CHAPTER

9 Circular Orbits

Edmund Bertschinger & Edwin F. Taylor †

How happy is the little Stone
That orbits a Black Hole alone*
And doesn't care about Careers
And Exigencies never fears –
Whose Coat of elemental Brown
A passing Universe put on
And independent as the Sun
Associates or glows alone
Fulfilling absolute Decree
In casual simplicity –

—Emily Dickinson

*Line two in the original reads:
That rambles in the Road alone

■ STEP OR ORBIT?

"Go straight!" implies maximal aging. Maximal aging implies that energy and angular momentum are constants of motion.

Nature shouts at the stone "Go straight!"

A stone in orbit streaks around a black hole—or around Earth. What tells the stone how to move? Spacetime grips the stone, giving it the simplest possible command: "Go straight!" or in the more legalistic language of the Principle of Maximal Aging, "Follow the worldline of maximal aging across the next two adjoining local inertial frames." From instant to instant this directive is enough to tell the stone what to do next, the next step to take in its motion.

Constants of motion: map energy and map angular momentum

Instructions for its next step—its travel across the next pair of adjoining frames—is enough for the stone, but it is not enough for us. We want more: We seek a global description of the trajectory of the stone through spacetime. In this chapter we begin to win from the metric and the Principle of Maximal Aging a global account of the orbit of a stone around a spherically symmetric center of attraction. This global view arises from map quantities that do not

†Draft of Second Edition of *Exploring Black Holes: Introduction to General Relativity* Latest drafts at dropsite exploringblackholes.com, with a request for comments.

change as the motion progresses. These unchanging map quantities—constants of motion—are energy and angular momentum.

In Chapter 6, Plunging, and Chapter 7, Inside the Black Hole, we used Schwarzschild map energy as a constant of motion to describe radial motion of stone and light flash:

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} \qquad \text{(Schwarzschild map energy)} \qquad (1)$$

This Chapter 9, Circular Orbits

Those chapters showed that this expression for map energy is general, not limited to radial motion. In the present chapter we derive the second constant of orbital motion—map angular momentum—and use it, together with map energy, to describe circular orbits around a nonspinning black hole. We shall find that a free stone can execute circular orbits only at radii greater than that of the light sphere at $r = 3M$.

Next Chapter 10, General Orbits

The following Chapter 10, Orbiting, describes trajectories used to insert our spaceship into any circular orbit, to transfer it from one circular orbit to another one, and to cross the event horizon.

2 ■ MAP ANGULAR MOMENTUM FROM MAXIMAL AGING

Fix end events of a worldline segment; vary an intermediate event to find angular momentum.

Special-relativistic expressions become Newtonian at low velocity.

When we reviewed special relativity in Chapter 1, we were forced to use relativistic expressions for energy and linear momentum, expressions different from those of Newton. Why did we accept these unfamiliar formulas? Because only relativistic expressions satisfy laws of conservation of both total energy and total linear momentum in high-speed particle collisions (and many other interactions) in an isolated system. What consolation did we have for leaving the old familiar territory? The consolation that in the limit of low velocity, relativistic expressions reduce to the Newtonian ones (provided we include the rest energy—the mass m—in the total energy of each isolated particle).

General-relativistic expressions become special-relativistic in remote, flat spacetime.

Now we shift to general-relativistic expressions for the energy and angular momentum of a stone moving near a nonspinning black hole. Why accept these new expressions? Because as the stone dips and swoops around the uncharged, non-spinning center of attraction, these expressions describe constant quantities: values for energy and angular momentum. What consolation do we have for leaving the familiar territory of special relativity? The consolation that in the limit of large radius—that is, in flat spacetime at a great distance from the black hole—the new expressions reduce to those of special relativity and further, for slow speeds, to Newtonian expressions.

Box 1 uses the now-familiar argument to derive the expression for map angular momentum from the Schwarzschild metric and the Principle of Maximal Aging. It starts with the approximate Schwarzschild metric in a small spacetime region of average map radius $\bar{r}$:

$$\Delta\tau^2 \approx \left(1 - \frac{2M}{\bar{r}}\right)\Delta t^2 - \frac{\Delta r^2}{1 - \dfrac{2M}{\bar{r}}} - \bar{r}^2\Delta\phi^2 \qquad (2)$$

Map angular momentum

The resulting constant of motion in Box 1 is $r^2 d\phi/d\tau$. We identify this constant of motion as L/m, the **map angular momentum** L of the stone divided by its mass m:

$$\frac{L}{m} \equiv r^2\frac{d\phi}{d\tau} \qquad \text{(Schwarzschild map angular momentum)} \qquad (3)$$

Symbols on the right side of this equation tell us that the units of L/m are meters. Recognition that (3) expresses angular momentum follows from noticing that this equation has the same form as in Newtonian mechanics except for wristwatch time τ in the denominator instead of Newton's universal time t. The presence of wristwatch time is not surprising, since the relativistic expression for linear momentum in special relativity, $m\,ds/d\tau$, also has wristwatch time in the denominator (Figure 2).

QUERY 1. Angular momentum and energy in global rain coordinates
Which map coordinates do Schwarzschild and rain global coordinates have in common? which are different? Make two predictions: (a) Is the expression for L/m in rain coordinates identical to (3) or different from it? (b) Is the expression for E/m in rain coordinates identical to (1) or different from it? Check your answer to part (b) by looking at equation (24) in Chapter 7, Inside the Black Hole.

In your derivation of E/m in Chapter 6, you made use of different wristwatch rates at different radial heights. I do not see any of that in this later derivation of L/m.

In the calculus limit, all the circular segments in Figure 1 have the same radius, so frame time runs at the same rate for both segments. For angular momentum, what changes aging is not variation of frame time rate with radius r. Rather it is the variation of wristwatch rate with tangential velocity, an effect of special relativity. Set $M = 0$ (no black hole!) and get exactly the same expression for angular momentum, $L/m = r^2 d\phi/d\tau$, about any arbitrary center that you may freely choose.

Our motto: *Think globally; measure locally!*

Angular momentum in (3) and energy in (1) express themselves in Schwarzschild map coordinates, which no observer near a black hole measures directly. L/m as well as E/m are mythical beasts, like unicorns. The big advantage of these expressions is that their values stay constant in map

BOX 1. Derivation of Expression for Map Angular Momentum

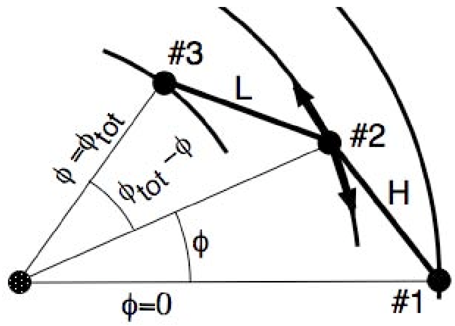

FIGURE 1 Find the intermediate angle ϕ such that the wristwatch time from event #1 to event #3 is a maximum.

Strategy: Apply the Principle of Maximal Aging to maximize the wristwatch time of a free stone flying across two adjoining worldline segments labeled H and L for "higher" and "lower." The stone emits flashes at events #1, #2, and #3, marking off the segments. Fix the *times* of all three flashes and the *positions* of flashes #1 and #3. Vary the *angle* of event #2 along a circle (two-headed arrow in Figure 1) in order to maximize the total wristwatch time between flashes #1 and #3. The result is an expression that is a constant of motion. Now the details.

Maximize the stone's total wristwatch time across the two segments by setting the derivative of $\Delta\tau$ with respect to $\Delta\phi$ equal to zero. To simplify the look of this derivative, temporarily replace increments $\Delta\tau$ and $\Delta\phi$ with τ and ϕ respectively.

Set the fixed angle ϕ of event #1 equal to zero and call ϕ_{tot} the fixed angle for event #3. Change the angle ϕ of event #2 by moving it either way along its circle, shown by the double-headed arrow in the figure. Let $\bar{r}_{\mathrm{H}}$ and $\bar{r}_{\mathrm{L}}$ be appropriate average values of the radii for segments H and L, respectively, and let τ_{H} and τ_{L} be the corresponding lapses of wristwatch time of the stone moving along these segments. With these substitutions, the approximate Schwarzschild metric (2) for higher Segment H becomes:

$$\tau_{\mathrm{H}} \approx [-\bar{r}_{\mathrm{H}}^2\phi^2 + (\text{terms without }\phi)]^{1/2} \qquad (4)$$

To prepare for the derivative that leads to maximal aging, take the derivative of this expression with respect to ϕ:

$$\frac{d\tau_{\mathrm{H}}}{d\phi} \approx -\frac{\bar{r}_{\mathrm{H}}^2\phi}{\tau_{\mathrm{H}}} \qquad (5)$$

Similarly for lower Segment L,

$$\tau_{\mathrm{L}} \approx [-\bar{r}_{\mathrm{L}}^2(\phi_{\mathrm{tot}}-\phi)^2 + (\text{terms without }\phi)]^{1/2} \qquad (6)$$

$$\frac{d\tau_{\mathrm{L}}}{d\phi} \approx \frac{\bar{r}_{\mathrm{L}}^2(\phi_{\mathrm{tot}}-\phi)}{\tau_{\mathrm{L}}} \qquad (7)$$

The total wristwatch time for both segments is $\tau = \tau_{\mathrm{H}} + \tau_{\mathrm{L}}$. Take the derivative of this expression with respect to ϕ, substitute from (5) and (7), and set the resulting derivative equal to zero in order to apply the Principle of Maximal Aging:

$$\frac{d\tau}{d\phi} \approx \frac{d\tau_{\mathrm{H}}}{d\phi} + \frac{d\tau_{\mathrm{L}}}{d\phi} \approx -\frac{\bar{r}_{\mathrm{H}}^2\phi}{\tau_{\mathrm{H}}} + \frac{\bar{r}_{\mathrm{L}}^2(\phi_{\mathrm{tot}}-\phi)}{\tau_{\mathrm{L}}} = 0 \qquad (8)$$

The condition for maximal lapse of wristwatch time becomes

$$\frac{\bar{r}_{\mathrm{H}}^2\phi}{\tau_{\mathrm{H}}} \approx \frac{\bar{r}_{\mathrm{L}}^2(\phi_{\mathrm{tot}}-\phi)}{\tau_{\mathrm{L}}} \qquad (9)$$

or in our Δ notation:

$$\frac{\bar{r}_{\mathrm{H}}^2\Delta\phi_{\mathrm{H}}}{\Delta\tau_{\mathrm{H}}} \approx \frac{\bar{r}_{\mathrm{L}}^2\Delta\phi_{\mathrm{L}}}{\Delta\tau_{\mathrm{L}}} \qquad (10)$$

The left side contains quantities for Segment H only; the right side quantities for Segment L only. We have discovered a quantity that is the same for both segments, a *constant of motion* for the free stone across *every* pair of adjoining segments along the worldline of the stone. In deriving this quantity, we assumed that each segment of the worldline is small. To guarantee this smallness, go to the calculus limit in (10), for which $\bar{r} \to r$; the constant of motion becomes

$$r^2\frac{d\phi}{d\tau} = \text{a constant of motion} \qquad (11)$$

The text identifies this constant of motion as L/m, the angular momentum of the stone per unit mass.

coordinates as a free stone streaks around a black hole—or around our Sun. These constants have global reach.

UNITLESS QUANTITIES

Unitless quantities

It is time to streamline our notation in order to simplify all expressions and their derivations. Unitless quantities allow us to apply *every* result immediately to *every* stone orbiting *every*

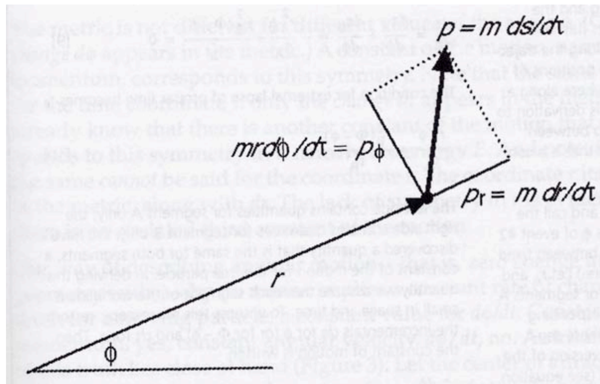

FIGURE 2 In flat spacetime angular momentum L is the product of r and the component of linear momentum p_ϕ in the tangential or ϕ direction, yielding $L = mr^2 d\phi/d\tau$, where $d\tau$ is the advance of wristwatch time of the stone whose momentum is being determined. The same formula applies to those spherically symmetric curved spacetimes whose metrics are written using reduced circumference r.

nonspinning black hole. For these purposes we use the following unitless quantities, labeled with asterisks. (Section 4 introduces the effective potential V_L.)

$$r^* \equiv \frac{r}{M}, \qquad (\text{time})^* \equiv \frac{(\text{time})}{M}, \qquad E^* \equiv \frac{E}{m} \tag{12}$$
$$V_\mathrm{L}^* \equiv \frac{V_\mathrm{L}}{m}, \qquad L^* \equiv \frac{L}{mM} = r^{*2}\frac{d\phi}{d\tau^*}$$

The symbol "(time)" means *any* measure of time. For example, "(time)" can be Schwarzschild global time t or rain global time T or shell time Δt_shell or time τ read on the wristwatch of the orbiter. Any measure of speed, such as v or v_shell, is already a unitless quantity—the fraction of light speed—so we do not add an asterisk to its symbol.

QUERY 2. Schwarzschild in unitless coordinates
Convert some central equations into unitless form:

A. Write the differential form of the Schwarzschild metric (2) in unitless coordinates:

$$d\tau^{*2} = (1 - 2/r^*)\, dt^{*2} - (1 - 2/r^*)^{-1}\, dr^{*2} - r^{*2} d\phi^2 \qquad \text{(Schwarzschild metric)} \tag{13}$$

B. Show that the unitless form of energy (1) is

$$E^* = (1 - 2/r^*)\, dt^*/d\tau^* \qquad \text{(Schwarzschild map energy)} \tag{14}$$

C. and the unitless form of angular momentum (3) is

$$L^* \equiv r^{*2}\,(d\phi/d\tau^*) \qquad \text{(Schwarzschild map angular momentum)} \tag{15}$$

BOX 2. Computing the Orbit

STRATEGY
Step 1: Use the constants of motion E^* and L^* to express dt^* and $d\phi$ in terms of the differential advance of satellite wristwatch time $d\tau^*$.
Step 2: Substitute these results into the Schwarzschild metric to find dr^* as a function of $d\tau^*$, r^*, E^*, and L^*.
Result: All map increments dt^*, $d\phi$, and dr^* are now locked to satellite time increment $d\tau^*$.
Computer starts with the initial position, advances satellite wristwatch time by $d\tau^*$as it updates values of t^*, ϕ, and r^*.
Now for the details.

Step 1: Relate dt^* to $d\tau^*$ using energy as a constant of motion (14):

$$dt^* = (1 - 2/r^*)^{-1}\, E^* d\tau^* \qquad (16)$$

Similarly find the advance of angle ϕ from the constant of motion angular momentum (15):

$$d\phi = (L^*/r^{*2}) d\tau^* \qquad (17)$$

Step 2: With dt^* and $d\phi$ now known in terms of $d\tau^*$, we lack only dr^* to specify completely the differential displacement of the satellite in space and time in one tick, $d\tau^*$, of satellite time. But dr^* appears, along with the three knowns, dt^*, $d\phi$, and $d\tau^*$, in the Schwarzschild metric (13). Into this metric substitute dt^* from (16) and $d\phi$ from (17) and solve for dr^*. The result is an equation that relates dr^* to $d\tau^*$:

$$dr^* = \pm\left[E^{*2} - \left(1 - \frac{2}{r^*}\right)\left\{1 + \left(\frac{L^*}{r^*}\right)^2\right\}\right]^{1/2} d\tau^* \qquad (18)$$

Starting with initial values of r^* and ϕ, equations (17) and (18) tell how each map polar coordinate changes as the satellite wristwatch ticks. If the Schwarzschild mapmaker demands that the increments be expressed also in terms of map time t^*, then (16) provides the corresponding change dt^*.

---

3 ■ FORECASTING THE ORBIT

Satellite wristwatch ticks off $d\tau$. From $d\tau$ find the resulting changes in dr, $d\phi$, and dt.

How do ϕ, r, and t change with change in satellite time τ?

We now have in hand the tools needed to calculate the step-by-step advance of the satellite through the world of space and time. Advance? Yes, (a) advance dt of map time, (b) advance $d\phi$, and (c) advance dr of map radius, all orchestrated—at our choice and for our convenience—to the time lapse $d\tau$ between ticks of the wristwatch aboard the satellite. Box 2 carries out this analysis.

Computing the orbit

Energy and angular momentum—constants of motion—plus the metric give us three equations in the three map unknowns dt, dr, and $d\phi$, expressed as functions of the advance $d\tau$ of the satellite's wristwatch. Starting from an initial event, the computer advances wristwatch time and calculates the consequent advance of map coordinates, summing results of a series of these steps to reckon the orbit, as spelled out in Box 2.

? *Hold on! Before you go any further—I notice that all the analysis so far in this chapter is in Schwarzschild map coordinates. What good is a global map description of orbits which nobody verifies by direct observation? You admit that map energy and map angular momentum are unicorns: mythical beasts. I want observable orbits!*

Guilty as charged! The first eleven pages of this chapter describe motion in map coordinates. After that, you will predict measurements made by each local shell observer and each orbiting astronaut. In curved spacetime no single local observer can measure or view a global orbit in its entirety. Remember our motto: "Think *globally*; measure *locally*!"

4 ■ EFFECTIVE POTENTIAL

The orbit at a single glance!

Box 2 puts in our hands powerful tools to describe any orbit of any free stone around a spherically symmetric center of attraction. Indeed, the wealth of possible orbits is so great that we need a simplifying strategy that allows us to quickly grasp many different orbits at a glance. Such a strategy makes use of the so-called **effective potential**, which focuses on radial motion alone (Section 4). Clearer even than our computed orbits, the **effective potential** lets us see immediately the central features of the motion of a stone.

DEVELOP INTUITION WITH INTERACTIVE COMPUTER MODELING

Another way to develop intuition is to let the computer draw orbits for you as it uses a much more sophisticated analytic procedure than the crude one outlined in Box 2. Slavomir Tuleja has developed just such a computer program, called GRorbits. See the refrences.

Preview: a pit in the potential

Vicious gravitational effects close to a black hole dominate the effective potential in which satellites move in its vicinity. The principal results can be simply stated: In addition to the attractive potential of gravity at great distances and the repulsive effects of angular momentum at intermediate distances, Einstein's theory adds at still shorter distances a pit in the potential, shown at the left in Figure 3 and some later figures.

The pit comes from constants of motion.

The potential? A pit in this potential? A potential attractive at large distances, repulsive at intermediate distances, and attractive again at yet smaller distances? Can we get this potential from principles that are simple, clear, and solid? Yes, from two principles: Energy as a constant of motion! Angular momentum as a constant of motion! And both of these from the single Principle of Maximal Aging.

Both the constant energy and the effective potential function become apparent when we derive from (18) (Box 2) the square of the radial velocity as registered in satellite wristwatch time:

$$\left(\frac{dr^*}{d\tau^*}\right)^2 = E^{*2} - \left(1 - \frac{2}{r^*}\right)\left[1 + \frac{L^{*2}}{r^{*2}}\right] \qquad (19)$$

QUERY 3. *Optional:* **Newtonian orbital motion**

The right side of (19) tells us a great deal about the difference between the stone's global motion described in Schwarzschild map coordinates and its motion described by Newton.

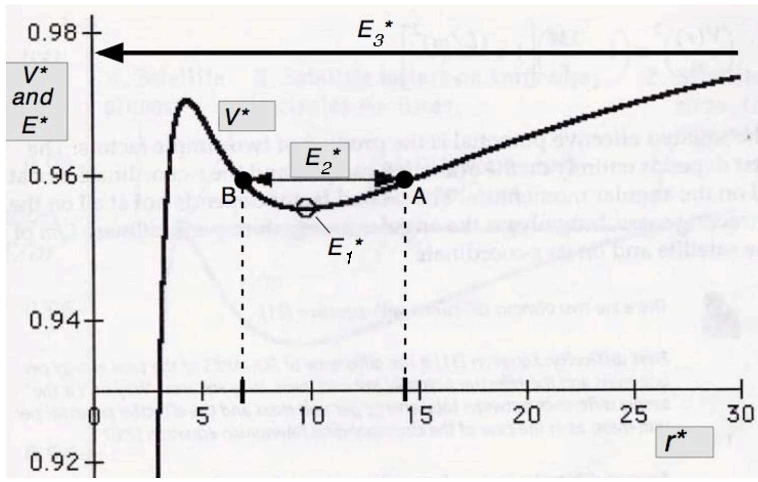

FIGURE 3 Effective potential curve for a stone orbiting a black hole with angular momentum $L^* = 3.75$. Stone energy is a constant of motion—independent of radius—so plots as a horizontal line. When the stone's energy equals the minimum of the potential, E_1^*, (open circle) the stone remains at a constant radius and orbits the black hole in a circle. A stone with slightly greater energy, E_2^*, (line with double arrow) oscillates back and forth between radial limits labeled A and B while orbiting around the center of attraction. When the stone's energy is greater than the peak of the effective potential, E_3^*, (upper horizontal line, with one arrow), the stone moves inward across the event horizon, all the while circulating with constant angular momentum.

A. Multiply out the right side of (19) and divide through by 2 to yield

$$\frac{1}{2}\left(\frac{dr^*}{d\tau^*}\right)^2 = \frac{1}{2}\left(E^{*2}-1\right) - \left(-\frac{1}{r^*} + \frac{L^{*2}}{2r^{*2}} - \frac{L^{*2}}{r^{*3}}\right) \qquad \text{(Schwarzschild)} \qquad (20)$$

B. In unitless coordinates, the Newtonian expression for angular momentum (with universal time t^* in unitless coordinates) is:

$$L^* \equiv r^{*2}\left(d\phi/dt^*\right) \qquad \text{(Newton)} \qquad (21)$$

Show that the Newton expression for the square of the velocity of the stone is, in unitless coordinates:

$$v^2 = \left(\frac{dr^*}{dt^*}\right)^2 + r^{*2}\left(\frac{d\phi}{dt^*}\right)^2 = \left(\frac{dr^*}{dt^*}\right)^2 + \frac{L^{*2}}{r^{*2}} \qquad \text{(Newton)} \qquad (22)$$

C. The Newtonian expression for gravitational potential energy per unit mass—chosen to go to zero far from the center of attraction— is:

$$V^* \equiv -\frac{1}{r^*} \qquad \text{(Newton)} \qquad (23)$$

Write down the Newtonian conservation of energy equation and solve it for the radial velocity. Write the result as:

$$\frac{1}{2}\left(\frac{dr^*}{dt^*}\right)^2 = E^* - \left(-\frac{1}{r^*} + \frac{L^{*2}}{2r^{*2}}\right) = E^* - V_{\rm L}^*(r^*) \qquad \text{(Newton)} \tag{24}$$

where $V_{\rm L}^*(r^*)$ is the **Newtonian effective potential**.

D. Sketch for the Newtonian case a $V_{\rm L}^*(r^*)$ and E^* *vs.* r^* diagram like that of Figure 3, along with horizontal lines for different values of E^*. Describe the resulting orbits and contrast them to those for motion in Schwarzschild spacetime.

Of course general relativity expression (20) is not just a version of Newton's equation (24)—general relativity is *not* the same as Newtonian mechanics! Look instead at the basic similarity of the right-hand sides of these two equations: a constant term from which we subtract a function of radius—the "effective potential"—that varies with the value of angular momentum L^*.

Conclusion of this analysis: It is the negative third term in the effective potential on the right side of (20), with r^{*3} in its denominator, that drives the effective potential downward as r^* approaches the event horizon—thereby creating the pit in the potential. This third term is the child of curved spacetime described by Schwarzschild.

Use equation (19) to introduce a new quantity $V_{\rm L}^*(r^*)$, the **Schwarzschild effective potential**. The subscript L reminds us that this effective potential is different for different values of the angular momentum L^*.

$$\left(\frac{dr^*}{d\tau^*}\right)^2 = E^{*2} - V_{\rm L}^{*2}(r^*) \tag{25}$$

Use a square of $V_{\rm L}^*$ in (25), in contrast with its first power in (24), because E^* is squared. This definition is purely cosmetic; see Part E in Query 4. The square of the effective potential $V_{\rm L}^{*2}(r^*)$ is what we have to take away from the constant squared energy term to get the square of the radial velocity.

Schwarzschild effective potential

$$\left[\frac{V_{\rm L}(r)}{m}\right]^2 \equiv V_{\rm L}^{*2}(r^*) \equiv \left(1 - \frac{2}{r^*}\right)\left(1 + \frac{L^{*2}}{r^{*2}}\right) \tag{26}$$

QUERY 4. Difference of squares?

The right-hand side of Newton's equation (24) is the difference between E^* and V^*, whereas the Schwarzschild equation (25) is the difference between the corresponding *squares* E^{*2} and V^{*2}. Why this difference? Investigate using the following outline:

A. Recast (26) in non-unitless form. Show that the result is

$$V_{\rm L}^2(r) \equiv \left(1 - \frac{2M}{r}\right)\left[1 + \frac{(L/m)^2}{r^2}\right] \tag{27}$$

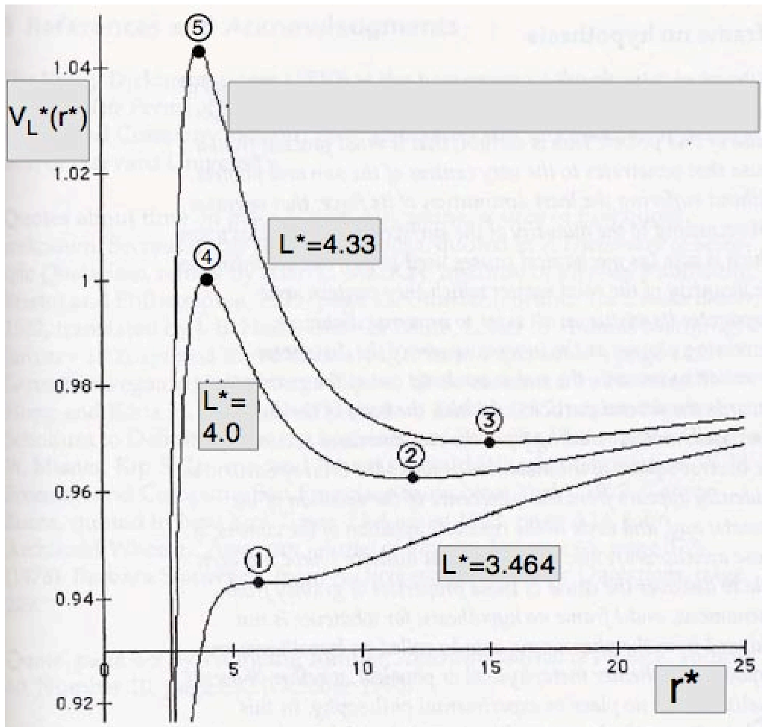

FIGURE 4 Radii of stable and unstable circular orbits around a black hole for a stone or spaceship with different values of angular momentum L^*. The radius of each *stable* circular orbit (circled numbers 1, 2, 3) lies at an effective potential *minimum*. The stable circular orbit of smallest radius lies at $r = 6M$ (circled number 1). The stone cannot be in a stable circular orbit for $r < 6M$. The radius of each *unstable* circular orbit (circled numbers 4 and 5) lies atop an effective potential *maximum*. A spaceship circling at either of these radii needs fine-tuning rockets to keep it balanced on that peak. Otherwise the slightest perturbation will push the orbiter to the left in the diagram, plunging into the black hole, or to the right, moving it outward.

B. Evaluate this expression in the limit that the black hole fades away: $M \to 0$, so that spacetime becomes flat. Angular momentum still exists with respect to any point we choose, for example the former location of the black hole. Show that energy will have a different numerical value in flat spacetime, but is still a constant of motion. Show that relativistic equation (20) becomes, for $M \to 0$,

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}\left(\frac{E}{m}\right)^2 - \frac{1}{2} - \frac{(L/m)^2}{2r^2} \qquad \text{(Schwarzschild, } M \to 0) \qquad (28)$$

Surprise: In flat spacetime the difference of squares in (26) remains. This shows that the difference in form between equations (24) and (25) results from special relativity rather than general relativity.

C. Show that, for $M \to 0$, Newtonian expression (24) becomes:

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \frac{E}{m} - \frac{(L/m)^2}{2r^2} \qquad \text{(Newton, } M \to 0) \tag{29}$$

D. Special relativistic equation (28) is still not the same as Newtonian equation (29). And why should it be? Special relativity is not Newtonian mechanics! However, if we let the speed of the stone become small compared with the speed of light, then, first of all, $d\tau \to dt$ on the left of (28). Show that the right sides of the two equations become equal in the limit of low velocity.

E. *Optional:* From the above analysis, show that it is purely a matter of convention whether we define the Schwarzschild effective potential in (26) to be the first power or the second power of $V_{\rm L}$. Equation (25) would look a little screwy if the right-hand side had E^* to the second power and $V_{\rm L}$ to the first power, but the content would be exactly the same, given the corresponding power on $V_{\rm L}$ in its definition (26),

---

Shape of effective potential curve

The effective potential has this wonderful feature: its radial dependence is determined only by the angular momentum of the satellite, not at all by its energy. Draw energies E^* of different values as horizontal lines on the effective potential graph, as in Figure 3. Then use the resulting combined plot to predict the qualitative shape of various orbits in a given potential, as analyzed in the caption to that figure. Note especially that a stone with energy E_2^* and initial position given by the point labeled A in Figure 3 will thereafter move back and forth between points A and B in that figure.

Note that $dr^*/d\tau^*$ in equation (25) is real only where E^{*2} has a value greater than $V_{\rm L}^{*2}(r^*)$. Therefore both ends of the horizontal energy line labeled E_2^* in Figure 3 terminate where it meets the $V_{\rm L}^*(r^*)$ curve. At these points—called *turning points*—$E^{*2} = V_{\rm L}^{*2}$, so $dr/d\tau = 0$ in (25). At a turning point the radial motion stops for an instant (while tangential motion continues), then reverses direction. In Figure 3 radial motion oscillates back and forth between turning points labeled A and B.

In this chapter we focus on the circular orbit, illustrated by the radius of the open circle labeled E_1^* in Figure 3. In that circular orbit the stone's energy rests at the minimum of the effective potential curve and the stone rides round and round the black hole without changing radius.

A circular orbit at the radius of a minimum of the effective potential is called a **stable circular orbit**, because a slight radial displacement puts the stone into a region where the slope of the potential urges it back toward the equilibrium point. Earth and each solar planet rests near such a minimum of the effective potential for its unique angular momentum. In fact each planet oscillates radially back and forth across this minimum in a radial motion similar to that labeled E_2^* in Figure 3. (Of course, these motions are almost exactly those predicted by Newton, but not quite: see Chapter 12, Advance of Mercury's Perihelion.) Several such stable radial positions for the black hole case are numbered 1 through 3 in Figure 4.

Stable orbit at potential minimum.

Einstein opens up a second set of radial locations where the effective potential has zero slope, illustrated by points labeled 4 and 5 in Figure 4. Each of these is a *maximum* of the effective potential curve; *exactly* at this radius the stone experiences no tendency to move either to larger radius or to smaller radius, so will stay put radially, riding round and round the black hole at constant radius. We call these **unstable circular orbits**.

Knife-edge orbit at potential maximum

But look out! We also call these **knife-edge orbits**. Slight cosmic wind, firing of a projectile, or ejection of the day's trash may give our spaceship a tiny radial motion. Once displacement from the peak of effective potential occurs, the slope of the effective potential urges the spaceship *away* from the point of zero slope. Departure from the knife-edge radius leads to decisive motion either radially outward, or else—horrors!—inward toward a dark fate. "Why, oh why," our captain cries, "didn't I carry along a booster rocket! A tiny rocket boost pushing us outward could have reversed our slow inward motion, allowing us to escape. But now it's too late!"

5 ■ PROPERTIES OF CIRCULAR ORBITS

Details! Details!

QUERY 5. Map radii of circular orbits

A. Circular orbits are possible at radii where the effective potential has zero slope. Zero-slope of the effective potential occurs at the same radius as zero-slope of the *square* of the effective potential. Take the radial derivative of both sides of (26), set this derivative equal to zero, and show the result:

$$r^{*2} - L^{*2} r^* + 3L^{*2} = 0 \qquad \text{(circular orbit)} \tag{30}$$

B. Solve (30) for the radius r^* of circular orbits:

$$r^* = \frac{L^{*2}}{2}\left[1 \pm \left(1 - \frac{12}{L^{*2}}\right)^{1/2}\right] \qquad \text{(circular orbit)} \tag{31}$$

Refer to Figure 4. The plus sign in (31) yields the radius at the minimum of the effective potential and therefore yields the radii of stable circular orbits. The minus sign yields the radius at the maximum of effective potential and thus the radii of unstable circular orbits of smaller radius. (*Optional:* Verify the statements about minima and maxima by taking the second derivative of V_L^{*2} with respect to r^* to determine whether this second derivative is positive or negative at the given values of r^*.)

C. Show that there are no circular orbits of any kind for angular momentum given by the inequality $L^* < (12)^{1/2}$. Show that for the minimum angular momentum, the radius of the circular orbit is $r^* = 6M$. This is the *stable* circular orbit of smallest radius. See the case with the circled number 1 in Figure 4.

QUERY 6. Angular momentum of satellite in circular orbit
Solve (30) to find L^{*2} for a circular orbit at map radius r^*:

$$L^{*2} = \frac{r^{*2}}{r^* - 3} \qquad \text{(circular orbit, } r^* > 3) \tag{32}$$

Note that this expression is valid for both unstable and stable circular orbits.

QUERY 7. Satellite shell speed in a circular orbit

Compute the shell speed of the satellite in a circular orbit.

A. Consider two ticks of the satellite clock, separated by wristwatch time $\Delta\tau^*$ and by zero distance in the satellite frame, but separated by time $\Delta t^*_{\text{shell}}$ and by distance $\bar{r}^*\Delta\phi$ in the shell frame. The relation between $\Delta t^*_{\text{shell}}$ and $\Delta\tau^*$ is just the special-relativity expression

$$\Delta t^*_{\text{shell}} = \gamma_{\text{shell}}\Delta\tau^* = (1 - v^2_{\text{shell}})^{-1/2}\Delta\tau^* \tag{33}$$

where γ_{shell} has an obvious definition. Knowing angular momentum, we can now use (33) to reckon shell speed:

$$\begin{aligned} v_{\text{shell}} &= \lim_{\Delta t^*_{\text{shell}} \to 0} \left(\frac{\bar{r}^*\Delta\phi}{\Delta t^*_{\text{shell}}} \right) = (1 - v^2_{\text{shell}})^{1/2} \, \frac{r^{*\,2} d\phi}{r^* d\tau^*} \\ &= (1 - v^2_{\text{shell}})^{1/2} \, \frac{L^*}{r^*} \end{aligned} \tag{34}$$

From this equation, show that

$$v^2_{\text{shell}} = \left(1 + r^{*2}/L^{*2}\right)^{-1} \qquad \text{(circular orbit)} \tag{35}$$

From equation (32) show that

$$r^{*2}/L^{*2} = r^* - 3 \qquad \text{(circular orbit)} \tag{36}$$

Substitute this into equation (35) to find

$$v^2_{\text{shell}} = \frac{1}{r^* - 2} \qquad \text{(circular orbit)} \tag{37}$$

Equation (37) is valid for both stable and unstable circular orbits.

B. What is the value of the shell speed v_{shell} in the stable orbit of smallest radius, $r^* = 6$ (part D of Query 4)?

C. Verify that the minimum map radius of the unstable circular orbit is $r^* = 3$. (*Hint:* What is the upper limit of the shell speed of a stone?)

D. From (32) show that, as a limiting case, the angular momentum L^* is infinite for the unstable circular orbit of minimum radius. *Discussion:* How can the angular momentum possibly go to infinity? It does so only as a limiting case. Recall that the angular momentum is equal to $L^* = r^{*2} d\phi/d\tau^*$. The relation between wristwatch time $d\tau^*$ and shell time $dt^*_{\rm shell}$ is given by (33), the usual time-stretch formula of special relativity. As the satellite speed approaches the speed of light, the advance of wristwatch time, the proper time increment, becomes smaller and smaller. Then it takes nearly zero wristwatch time for the satellite—still of mass m—to circulate once around the black hole. Because $d\tau^*$ is in the denominator of the expression for angular momentum, the angular momentum L^* grows large without limit. The speed of light is the limiting speed of a stone, so this is a limiting case. This analysis has consequences for time travel using unstable circular orbits around a black hole, Exercise 6 at the end of this chapter.

QUERY 8. Shell and map energies in a circular orbit

A. Using the definition of $E_{\rm shell}$ for the locally-flat shell spacetime and (37), show that

$$E^*_{\rm shell} \equiv \left(1 - v^2_{\rm shell}\right)^{-1/2} = \left(\frac{r^* - 2}{r^* - 3}\right)^{1/2} \qquad \text{(circular orbit)} \qquad (38)$$

B. Show that the general equation connecting shell and map energies from equation (22) of Chapter 6, Plunging, in unitless coordinates is:

$$E^*_{\rm shell} = \left(1 - \frac{2}{r^*}\right)^{-1/2} E^* \qquad (39)$$

C. Using (38) and (39), show that for a circular orbit:

$$E^* = \frac{r^* - 2}{r^{*1/2}(r^* - 3)^{1/2}} \qquad \text{(circular orbit)} \qquad (40)$$

All of these equations are true for both stable and unstable circular orbits.

D. Find numerical values for both E^* and $E^*_{\rm shell}$ at the following values of r^*:

(a) at infinity,

(b) at the minimum radius for a stable circular orbit,

(c) at the minimum radius for an unstable circular orbit. Analyze and justify the apparently crazy result of this Item (c).

QUERY 9. Satellite map speed in a circular orbit

A satellite in circular orbit has tangential map speed equal to the tangential distance covered per unit map time. We *defined* map radius r as reduced circumference, that is, the measured circumference of a circle concentric to the black hole divided by 2π. From this definition, the map differential of tangential motion is $rd\phi$, which is the numerator of the instantaneous tangential speed of a stone in circular orbit whose denominator is map differential dt

$$v_\phi \equiv \frac{rd\phi}{dt} \qquad \text{(map tangential speed, circular orbit)} \tag{41}$$

A. Find an expression for the tangential map speed in a circular orbit as follows: Use expression (3) for the angular momentum to replace $d\tau$ in the Schwarzschild metric (13) by an expression in $d\phi$. From definition (41) show that—for $dr = 0$—the square of tangential map speed is.

$$v_\phi^2 = \frac{1 - 2/r^*}{1 + r^{*2}/L^{*2}} = \frac{1}{r^*} \qquad \text{(circular orbit)} \tag{42}$$

The final step in (42) uses equation (36).

B. What is the value of the map speed of a satellite in the stable circular orbit of smallest radius?

C. Is equation (42) valid for *unstable* circular orbits? If so, what is the value of the map speed in the unstable circular orbit of minimum radius?

D. In a circular orbit of given radius r^*, is map satellite speed greater or less than shell satellite speed?

---

6 ■ MODEL OF A QUASAR

Beacon of the heavens

A **quasar** ("quasi-stellar object") is a distant astronomical object that pours out radiation at a prodigious rate. Quasars are steady sources of light that we can see at the greatest distance. Almost certainly a quasar is a spinning black hole (Chapter 14, Spinning: The Kerr Map), but here we take a first stab at modeling a quasar using a non-rotating black hole.

A likely model of quasar emission postulates an **accretion disk**, a swirling plane of gas in orbit around a black hole. Gas atoms and ions are heated to high temperature by friction among nearby atoms and ions in the accretion disk and emit electromagnetic radiation, which we observe at a great distance from the quasar. The radiated energy comes from change in orbital energy of each atom as it moves successively to orbits of lower and lower energy. What causes this change in orbit?

Equation (31) tells us that angular momentum alone, not energy, determines the radius of a circular orbit. Figure 4 shows that "our atom" decreases angular momentum as moves from a higher circular orbit to a lower circular orbit. How is this angular momentum extracted? A crude model notes that atoms in an adjacent slightly higher orbit circulate more slowly than "our atom." Friction between atoms in these two orbits tends to increase velocity—and therefore angular momentum—of atoms in the higher orbit. In turn, atoms moving faster in the adjacent smaller orbit transfer angular momentum to "our atom." This mechanism, if correct, moves angular momentum outward through the accretion disk. The vertical axis in Figure 4 also indicates the change in energy that accompanies this decrease in orbit radius, energy that is radiated away for us to see at a great distance.

Sooner or later our atom reaches radius $r = 6M$, the minimum radius of stable circular orbit, labeled one in Figure 4. At this point our atom continues to lose angular momentum to atoms in the next-higher orbit, but now there are no atoms in a smaller radius. Any further loss of angular momentum plunges our atom into the black hole. Once it crosses the horizon, no further radiation can reach any external observer.

QUERY 10. Map energy given up by "our atom."

A. Start with an atom in a circular orbit at such a great distance, so that its initial energy is approximately equal to its mass, $E^* \approx 1$. Now think of its energy later, as it moves in the stable circular orbit of minimum radius, $r_{\min} = 6M$. Using (40), find the map energy E of the atom in this minimum-radius circular orbit. How much map energy has the atom given up during the process of dropping gradually from a large distance to the minimum-radius stable circular orbit?

B. Suppose that all the map energy given up according to Item A is emitted in electromagnetic radiation on the way down to the orbit at $r^* = 6$. To one significant digit, the total radiated map energy is $E^* = 0.06$. What percentage is this of initial map energy? Find this value to two significant digits.

NOTE: No nuclear reaction on Earth releases as much as one percent of the rest energy of its constituent particles. Chapter 14, Spinning, shows that for a black hole of maximum spin, the fraction of initial mass emitted from a similar minimum-stable orbit approaches half of its rest mass. No wonder quasars are such efficient emitters!

? *Wait a minute! Map energy is not directly measurable. Yet in this model of a quasar we equate the energy of radiation detected at a distance with the drop in Schwarzschild map energy.*

! Far from the black hole, shell energy has the same value as map energy. The observer, remote from the black hole, receives energy equal to the change in map energy. This energy is smaller than the shell energy of radiation emitted near the black hole, as you show in the following Query. Gravitational red shift decreases the energy of this radiation as it climbs out of the gravitational well.

QUERY 11. Radiation climbs the ladder

A. Use (38) to calculate the *shell* energy E^*_{shell} of the atom (a) at a great distance from the black hole, and (b) in the circular orbit at $r^* = 6$. Answer of (b) to two significant digits is $E^*_{\text{shell}} = 1.2$. Calculate this answer to three significant digits.

B. Can we correctly calculate the radiated energy detected at a great distance by subtracting the shell energy at $r^* = 6$ from the initial shell energy at a great distance? Explain your answer.

---

7 ■ REFERENCES

Emily Dickinson poem at the beginning of the chapter is from R. W. Franklin, *The Poems of Emily Dickinson, Variorum Edition* 1998, The Belknap Press of Harvard University. This poem is variation E of the poem with Franklin number 1570, written about 1882. Reprinted and modified with permission of Harvard University.

GRorbits software program that displays orbits of a stone and light flash available at `http://stuleja.org/grorbits/`

PROBLEMS

Use unitless quantities in the following exercises, but practice converting results to everyday units such as meters and seconds.

1. Three Views of a Circular Orbit

A shell observer on the shell of average map radius $\bar{r}$ compares measurements with an observer in a circular orbit moving tangentially past him with speed v_{shell}. The orbiter is in a local inertial frame, and so, by definition, is the shell observer. Therefore, special relativity applies, and we can use Lorentz transformations between the two frames, calling the shell frame "laboratory," the orbiting frame "rocket," and the measured relative speed v_{shell}. Choose the positive tangential x-axis to be the direction of motion of the orbiter "rocket" frame with respect to the shell "laboratory" frame.

A. The orbiter does one circuit and returns to the same shell clock. What time lapse $\Delta t^*_{\text{shell}}$ does this shell clock record for one circuit, in terms of v_{shell}?

B. During one complete circuit of the shell, the orbiting clock "runs slow" by the usual stretch factor γ of special relativity when compared with the shell clock to which it returns. What time lapse $\Delta t^*_{\text{orbiter}}$ does the orbiter's clock record between sequential passes over the recording shell clock, in terms of v_{shell}?

C. What is the map angular momentum L^* of the orbiter in terms of v_{shell}? (The answer is *not* $r^* v_{\text{shell}}$.)

D. What time lapse Δt^* does the Schwarzschild mapmaker record for one circuit of the orbiter, in terms of v_{shell}?

E. Part D of Query 4 shows that the smallest radius for a stable circular orbit is $r^* = 6$; equation (37) determines that in this orbit the orbiter's shell speed $v_{\text{shell}} = 0.5$, half the speed of light. Assume the central attractor to be Black Hole Alpha, with $M = 5000$ meters. Following, to one significant digit, are L/m and the times of one orbit for the shell observer, orbiter, and mapmaker. Find the value of L/m and these times to three significant digits. (Notice that this orbiter completes one circuit in approximately 1 millisecond on her wriswatch!)

$$\begin{aligned} L/m &\approx 2 \times 10^4 \text{meters} && (43)\\ \Delta t_{\text{shell}} &\approx 4 \times 10^5 \text{meters} && \text{(one orbit)}\\ \Delta t_{\text{orbiter}} &\approx 3 \times 10^5 \text{meter} && \text{(one orbit)}\\ t &\approx 5 \times 10^5 \text{meters} && \text{(one orbit)} \end{aligned}$$

F. How do you respond to the objection that a complete orbit vastly exceeds the dimension of an inertial frame and, at least for Parts A through E of this exercise, one orbit can take longer than the shell time during which the shell frame remains inertial?

2. When are Newton's Circular Orbits Almost Correct?

Your analysis of the Global Positioning System (GPS) in Chapter 4 calculated values of radius and orbital speed of a GPS satellite in circular orbit using Newtonian mechanics, with the prediction that the general relativistic analysis gives essentially the same values of radius and speed for this application. Under what circumstances are circular orbits predicted by Newton indistinguishable from circular orbits predicted by Einstein? Answer this question using the following outline or some other method.

A. Find the Newtonian expression similar to equation (31) for the radius of a stable circular orbit, starting with equation (24).

B. Recast equation (31) for the general-relativistic prediction of r^* for stable orbits in the form

$$r^* = r^*_{\text{Newt}}(1 - \epsilon) \tag{44}$$

where r^*_{Newt} is the radius of the orbit predicted by Newton and ϵ is the small fractional deviation of the actual orbit from the Newtonian prediction. This expression neglects differences between the Newtonian and relativistic values of L^* when expressed in the same units. Use the approximation inside the front cover to derive a simple algebraic expression for ϵ as a function of r^*_{Newt}.

C. Set your expression for ϵ equal to 0.01 as a criterion for good-enough equality of the radius according to both Newton and Einstein. Find an expression for r^*_{min}, the smallest value of the radius for which this approximation is valid.

D. Find a numerical value for $r^*_{\rm min}$ in meters for our Sun. Compare the value of $r{*}_{\rm min}$ with the radius of the Sun.

E. What is the value of ϵ for the radius of the orbit of the planet Mercury, whose orbit has an average radius 0.387 times that of Earth?

3. Kepler's Laws of Planetary Motion

Johannes Kepler (1571-1630) provided a milestone in the history of mechanics with his **Three Laws of Planetary Motion**, deduced from a huge stack of planetary observations made by Tycho Brahe.

1. A planet orbits around the Sun in an elliptical orbit with the Sun at one focus of the ellipse.

2. The radius vector from the Sun to the planet sweeps out equal areas in equal times.

3. The square of the period of the planet is proportional to the cube of the planet's mean distance from the Sun.

A. Show by a simple symmetry argument that Kepler's Second Law is true for circular orbits around a black hole.

B. From equation (42) show that for *circular* orbits the period squared is given by the expression

$$t^{*2} = (2\pi)^2 r^{*3} \qquad (t^* = \text{period of one circular orbit}) \tag{45}$$

so that Kepler's Third Law (when expressed in Schwarzschild map coordinates) is also valid for circular orbits around a black hole.

C. Kepler's Third Law is sometimes called **The 1-2-3 Law** from the exponents in the following equation. Show that for circular orbits, in our regular geometric notation,

$$M \equiv M^1 = \omega^2 r^3 \tag{46}$$

where $\omega \equiv 2\pi/t$, with t the map time for one orbit.

D. *Preview:* Is Kepler's First Law true for noncircular orbits near a black hole? We shall see that the planet Mercury departs slightly from this law: Chapter 11, Advance of Mercury's Perihelion. Look at Figure 9 in Chapter 10. State your conclusion about the validity of Kepler's First Law for orbits about a black hole.

4. Time Travel Using Black Hole *Stable* Circular Orbits

You are on a panel of experts called together to evaluate a proposal from the Space Administration to travel forward in time using the difference in rates between a clock in a stable circular orbit around a black hole and our clocks remote from the black hole. Give your advice about the feasibility of the scheme, based on the following analysis or on your own.

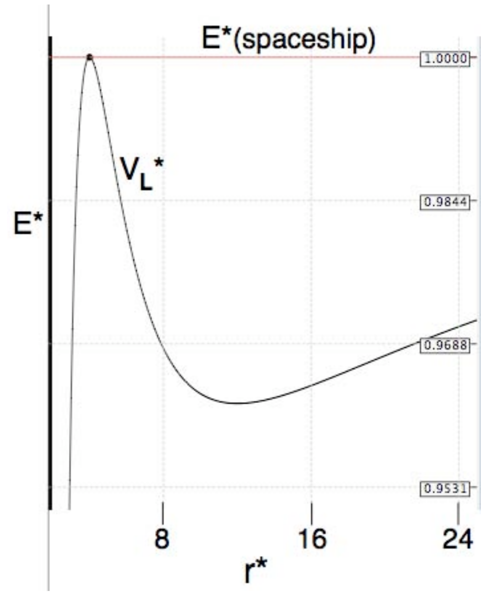

FIGURE 5 Insertion into a knife-edge orbit at radius $r^* = 4$ with energy $E^* \approx 1$ equal to that of a spaceship moving slowly at large radius in a direction chosen to give it the value of L^* required to establish the peak value for V_{L}^*.

A. Consider two sequential ticks of the clock of a satellite in a stable circular orbit around a black hole. We want to find the ratio $d\tau^*/dt^*$. The numerator in this fraction is equal to the wristwatch time $d\tau$ between the ticks in the frame of the satellite; the denominator is the map time lapse dt^*, which is also the wristwatch time lapse of a distant observer at rest. Use the expression for angular momentum to eliminate $d\phi$ from the Schwarzschild metric in this case to obtain

$$\left(\frac{d\tau}{dt}\right)^2 \equiv \left(\frac{d\tau^*}{dt^*}\right)^2 = \frac{1-(2/r^*)}{1+(L^*/r^*)^2} = \left(1-\frac{3}{r^*}\right)^{1/2} \qquad (47)$$

where the final step uses equation (36).

B. What is the value of the ratio $d\tau/dt$ for the stable circular orbit of smallest possible radius, $r^* = 6$?

C. What rocket speed in flat spacetime gives the same ratio of rocket clock time to "laboratory" time" as the stable circular orbit of smallest radius?

D. Does the proposed time travel method require rocket fuel?

E. Based on this analysis, do you recommend in favor of—or against—the Space Administration's proposal for time travel using stable circular orbits around a black hole?

5. Time Travel Using the Black Hole *Unstable* Circular Orbits

Whatever your vote on the time travel proposal of Exercise 5, the majority on your panel rejects the proposal because it requires extra rocket power for insertion into and extraction from the circular orbit at $r^* = 6$. The Space Administration comes back with a new proposal that uses an unstable circular orbit, assuming that an automatic device can fire small rockets to maintain the satellite safely on the radial knife-edge of the effective potential. The Space Administration notes that such an orbit can be set up to require *zero* rocket burns, either for insertion or extraction from unstable circular orbit. As an example, they present Figure 5 for the case of nonrelativistic distant velocity, so that the map energy of the satellite is $E^* \approx 1$. The direction of the remote velocity of the satellite is adjusted to achieve the value of L^* required so that $V_{\mathrm{L}}^* = E^* = 1$ at the peak, as shown in Figure 5. They boast that the time stretch factor is increased enormously by high satellite speed in the unstable orbit without the need for rocket burns to achieve that speed.

A. The condition shown in Figure 5 means that $V_{\mathrm{L}}^* = 1$ at the peak shown in equation (26). The resulting equation plus equation (31) with the minus sign are two source equations in the two unknowns r^* and L^*. To solve them is an algebraic mess, so we give you the results: $r^* = 4$ and $L^* = 4$. Verify these results by substituting them into either or both of the source equations.

B. *Optional:* Describe in words how the commander of the satellite sets the desired value of L^* while still at a great distance, without changing the remote nonrelativistic speed v_{far}.

C. What is the factor $d\tau/dt$ for the satellite in this orbit? What speed in flat spacetime gives the same time-stretch ratio?

D. Does the satellite require a significant rocket burn to leave its unstable circular orbit and return remote position? If so, what will be its speed at that distant location?

E. After its long interstellar trip, the satellite approaches the black hole at relativistic speed. The crew does not want to use a rocket burn to change satellite speed, but rather only its direction of motion to enter an unstable circular orbit with the same map energy it already has. Draw a figure similar to Figure 5 for this case. Can the astronauts find an unstable circular orbit on which to perch, no matter how large is its relativistic energy of approach?

The small size of the ratio $d\tau/dt$ for the case shown in Figure 5 and analyzed in Item C leads the review panel to reject the proposal to use an unstable circular orbit for the purpose of time travel. However, the review panel approves the use of an unstable circular orbit as an essentially zero-cost parking orbit from which to take data on a black hole over an extended time. Little rocket power is required to put the approaching spaceship into that orbit, only the thrust to change direction in order to give the spaceship the correct value of L^*. After they finish collecting data, the astronauts can choose the time of radially-outward push-off so that they return toward home base at the same speed at which they approached, even if this speed is relativistic (Item E). In summary, the explorers need almost no rocket power to study and return from a nonspinning black hole. Further details in Chapter 10, General Orbits.