rest in map coordinates; it
 320 does not deviate from the vertical worldline in Figure 6. Now for the details.

321 Write the metric (6) in approximate form for one of the segments:

$$\Delta\tau^2 \approx \Delta t^2 - (1 + \bar{h})\Delta x^2 \quad (12)$$

322 where \bar{h} is an average value of h across that segment. Apply (12) first to
 323 Segment A in Figure 6, then to Segment B. We are going to take derivatives of
 324 these expressions, which will look awkward applied to Δ symbols. Therefore
 325 we temporarily ignore the Δ symbols in (12) and let τ stand for $\Delta\tau$, t for Δt ,
 326 and x for Δx , holding in mind that these symbols represent increments, so
 327 equations in which they appear are approximations.

328 With these substitutions, equation (12) becomes, for the two adjoining
 329 worldline segments:

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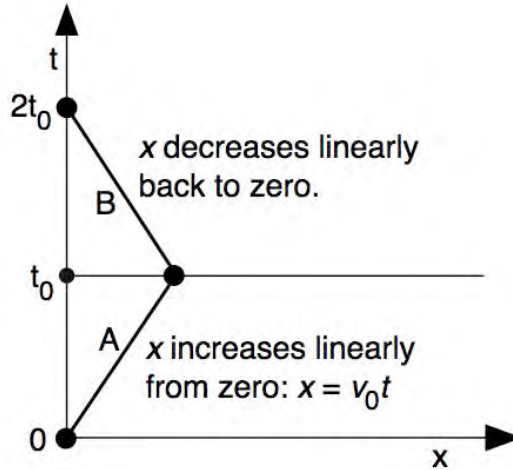


FIGURE 6 Trial worldline for a test mass; incremental departure from vertical line of a particle at rest. Segments A and B are very short.

$$\tau_A \approx \left[t_0^2 - (1 + \bar{h}_A) (v_0 t_0)^2 \right]^{1/2} \quad \text{Segment A} \quad (13)$$

$$\tau_B \approx \left[t_0^2 - (1 + \bar{h}_B) (v_0 t_0)^2 \right]^{1/2} \quad \text{Segment B}$$

330 so that the total wristwatch time along the bent worldline from $t = 0$ to
 331 $t = 2t_0$ is the sum of the right sides of equations (13).

332 We want to know what value of v_0 (the out-and-back speed of the test
 333 mass) will lead to a maximal value of the total wristwatch time. To find this,
 334 take the derivative with respect to v_0 of the sum of individual wristwatch
 335 times and set the result equal to zero.

$$\frac{d\tau_A}{dv_0} + \frac{d\tau_B}{dv_0} \approx -\frac{(1 + \bar{h}_A)v_0 t_0^2}{\tau_A} - \frac{(1 + \bar{h}_B)v_0 t_0^2}{\tau_B} = 0 \quad (14)$$

336 so that

$$\frac{(1 + \bar{h}_A)v_0 t_0^2}{\tau_A} = -\frac{(1 + \bar{h}_B)v_0 t_0^2}{\tau_B} \quad (15)$$

Initially at rest
 in map coordinates?
 Then stays at rest
 in map coordinates.

337 Worldline segments A and B in Figure 6 are identical except in the
 338 direction of motion in x . In equation (15), v_0 is our proposed speed in global
 339 coordinates, a positive quantity. The only way that (15) can be satisfied is if
 340 $v_0 = 0$. *The test mass initially at rest does not change its map x -coordinate as*
 341 *the gravitational wave passes over.*

342 Our result seems rather specialized in two senses: First, it treats only the
 343 vertical worldline in Figure 6 traced out by a test mass at rest. Second, it deals

Section 16.6 Detection of a gravitational wave by LIGO 16-13

344 only with a very short segment of the worldline, along which \bar{h} is considered to
 345 be nearly constant. Concerning the second point, you can think of (14) as a
 346 tiny out-and-back “jog” *anywhere* on a much longer vertical worldline. Then
 347 our result implies that *any* jog in the vertical worldline does not lead to an
 348 increased value of the wristwatch time, even if h varies a lot over a longer
 349 stretch of the worldline.

Not at rest in map
 coordinates? Maybe
 kink in map worldline.

350 The first specialization, the vertical worldline in Figure 6, is important:
 351 The gravitational wave does not cause a kink in a *vertical* map worldline. The
 352 same is typically *not* true for a particle that is moving in map coordinates
 353 before the gravitational wave arrives. (We say “typically” because the kink
 354 may not appear for some directions of motion of the test mass and for some
 355 polarization forms and directions of propagation of the gravitational wave.) In
 356 this more general case, a kink in the worldline corresponds to a change of
 357 velocity. In other words, a passing gravitational wave can change the map
 358 velocity of a moving particle just as if it were a velocity-dependent force. If the
 359 particle velocity is zero, then the force is zero: a particle at rest in map
 360 coordinates remains at rest.

QUERY 4. Disproof of relativity? (optional)

361 “Aha!” exclaims Kristin Burgess. “Now I can disprove relativity once and for all. If the test mass
 362 *moves*, a passing gravitational wave can cause a kink in the worldline of the test mass as observed in
 363 the local inertial Earth frame. No kink appears in its worldline if the test mass is at rest. But if a
 364 worldline has a kink in it as observed in one inertial frame, it will have a kink in it as observed in all
 365 overlapping relatively-moving inertial frames. An observer in any such frame can detect this kink. So
 366 the *absence* of a kink tells me *and every other inertial observer* that the test mass is ‘at rest’? We have
 367 found a way to determine absolute rest using a local experiment. Goodbye relativity!” Is Kristin right?
 368 (A detailed answer is beyond the scope of this book, but you can use some relevant generalizations
 369 drawn from what we already know to think about this paradox. As an analogy from flat-spacetime
 370 electromagnetism, think of a charged particle at rest in a purely magnetic field: The particle
 371 experiences no magnetic force. In contrast, when the same charged particle moves in the same frame, it
 372 may experience a magnetic force for some directions of motion.)

At rest in map
 coordinates?
 Still can move
 in Earth coordinates.

373
 374
 375
 376 In this book we make every measurement in a local inertial frame, not
 377 using differences in global map coordinates. So of what possible use is our
 378 result that a particle at rest in global coordinates does not move in those
 379 coordinates when a gravitational wave passes over it? Answer: Just because
 380 something is at rest in map coordinates does not mean that it is at rest in
 381 local inertial Earth coordinates. In the following section we find that a
 382 gravitational wave *does* move a test mass as observed in the Earth coordinates.
 383 LIGO—attached to the Earth—can detect gravitational waves!

16.6 ■ DETECTION OF A GRAVITATIONAL WAVE BY LIGO

385 *Make measurement in the local Earth frame.*

16-14 Chapter 16 Gravitational Waves

386 Suppose that the gravitational wave that satisfies metric (1) passes over the
 387 LIGO detector oriented as in Figure 5. We know how the test masses at the
 388 two ends of the legs of the detector respond to the gravitational wave: they
 389 remain at rest in map coordinates (Section 16.5). We know how light
 390 propagates along both legs: as the gravitational wave passes through, the map
 391 speed of light varies slightly from the value one, as given by equations (9)
 392 through (11) in Section 16.4.

Earth frame
 tied to LIGO slab

393 The trouble with map coordinates is that they are arbitrary and typically
 394 do not correspond to what an observer measures. Recall that we require all
 395 measurements to take place in a local inertial frame. So think of a local inertial
 396 frame anchored to the concrete slab on which LIGO rests. (Section 16.1
 397 insisted that the gravitational wave has essentially no effect on this slab.) Call
 398 the coordinates in the resulting local coordinate system **Earth coordinates**.
 399 Earth coordinates are analogous to shell coordinates for the Schwarzschild
 400 black hole: useful only locally but yielding the numbers that predict results of
 401 measurements. The metric for the local inertial frame then has the form:

$$\Delta\tau^2 \approx \Delta t_{\text{Earth}}^2 - \Delta x_{\text{Earth}}^2 - \Delta y_{\text{Earth}}^2 - \Delta z_{\text{Earth}}^2 \quad (16)$$

402 Compare this with the approximate version of (1):

$$\Delta\tau^2 \approx \Delta t^2 - (1 + h)\Delta x^2 - (1 - h)\Delta y^2 - \Delta z^2 \quad (h \ll 1) \quad (17)$$

403 Legally, in order to make the coefficients in (17) constants we should use
 404 the symbol \bar{h} , with a bar over the h , to indicate the average value of the
 405 gravitational wave amplitude over the detector. However, in Query 1 you
 406 showed that for the frequencies at which LIGO is sensitive, the wavelength is
 407 very much greater than the dimensions of the detector, so the amplitude h of
 408 the gravitational wave is effectively uniform across the LIGO detector.
 409 Therefore it is not necessary to take an average, and we use the symbol h
 410 without a superscript bar.

Earth frame
 coordinate
 differences

411 Compare (16) with (17) to yield:

$$\Delta t_{\text{Earth}} = \Delta t \quad (18)$$

$$\Delta x_{\text{Earth}} = (1 + h)^{1/2} \Delta x \approx (1 + \frac{h}{2}) \Delta x \quad h \ll 1 \quad (19)$$

$$\Delta y_{\text{Earth}} = (1 - h)^{1/2} \Delta y \approx (1 - \frac{h}{2}) \Delta y \quad h \ll 1 \quad (20)$$

$$\Delta z_{\text{Earth}} = \Delta z \quad (21)$$

412 where we use approximation (8). Notice, first, that the lapse Δt_{Earth} between
 413 two events is identical to their lapse Δt and the z component of their
 414 separation in Earth coordinates, Δz_{Earth} , is identical to the z component of
 415 their separation in map coordinates, Δz .
 416

Section 16.6 Detection of a gravitational wave by LIGO 16-15

417 Now for the differences! Let Δx be the map x -coordinate separation
 418 between the pair of mirrors in the x -leg of the LIGO interferometer and Δy be
 419 the map separation between the corresponding pair of mirrors in the y -leg. As
 420 the z -directed wave passes through the LIGO detector, the test masses at rest
 421 at the ends of the legs stay at rest in map coordinates, as Section 16.5 showed.
 422 Therefore the value of Δx remains the same during this passage, as does the
 423 value of Δy . But the presence of varying $h(t)$ in (19) and (20) tell us that
 424 these test masses move when observed in Earth coordinates. *More:* When
 425 Δx_{Earth} between test masses increases (say) along the Earth x -axis, it
 426 decreases along the perpendicular Δy_{Earth} ; and vice versa. Perfect for
 427 detection of a gravitational wave by an interferometer!

Test masses move
in Earth coordinates.

Light speed = 1
in local Earth
frame.

428 Earth metric (16) is that of an inertial frame in which the speed of light
 429 has the value one in whatever direction it moves. With light we have the
 430 opposite weirdness to that of the motion of test masses initially at rest: In
 431 map coordinates light moves at map speeds different from unity in the
 432 presence of this gravitational wave—equations (9) through (11)—but in Earth
 433 coordinates light moves with speed one. This is reminiscent of the
 434 corresponding case near a Schwarzschild black hole: In Schwarzschild map
 435 coordinates light moves at speeds different from unity, but in local inertial
 436 shell coordinates light moves at speed one.

Different Earth
times along
different legs

437 *In summary* the situation is this: As the gravitational wave passes over the
 438 LIGO detector, the speed of light propagating down the two legs of the
 439 detector has the usual value one as measured by the Earth observer. However,
 440 for the Earth observer the separations between the test masses along the x -leg
 441 and the y -leg change: one increases while the other decreases, as given by
 442 equations (19) and (20). The result is a t -difference in the round-trip of light
 443 along the two legs. It is this difference that LIGO is designed to measure and
 444 thereby to detect the gravitational wave.

445 What will be the value of this difference in round-trip t between light
 446 propagation along the two legs? Let D be the Earth-measured length of each
 447 leg in the absence of the gravitational wave. The round-trip t is twice this
 448 length divided by the speed of light, which has the value one in Earth
 449 coordinates. Equations (19) and (20) tell us that the difference in round-trip t
 450 between light propagated along the two legs is

$$\Delta t_{\text{Earth}} = 2D \left(\frac{h}{2} + \frac{h}{2} \right) = 2Dh \quad (\text{one round trip of light}) \quad (22)$$

Time difference
after N round trips.

451 Using the latest interferometer techniques, LIGO reflects the light back
 452 and forth down each leg approximately $N = 300$ times. That is, light executes
 453 approximately 300 round trips, which multiplies the detected delay, increasing
 454 the sensitivity of the detector by the same factor. Equation (22) becomes

$$\Delta t_{\text{Earth}} = 2NDh \quad (N \text{ round trips of light}) \quad (23)$$

455 Quantities N and h have no units, so the unit of Δt_{Earth} in (23) is the same as
 456 the unit of D , for example meters.

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457

QUERY 5. LIGO fast enough?

Do the 300 round trips of light take place much faster than one period of the gravitational wave being detected? (If it does not, then LIGO detection is not fast enough to track the *change* in h .)

461

462

QUERY 6. Application to LIGO.

Each leg of the LIGO interferometer is of length $D = 4$ kilometers. Assume that the laser emits light of wavelength 1064 nanometer, $\approx 10^{-6}$ meter (infrared light from a NdYAG laser). Suppose that we want LIGO to reach a sensitivity of $h = 10^{-23}$. For $N = 300$, find the corresponding value of Δt_{Earth} . Express your answer as a decimal fraction of the period T of the laser light used in the experiment.

468

469

QUERY 7. Faster derivation?

In this book we insist that global map coordinates are arbitrary human choices and do not treat map coordinate differences as measurable quantities. However, the value of h in (1) is so small that the metric differs only slightly from an inertial metric. This once, therefore, we treat map coordinates as directly measurable and ask you to redo the derivation of equations (22) and (23) using only map coordinates.

475

Remember that test masses initially at rest in map coordinates do not change their coordinates as the gravitational wave passes over them (Section 16.4), but the gravitational wave alters the map speeds of light, differently in the x -direction, equation (9), and in the y -direction, equation (10). Assume that each leg of the interferometer has the length D_{map} in map coordinates.

- A. Find an expression for the difference Δt between the two legs for one round trip of the light.
- B. How great do you expect the difference to be between Δt and Δt_{Earth} and the difference between D (in Earth coordinates) and D_{map} ? Taken together, will these differences be great enough so that the result of your prediction and that of equation (23) can be distinguished experimentally?

484

485

QUERY 8. Different directions of propagation of the gravitational wave

Thus far we have assumed that the gravitational plane wave of the polarization described by equation (1) descends vertically onto the LIGO detector, as shown in Figure 5. Of course the observers cannot prearrange in what direction an incident gravitational wave will move. Suppose that the wave propagates along the direction of, say, the y -leg of the interferometer, while the x -direction lies along the other leg, as before. What is the equation that replaces (23) in this case?

492

493

QUERY 9. LIGO fails to detect a gravitational wave?

Section 16.7 Binary System as a Source of Gravitational Waves 16-17

Think of various directions of propagation of the gravitational wave pictured in Figure 3, together with different directions of x and y in equation (1) with respect to the LIGO detector. Give the name **orientation** to a given set of directions x and y —the transverse directions in (1)—plus z (the direction of propagation) in (1) relative to the LIGO detector. How many orientations are there for which LIGO will detect *no signal whatever*, even when its sensitivity is 10 times better than that needed to detect the wave arriving in the orientation shown in Figure 5? Are there zero such orientations? one? two? three? some other number less than 10? an unlimited number?

502

16.7 ■ BINARY SYSTEM AS A SOURCE OF GRAVITATIONAL WAVES

504 “Newtonian” source of gravitational waves

Unequal masses,
each in circular
orbit

505 The gravitational wave detected on 15 September 2015 came from the merging
506 of two black holes; assume that each is initially in a circular orbit around their
507 center of mass. The binary system is the only known example for which we can
508 explicitly calculate the emitted gravitational waves. Let the M_1 and M_2
509 represent the masses of these two black holes that initially orbit at a value r
510 apart, as shown in Figure 7.

Energy of the system.

511 The basic parameters of the orbit are adequately computed using
512 Newtonian mechanics, according to which the energy of the system in
513 conventional units is given by the expression:

$$E_{\text{conv}} = -\frac{GM_{1,\text{kg}}M_{2,\text{kg}}}{2r} \quad (\text{Newtonian circular orbits}) \quad (24)$$

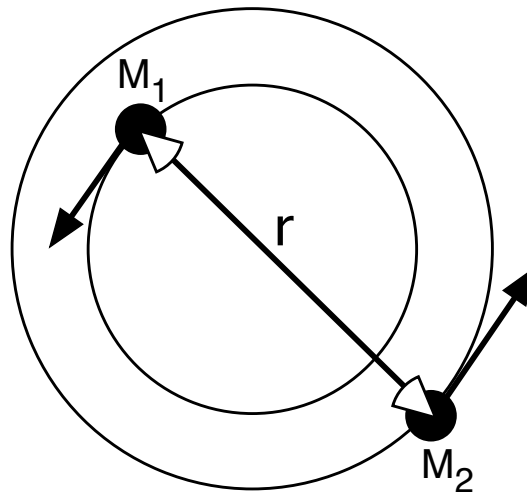


FIGURE 7 A binary system with each object in a circular path.

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Rate of energy loss . . . As these black holes orbit, they generate gravitational waves. General relativity predicts the rate at which the orbital energy is lost to this radiation. In conventional units, this rate is:

$$\frac{dE_{\text{conv}}}{dt_{\text{conv}}} = -\frac{32G^4}{5c^5r^5} (M_{1,\text{kg}}M_{2,\text{kg}})^2 (M_{1,\text{kg}} + M_{2,\text{kg}}) \quad (\text{Newtonian circular orbits}) \tag{25}$$

. . . derived from Einstein's equations. Equation (25) assumes that the two orbiting black holes are separated by much more than the r -values of their event horizons and that they move at nonrelativistic speeds. Deriving equation (25) involves a lengthy and difficult calculation starting from Einstein's field equations. The same is true for the derivation of the metric (1) for a gravitational wave. These are two of only three equations in this chapter that we simply quote from a more advanced treatment.

QUERY 10. Energy and rate of energy loss

Convert Newton's equations (24) and (25) to units of meters to be consistent with our notation and to get rid of the constants G and c . Use the sloppy professional shortcut, "Let $G = c = 1$."

A. Show that (24) and (25) become:

$$E = -\frac{M_1M_2}{2r} \quad (\text{Newton: units of meters}) \tag{26}$$

$$\frac{dE}{dt} = -\frac{32}{5r^5} (M_1M_2)^2 (M_1 + M_2) \quad (\text{Newton: units of meters}) \tag{27}$$

- B. Verify that in both of these equations E has the unit of length.
- C. Suppose you are given the value of E in meters. Show how you would convert this value first to kilograms and then to joules.

QUERY 11. Rate of change of radius

Derive a Newtonian expression for the rate at which the radius changes as a result of this energy loss. Show that the result is:

$$\frac{dr}{dt} = -\frac{64}{5r^3} M_1M_2 (M_1 + M_2) \quad (\text{Newton: circular orbits}) \tag{28}$$

16.8 ■ GRAVITATIONAL WAVE AT EARTH DUE TO DISTANT BINARY SYSTEM

How far away from a binary system can we detect its emitted gravitational waves?

Section 16.8 Gravitational Wave at Earth Due to Distant Binary System 16-19

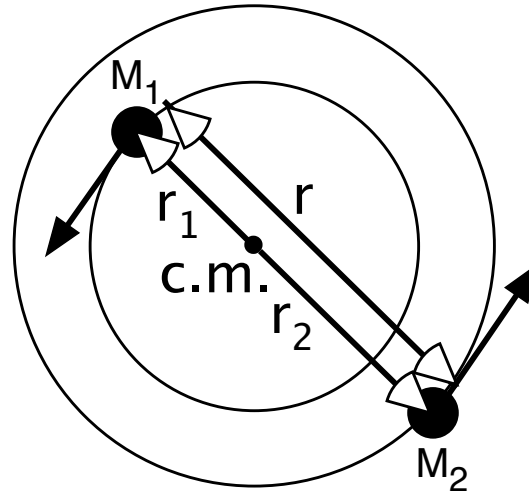


FIGURE 8 Figure 7 augmented to show the center of mass (c.m.) and orbital r -values of individual masses in the binary system.

541 LIGO on Earth's surface detects the gravitational waves emitted by the
 542 distant binary system of two black holes of Figure 7, augmented in Figure 8 to
 543 show the center of mass and individual r_1 and r_2 of the two black holes.

544 What is the amplitude of gravitational waves from this source measured
 545 on Earth? Here is the third and final result of general relativity quoted
 546 without proof in this chapter. The function $h(z, t)$ is given by the equation (in
 547 conventional units)

$$h(z, t) = -\frac{4G^2 M_1 M_2}{c^4 r z} \cos \left[\frac{2\pi f(z - ct)}{c} \right] \quad (\text{conventional units}) \quad (29)$$

548 where r is the separation of orbiters in Figures 7 through 9. Here z is the
 549 separation between source to detector, and—surprisingly— f is twice the
 550 frequency of the binary orbit (see Query 15). Convert (29) to units of meters
 551 by setting $G = c = 1$. Note that $h(z, t)$ is a function of z and t .

552 Figure 9 schematically displays the notation of equation (29), along with
 553 relative orientations and relative magnitudes assumed in the equation. This
 554 equation makes the Newtonian assumptions that

555 (a) the r separation between two the circulating black holes is
 556 much larger than either Schwarzschild r -value, and

557 (b) they move at nonrelativistic speeds.

558 Additional assumptions are:

16-20 Chapter 16 Gravitational Waves

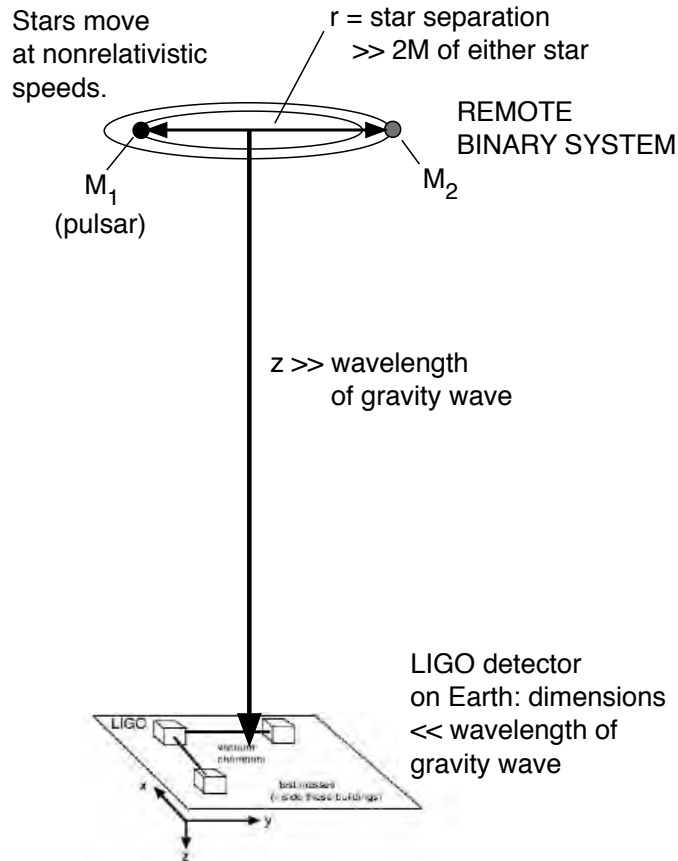


FIGURE 9 Schematic diagram, *not to scale*, showing notation and relative magnitudes for equation (29). The binary system and the LIGO detector lie in parallel planes.[Illustrator: See note in caption to Figure 5.]

559 (c) Separation z between the binary system and Earth is very
 560 much greater than a wavelength of the gravitational wave. This
 561 assumption assures that the radiation at Earth constitutes the
 562 so-called “far radiation field” where it assumes the form of a plane
 563 wave given in equation (5).

564 (d) The wavelength of the gravitational wave is much longer than
 565 the dimensions of the LIGO detector.

566 (e) The binary stars are orbiting in the xy plane, so that from
 567 Earth the orbits would appear as circles if we could see them
 568 (which we cannot).

569 Equation (29) describes only one linear polarization at Earth, the one
 ... for one case 570 generated by metric (1) and shown in Figure 3. The orthogonal polarization

Section 16.8 Gravitational Wave at Earth Due to Distant Binary System 16-21

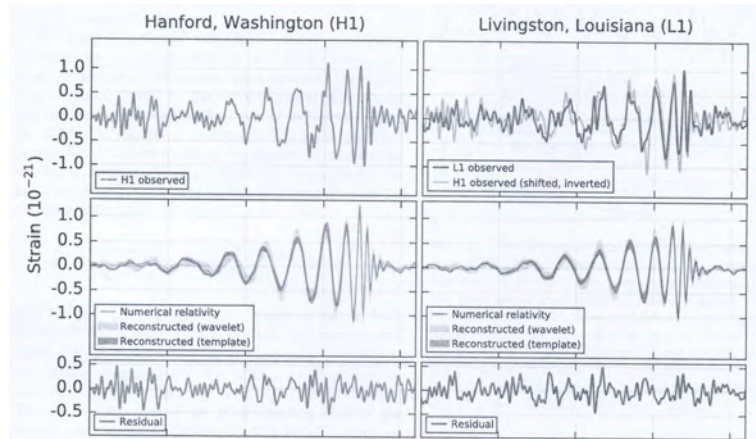


FIGURE 10 Detected “chirps” of the gravitational wave at two locations. The top row shows detected waveforms (superposed in the right-hand panel). The second row shows the cleaned-up image (again superposed). The bottom row displays “residuals,” the noise deducted from images in the first row.

571 shown in Figure 4 is also transverse and equally strong, with components
 572 proportional to $(1 \pm h)$. The formula for the magnitude of h in that
 573 orthogonally polarized wave is identical to (29) with a sine function replacing
 574 the cosine function. We have not displayed the metric for that orthogonal
 575 polarization.

Detection requirements

576 In order for LIGO to detect a gravitational wave, two conditions must be
 577 met: (a) the amplitude h of the gravitational wave must be sufficiently large,
 578 and (b) the frequency of the wave must be in the range in which LIGO is most
 579 sensitive (100 to 400 hertz). Query 14 deals with the amplitude of the wave.
 580 The frequency of gravitational waves, discussed in Query 15, contains a
 581 surprise.

QUERY 12. Amplitude of gravitational wave at Earth

- A. Use (29) to calculate the maximum amplitude of h at Earth due to the radiation from our “idealized circular-orbit” binary system.
- B. Can LIGO detect the gravitational waves whose amplitude is given in part A?
- C. What is the maximum amplitude of h at Earth just before coalescence, when the orbiting black holes are separated by $r = 20$ kilometers (but with orbits still described approximately by Newtonian mechanics)?

QUERY 13. Frequency of emitted gravitational waves

- A. In order LIGO to detect the gravitational waves whose amplitude is given in Query 14, the frequency of the gravitational wave must be in the range 100 to 400 hertz. In Figure 9 the point

16-22 Chapter 16 Gravitational Waves

C. M. is the stationary center of mass of the pulsar system. Using the symbols in this figure, fill in the steps to complete the following derivation.

$$\frac{v_1^2}{r_1} = \frac{GM_1}{r_1^2} \quad (\text{for } M_1, \text{ Newton, conventional units}) \quad (30)$$

$$\frac{v_2^2}{r_2} = \frac{GM_2}{r_2^2} \quad (\text{for } M_2, \text{ Newton, conventional units}) \quad (31)$$

$$M_1 r_1 = M_2 r_2 \quad (\text{center-of-mass condition}) \quad (32)$$

$$f_{\text{orbit}} \equiv \frac{1}{T_{\text{orbit}}} = \frac{v_1}{2\pi r_1} = \frac{v_2}{2\pi r_2} \quad (\text{common orbital frequency}) \quad (33)$$

where f_{orbit} and T_{orbit} are the frequency and period of the orbit, respectively. From these equations, show that for $r \equiv r_1 + r_2$ the frequency of the orbit is

$$f_{\text{orbit}} = \frac{1}{2\pi} \left[\frac{G(M_1 + M_2)}{r^3} \right]^{1/2} \quad (\text{conventional units}) \quad (34)$$

$$= \frac{1}{2\pi} \left[\frac{M_1 + M_2}{r^3} \right]^{1/2} \quad (\text{metric units}) \quad (35)$$

B. Next is a surprise: The frequency f of the gravitational wave generated by this binary pair and appearing in (29) is twice the orbital frequency.

$$f_{\text{gravity wave}} = 2f_{\text{orbit}} \quad (36)$$

Why this doubling? Essentially it is because gravitational waves are waves of tides. Just as there are two high tides and two low tides per day caused by the moon's gravity acting on the Earth, there are two peaks and two troughs of gravitational waves generated per binary orbit.

C. Approximate the average of the component masses in (34) by the value $M = 30M_{\text{Sun}}$. Find the r -value between the binary stars when the orbital frequency is 75 hertz, so that the frequency of the gravitational wave is 150 hertz.

D. Use results quoted earlier in this chapter to find an approximate expression for the time for the binary system to decay from the current radial separation to the radial separation calculated in part C.

ANS: $t_2 - t_1 \approx 5(r_2^4 - r_1^4)/(256M^3)$, every symbol in unit meter.

"Chirp" at
coalescence

612 Newtonian mechanics predicts the motion of the binary system
613 surprisingly accurately until the two components touch, a few milliseconds
614 before they coalesce. Newton tells us that as the separation r between the
615 orbiting masses decreases, their orbiting frequency increases. As a result the
616 gravitational wave sweeps upward in both frequency and amplitude in what is
617 called a **chirp**. Figure 1 is the predicted wave form for such a chirp.

16.9 ■ RESULTS FROM GRAVITATIONAL WAVE DETECTION; FUTURE PLANS619 *Unexpected details*

620 Investigators milked a surprising amount of information from the first
621 detection of gravitational waves. For example:

- 622 1. The initial binary system consisted of two black holes of mass
623 $M_1 = (36 + 5/ - 4)M_{\text{Sun}}$ (that is, uncertainty of $+5M_{\text{Sun}}$ and $-4M_{\text{Sun}}$)
624 and $M_2 = (29 \pm 4)M_{\text{Sun}}$.
- 625 2. The mass of the final black hole was $(62 \pm 4)M_{\text{Sun}}$.
- 626 3. Items 1 and 2 mean that the total energy of emitted gravitational
627 radiation was about $3M_{\text{Sun}}$. A cataclysmic event indeed!
- 628 4. The two detection locations are separated by 10 milliseconds of
629 light-travel time, or 3000 kilometers.
- 630 5. The signals were separated by $6.9 + 0.5/ - 0.4$ milliseconds, which
631 means that they did not come from overhead.

632 How did observations lead to these results?

- 633 Item 1 derives from two equations in two unknowns (27) and (34), with
634 validation in the small separation r -value at which merging takes place.
- 635 Item 2 follows from the frequency of ringing in the merged black hole.
- 636 Item 3 follows from Item 2.
- 637 Item 4 results from standard surveying.
- 638 Item 5 follows from direct comparison of synchronized clocks.

639 What are plans for future gravitational wave detections?

- 640 A. Increased sensitivity of each LIGO system
- 641 B. Increased number of LIGO detectors across the Earth, to measure the
642 source direction more accurately.
- 643 C. Installation of LISA (Laser Interferometer Space Antenna Project) in
644 space, which removes seismic noise at low frequencies in Figure 2).

16-24 Chapter 16 Gravitational Waves**16.10 ■ REFERENCES**

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