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Chapter 9. Circular Orbits

- *How are orbits around a black hole different from planetary orbits around our Sun?*
- *How close to a black hole can I move in a circular orbit?*
- *Can I reach the speed of light in a circular orbit around a black hole?*
- *Can I orbit a black hole to travel forward in time? backward in time?*
- *What is the source of radiation energy that a "quasar" pours out in such prodigious quantity?*

CHAPTER

9

Circular Orbits

Edmund Bertschinger & Edwin F. Taylor †

How happy is the little Stone
That orbits a Black Hole alone*
And doesn't care about Careers
And Exigencies never fears –
Whose Coat of elemental Brown
A passing Universe put on
And independent as the Sun
Associates or glows alone
Fulfilling absolute Decree
In casual simplicity –

—Emily Dickinson

*Line two in the original reads:
That rambles in the Road alone

9.1 ■ STEP OR ORBIT?

"Go straight!" implies maximal aging. Maximal aging implies that map energy and map angular momentum are constants of motion.

Nature shouts at the stone "Go straight!"

A stone in orbit streaks around a black hole—or around Earth. What tells the stone how to move? Spacetime grips the stone, giving it the simplest possible command: "Go straight!" or in the more legalistic language of the Principle of Maximal Aging, "Follow the worldline of maximal aging across the next two adjoining local inertial frames." From instant to instant this directive is enough to tell the stone what to do next, the next step to take in its motion.

Constants of motion: map energy and map angular momentum

Instructions for its next step—its travel across the next pair of adjoining frames—is enough for the stone, but it is not enough for us. We want more: We seek a global description of the trajectory of the stone through spacetime—its worldline. In this chapter we begin to win from the metric and the Principle of Maximal Aging a global account of the orbit of a stone around a spherically symmetric center of attraction. This global view derives from map quantities that do not change as the motion progresses. These unchanging

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 Latest drafts at dropsite exploringblackholes.com, with a request for comments.

map quantities—constants of motion—are map energy and map angular momentum.

Starting with equation (6.12), we used Schwarzschild map energy as a constant of motion to describe the motion of a free stone: (#eq:3)

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} \qquad \text{(Schwarzschild map energy of a stone)} \qquad (9.1)$$

This expression for map energy is valid for a free stone moving in any direction around the black hole, not just radially.

QUERY 9.1. *Optional:* **Create energy?**
Suppose that equation (9.1) is correct only for a stone that moves *radially* inward or outward, and that—in contrast—a stone moving at some oblique angle with respect to the radial direction has *less* map energy than that given by (9.1).

A. Design a machine that moves a stone repeatedly around a closed spatial path near a black hole with the purpose of giving out more map energy than is required to run the machine.
B. Define an **advanced civilization** as one that can carry out any engineering task that does not violate fundamental physical law. How could an advanced civilization transport to a space colony distant from the black hole the excess energy from the machine you designed in Item A?
C. Can you design a similar energy-creating machine in the case that the obliquely-moving stone has *more* energy than a radially-moving stone?
D. What does your (magic!) machine prove about equation (9.1)?

This Chapter 9, Circular Orbits

In the present chapter we derive the second constant of orbital motion—map angular momentum—and use it, together with map energy, to describe circular orbits around a nonspinning black hole. We find that a free stone can follow a *stable* circular orbit only at a radius greater than $r = 6M$ and an *unstable* circular orbit between that radius and the radius of the light sphere at $r = 3M$. No circular orbit exists for a radius smaller than $r = 3M$.

Next Chapter 10, General Orbits

The following Chapter 10, Orbiting, describes trajectories used to insert our spaceship into any circular orbit, to transfer it from one circular orbit to another one, and to cross the event horizon.

9.2 ■ MAP ANGULAR MOMENTUM FROM MAXIMAL AGING

Vary the map angle of an intermediate event to find map angular momentum.

Special-relativistic expressions become Newton's at low velocity.

When we reviewed special relativity in Chapter 1, Speeding, we were forced to use expressions for energy and linear momentum different from those of Newton. Why did we accept these unfamiliar formulas? Because in high-speed particle collisions (and many other interactions), only relativistic expressions satisfy laws of conservation of both total energy and total linear momentum. What consolation did we have for leaving the old familiar territory? The consolation that in the limit of low velocity, relativistic expressions reduce to the Newtonian ones (provided we include the rest energy—the mass m—in the total energy of each isolated particle).

General-relativistic expressions become special-relativistic in flat spacetime.

Now we shift to general-relativistic expressions for the map energy and map angular momentum of a stone moving near a nonspinning black hole. Why do we accept these new, unfamiliar expressions? Because as the stone dips and swoops around the uncharged, non-spinning center of attraction, these expressions describe constants of motion: map energy and map angular momentum. What consolation do we have for leaving the familiar territory of special relativity? The consolation that in the limit of large radius—that is, at a great distance from the black hole—the new expressions reduce to those of special relativity and, for slow speeds, to Newton's expressions.

Box 9.1 uses the now-familiar Principle of Maximal Aging to derive the expression for map angular momentum from the Schwarzschild metric. Start with the approximate Schwarzschild metric in a small spacetime region of average map radius $\bar{r}$: (#SchwarzB)

$$\Delta\tau^2 \approx \left(1 - \frac{2M}{\bar{r}}\right)\Delta t^2 - \frac{\Delta r^2}{1 - \dfrac{2M}{\bar{r}}} - \bar{r}^2\Delta\phi^2 \qquad (9.10)$$

Map angular momentum

The resulting constant of motion in Box 9.1 is $r^2 d\phi/d\tau$. We identify this constant of motion as L/m, the **map angular momentum** L of the stone divided by its mass m: (#eq:2)

$$\frac{L}{m} \equiv r^2\frac{d\phi}{d\tau} \qquad \text{(Schwarzschild map angular momentum)} \quad (9.11)$$

Symbols on the right side of this equation tell us that the units of L/m are meters. Recognition that (9.11) expresses map angular momentum follows from noticing that this equation has the same form as in Newton's mechanics except for the wristwatch time differential $d\tau$ in the denominator instead of the differential dt of Newton's universal time. The presence of wristwatch time is not surprising, since the relativistic expression for linear momentum in special relativity, $m\,ds/d\tau$, also has wristwatch time in the denominator—equation (1.36) and Figure 9.2.

QUERY 9.2. Map angular momentum and map energy in global rain coordinates
Which map coordinates do Schwarzschild and rain global coordinates have in common? Which are different? Make two predictions: (a) Is the expression for L/m in rain coordinates identical to (9.11) or different from it? (b) Is the expression for E/m in rain coordinates identical to (9.1) or different from it? Check your answer to part (b) with equation (7.24).

BOX 9.1. Derivation of Expression for Map Angular Momentum

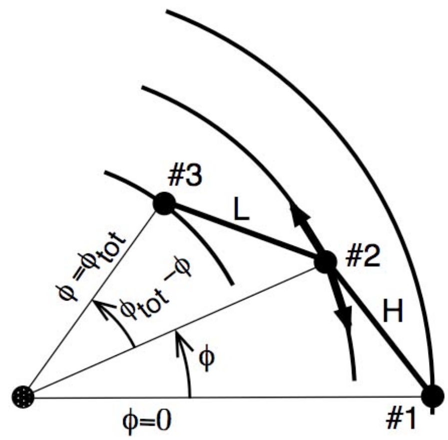

FIGURE 9.1 Find the intermediate angle ϕ such that the wristwatch time from event #1 to event #3 is a maximum.

Strategy: Apply the Principle of Maximal Aging to maximize the wristwatch time of a free stone flying across two adjoining worldline segments labeled H and L for "higher" and "lower" in Figure 9.1. The stone emits flashes at events #1, #2, and #3, marking off the segments. Fix the *times* of all three flashes and the *positions* of flashes #1 and #3. Vary the *position* of event #2 by sliding it along a circle (two-headed arrow in Figure 9.1) in order to maximize the total wristwatch time between flashes #1 and #3. We identify the resulting constant of motion as the map angular momentum. Now the details.

Maximize the stone's total wristwatch time $\Delta\tau$ across the two segments by setting its derivative with respect to $\Delta\phi$ equal to zero. To simplify the look of this derivative, temporarily replace increments $\Delta\tau$ and $\Delta\phi$ with τ and ϕ respectively, which remain incrementally small.

Set the fixed angle ϕ of event #1 equal to zero and call $\phi_{\rm tot}$ the fixed angle for event #3. Change the angle ϕ of event #2 by moving it either way along its circle, shown by the double-headed arrow in the figure. Let $\bar{r}_{\rm H}$ and $\bar{r}_{\rm L}$ be appropriate average values of the radii for segments H and L, respectively, and let $\tau_{\rm H}$ and $\tau_{\rm L}$ be the corresponding lapses of wristwatch time of the stone moving along these segments. With these substitutions, the approximate Schwarzschild metric (9.10) for higher Segment H becomes: (#eq:4)

$$\tau_{\rm H} \approx [-\bar{r}_{\rm H}^2\phi^2 + (\text{terms without } \phi)]^{1/2} \qquad (9.2)$$

To prepare for the derivative that leads to maximal aging, take the derivative of this expression with respect to ϕ: (#eq:5)

$$\frac{d\tau_{\rm H}}{d\phi} \approx -\frac{\bar{r}_{\rm H}^2\phi}{\tau_{\rm H}} \qquad (9.3)$$

Similarly for lower Segment L, (#eq:6 and #eq:7)

$$\tau_{\rm L} \approx [-\bar{r}_{\rm L}^2(\phi_{\rm tot}-\phi)^2 + (\text{terms without } \phi)]^{1/2} \qquad (9.4)$$

$$\frac{d\tau_{\rm L}}{d\phi} \approx \frac{\bar{r}_{\rm L}^2(\phi_{\rm tot}-\phi)}{\tau_{\rm L}} \qquad (9.5)$$

The total wristwatch time for both segments is $\tau = \tau_{\rm H} + \tau_{\rm L}$. Take the derivative of this expression with respect to ϕ, substitute from (9.3) and (9.5), and set the resulting derivative equal to zero in order to apply the Principle of Maximal Aging: (#eq:9)

$$\frac{d\tau}{d\phi} = \frac{d\tau_{\rm H}}{d\phi} + \frac{d\tau_{\rm L}}{d\phi} \approx -\frac{\bar{r}_{\rm H}^2\phi}{\tau_{\rm H}} + \frac{\bar{r}_{\rm L}^2(\phi_{\rm tot}-\phi)}{\tau_{\rm L}} = 0 \qquad (9.6)$$

The condition for maximal lapse of wristwatch time becomes (#eq:10)

$$\frac{\bar{r}_{\rm H}^2\phi}{\tau_{\rm H}} \approx \frac{\bar{r}_{\rm L}^2(\phi_{\rm tot}-\phi)}{\tau_{\rm L}} \qquad (9.7)$$

or in our original Δ notation: (#eq:10A)

$$\frac{\bar{r}_{\rm H}^2\Delta\phi_{\rm H}}{\Delta\tau_{\rm H}} \approx \frac{\bar{r}_{\rm L}^2\Delta\phi_{\rm L}}{\Delta\tau_{\rm L}} \qquad (9.8)$$

The left side contains quantities for Segment H only; the right side quantities for Segment L only. We have discovered a quantity that is the same for both segments, a *constant of motion* for the free stone across *every* pair of adjoining segments along the worldline of the stone. In deriving this quantity, we assumed that each segment of the worldline is small. To guarantee this smallness, go to the calculus limit in (9.8), for which $\bar{r} \rightarrow r$; the constant of motion becomes (#eq:11)

$$\lim_{\Delta\tau\rightarrow 0}\left(\bar{r}^2\frac{\Delta\phi}{\Delta\tau}\right) = r^2\frac{d\phi}{d\tau} = \text{a constant of motion} \qquad (9.9)$$

The text identifies this constant of motion as L/m, the map angular momentum of the stone per unit mass.

Objection 9.1. *In your derivation of E/m in Chapter 6, you made use of different wristwatch rates at different radial heights. I do not see any mention of wristwatch time in this later derivation of L/m.*

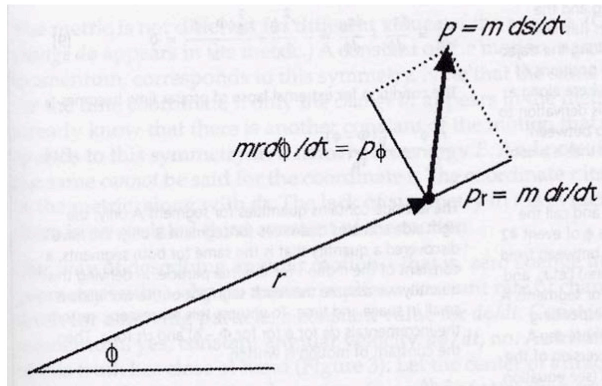

FIGURE 9.2 In flat spacetime angular momentum L is the product of r and the component of linear momentum p_ϕ in the tangential or ϕ direction, yielding $L = mr^2 d\phi/d\tau$, where $d\tau$ is the advance of wristwatch time of the stone whose momentum is being determined. The same formula applies to the stone's angular momentum around a black hole in Schwarzschild coordinates.

(#AngMomDef)

! Perceptive comment! In deriving map angular momentum, what changes aging is the variation of wristwatch rate with tangential velocity, an effect of special relativity. Set $M = 0$ (no black hole!) and get exactly the same expression for map angular momentum, $L/m = r^2 d\phi/d\tau$, about any arbitrary center that you may choose.

Our motto: Think *globally;* measure *locally!*

Map angular momentum L/m in (9.11) and map energy E/m in (9.1) are mythical beasts, like unicorns; no one near the black hole measures their values directly. The big advantage of these expressions is that each of their values stays constant in map coordinates as a free stone streaks around a black hole—or around our Sun. These constants have global reach.

Comment 9.1. DEFINITIONS: Unitless quantities

Unitless quantities

It's time to streamline our notation in order to simplify expressions and their derivations in the remainder of the book. Our streamlined equations contain only quantities that have no units: they are **unitless**. Unitless quantities are summarized for reference in Appendix 00.

We already use some unitless quantities:

- Any measure of speed, such as v or v_{shell}, is already unitless—the fraction of light speed.
- The global angle ϕ between two events is unitless. Section 3.3 *defined* the global radius r as the "reduced circumference" of a shell, that is $r \equiv (\text{measured circumference})/2\pi$, where 2π is the total angle around the circle in radians. So to find the angle ϕ between two points at the same radius, we measure the distance between them along the shell, divide by the circumference to get the fraction of the distance around the shell, then multiply by 2π, which yields ϕ. The ratio of lengths being unitless, so is ϕ.

To these we now add unitless quantities flagged with an asterisk, as follows: (#asteriskCoord)(#asteriskE) (#asteriskL)

$$\text{(coordinate)}^* \equiv \frac{\text{(coordinate)}}{M} \tag{9.12}$$

$$E^* \equiv \frac{E}{m} \qquad \text{(stone)} \tag{9.13}$$

$$L^* \equiv \frac{L}{mM} = r^{*2}\frac{d\phi}{d\tau^*} \qquad \text{(stone)} \tag{9.14}$$

In (9.12) the symbol (coordinate)* means *any* space or time coordinate (except ϕ), for example Schwarzschild coordinates r^* or t^* or global rain time T^* or shell coordinates $\Delta t^*_{\text{shell}}$ or $\Delta y^*_{\text{shell}}$ or $\Delta x^*_{\text{shell}}$.

Equation (9.1) shows that E/m is unitless (which requires that E and m be expressed in the same units); in (9.13) we define this ratio as E^*.

Equation (9.11) gives L/m the units of meters. In (9.14) we divide that quantity by M in meters; the result, L^*, is unitless.

QUERY 9.3. Schwarzschild in unitless coordinates
Convert some central equations into unitless form:

A. Write the differential form of the Schwarzschild metric (9.10) in unitless coordinates: (#Schwarzunitless)

$$d\tau^{*2} = \left(1 - \frac{2}{r^*}\right) dt^{*2} - \frac{dr^{*2}}{\left(1 - \dfrac{2}{r^*}\right)} - r^{*2} d\phi^2 \qquad \text{(Schwarzschild metric)} \tag{9.15}$$

This was your own earlier suggestion, equation (3.11) in Objection 3.4.

B. Show that unitless equations for map energy are, from (9.1) and (7.24):(#eq:3A) (#Erain)

$$E^* = \left(1 - \frac{2}{r^*}\right)\frac{dt^*}{d\tau^*} \qquad \text{(Schwarzschild coordinates)} \tag{9.16}$$

$$E^* = \left(1 - \frac{2}{r^*}\right)\frac{dT^*}{d\tau^*} - \left(\frac{2}{r^*}\right)^{1/2}\frac{dr^*}{d\tau^*} \qquad \text{(global rain coordinates)} \tag{9.17}$$

BOX 9.2. Computing the Orbit

STRATEGY

Step 1: Use the constants of motion E^* and L^* to express dt^* and $d\phi$ in terms of the differential advance of satellite wristwatch time $d\tau^*$.

Step 2: Substitute these results into the Schwarzschild metric to find dr^* as a function of $d\tau^*$, r^*, E^*, and L^*.

Result: All map increments dt^*, $d\phi$, and dr^* are now locked to satellite time increment $d\tau^*$.

Computer starts with the initial position, advances satellite wristwatch time by $d\tau^*$as it updates values of t^*, ϕ, and r^*. Now for the details.

Step 1: Relate dt^* to $d\tau^*$ using map energy as a constant of motion (9.16): (#eq:20)

$$dt^* = (1 - 2/r^*)^{-1}\, E^* d\tau^* \qquad (9.18)$$

Similarly find the advance of map angle ϕ from the constant of motion given by map angular momentum (9.14): (#eq:21)

$$d\phi = (L^*/r^{*2})d\tau^* \qquad (9.19)$$

Step 2: With dt^* and $d\phi$ now known in terms of $d\tau^*$, we lack only dr^* to specify completely the differential displacement of the satellite in map space and map time in one tick, $d\tau^*$, of satellite time. But dr^* appears, along with the three knowns, dt^*, $d\phi$, and $d\tau^*$, in the Schwarzschild metric (9.15). Into this metric substitute dt^* from (9.18) and $d\phi$ from (9.19) and solve for dr^*. The result is an equation that relates dr^* to $d\tau^*$: (#eq:22)

$$dr^* = \pm\left[E^{*2} - \left(1 - \frac{2}{r^*}\right)\left\{1 + \left(\frac{L^*}{r^*}\right)^2\right\}\right]^{1/2} d\tau^* \qquad (9.20)$$

Starting with initial values of r^* and ϕ, equations (9.19) and (9.20) tell how each map polar coordinate changes as the satellite wristwatch ticks. If the Schwarzschild mapmaker demands that the increments be expressed also in terms of map time t^*, then (9.18) provides the corresponding change dt^*.

9.3 ■ FORECASTING THE ORBIT

Satellite wristwatch ticks off $d\tau^*$. From $d\tau^*$ find the resulting changes dr^*, $d\phi$, and dt^*.

How do ϕ, r^*, and t^* change with change in satellite time τ^*?

We now have in hand the tools needed to calculate the step-by-step advance of the satellite through the world of space and time. Advance? Yes, (a) advance dt^* of map time, (b) advance $d\phi$ of map angle, and (c) advance dr^* of map radius, all correlated—at our choice and for our convenience—to the time lapse $d\tau^*$ between ticks of the satellite wristwatch.

Computing the orbit

Map energy and map angular momentum—constants of motion—plus the metric give us three equations in the three map unknowns dt^*, dr^*, and $d\phi$, expressed as functions of the advance $d\tau^*$ of the satellite's wristwatch. Starting from an initial event, the computer advances wristwatch time and calculates the consequent advance of map coordinates, then sums the results of these steps to reckon the orbit, as spelled out in Box 9.2.

?

Objection 9.2. *Hold on! Before you go any further—I notice that all the analysis so far in this chapter is in Schwarzschild map coordinates. What good is a global map description of orbits which nobody verifies by direct observation? You admit that map energy and map angular momentum are like unicorns: mythical beasts. I want observable orbits!*

Guilty as charged! Most of this chapter describes motion in map coordinates. We have no choice: In curved spacetime no single local observer can measure or view a global orbit in its entirety; global coordinates are required. In Chapter 10 we analyze conditions in local inertial frames. Remember our motto: "Think *globally*; measure *locally*!"

9.4 ■ EFFECTIVE POTENTIAL

Orbit features at a single glance!

Box 9.2 puts into our hands powerful tools to describe any orbit of any free stone around any spherically symmetric center of attraction. Indeed, the wealth of possible orbits is so great that we need a simplifying strategy that allows us to grasp many different orbits at a glance. One such strategy makes use of the so-called **effective potential**, which focuses on radial motion alone. Clearer even than our computed orbits, the effective potential lets us see immediately the central features of the stone's motion.

Comment 9.2. Develop intuition with computer-plotted orbits.
Another way to develop intuition is to let the computer draw orbits for you as it uses a more sophisticated analytic procedure than the crude one outlined in Box 9.2. Slavomir Tuleja has developed just such a computer program, called GRorbits. For access to GRorbits, see the references.

Preview: a pit in the potential

Vicious gravitational effects close to a black hole dominate the effective potential there. The principal results can be simply stated: In addition to the attractive potential of gravity at great distances and the repulsive effects of map angular momentum at intermediate distances, Einstein's theory adds at still shorter distances a pit in the potential, shown at the left in Figure 9.3 and some later figures.

The pit comes from constants of motion.

The potential? A pit in this potential? A potential attractive at large distances, repulsive at intermediate distances, and attractive again at yet smaller distances? Can we get this potential from principles that are simple, clear, and solid? Yes, from two principles: Map energy as a constant of motion! Map angular momentum as a constant of motion! And each of these from the Principle of Maximal Aging.

Both the constant map energy and the effective potential function become apparent when we derive from (9.20) (Box 9.2) the square of the radial velocity as registered in satellite wristwatch time: (#eq:30)

$$\left(\frac{dr^*}{d\tau^*}\right)^2 = E^{*2} - \left(1 - \frac{2}{r^*}\right)\left[1 + \left(\frac{L^*}{r^*}\right)^2\right] \qquad (9.21)$$

Use equation (9.21) to introduce a new unitless quantity $V_{\rm L}^{*2}(r^*)$, the **Schwarzschild effective potential**. (#eq:32)

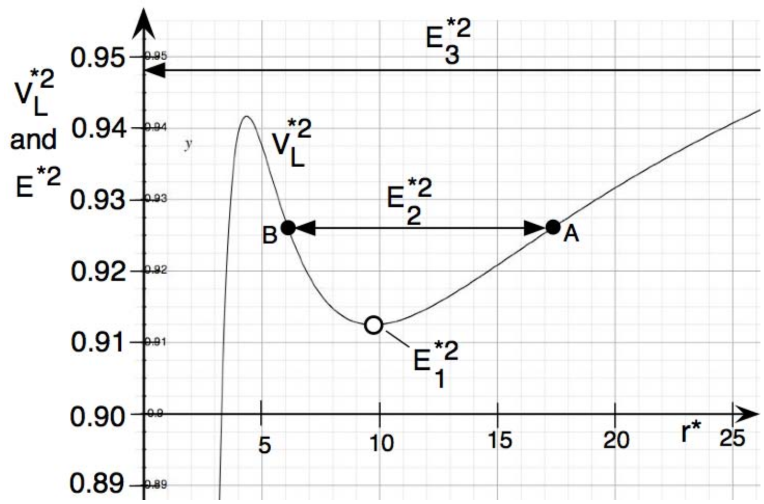

FIGURE 9.3 Effective potential curve for a stone orbiting a black hole with map angular momentum $L^* = 3.75$. Stone map energy is a constant of motion—independent of radius—so plots as a horizontal line. When the stone's map energy squared E_1^{*2} equals the minimum of the potential squared (open circle) the stone remains at a constant radius and orbits the black hole in a circle. A stone with slightly greater map energy squared, E_2^*, (line with double arrow) oscillates back and forth between radial limits labeled A and B while orbiting around the center of attraction. When the stone's map energy squared E_3^* is greater than the peak of the effective potential squared (upper horizontal line, with inward arrow), the stone moves inward across the event horizon, all the while circulating with constant map angular momentum.
(#MapEffPot)

$$V_{\rm L}^{*2}(r^*) \equiv \left(1 - \frac{2}{r^*}\right)\left[1 + \left(\frac{L^*}{r^*}\right)^2\right] \qquad (9.22)$$

The subscript L on the function $V_{\rm L}^{*2}(r^*)$ reminds us that this effective potential is different for different values of the map angular momentum L^*. Use (9.22) to simplify (9.23): (#eq:30)

$$\left(\frac{dr^*}{d\tau^*}\right)^2 = E^{*2} - V_{\rm L}^{*2}(r^*) \qquad (9.23)$$

Effective potential for a stone

Use a square $V_{\rm L}^{*2}(r^*)$ in (9.23)—instead of its first power—because E^* is squared. This definition with a square is purely cosmetic; see Query 9.6. The square of the effective potential $V_{\rm L}^{*2}(r^*)$ is what we have to take away from the constant squared map energy term to get the square of the radial velocity.

QUERY 9.4. *Optional:* **Compare Newtonian and general-relativistic orbital motion**
The right side of (9.21) tells us a great deal about the difference between the stone's global motion described in Schwarzschild map coordinates and its motion described by Newton.

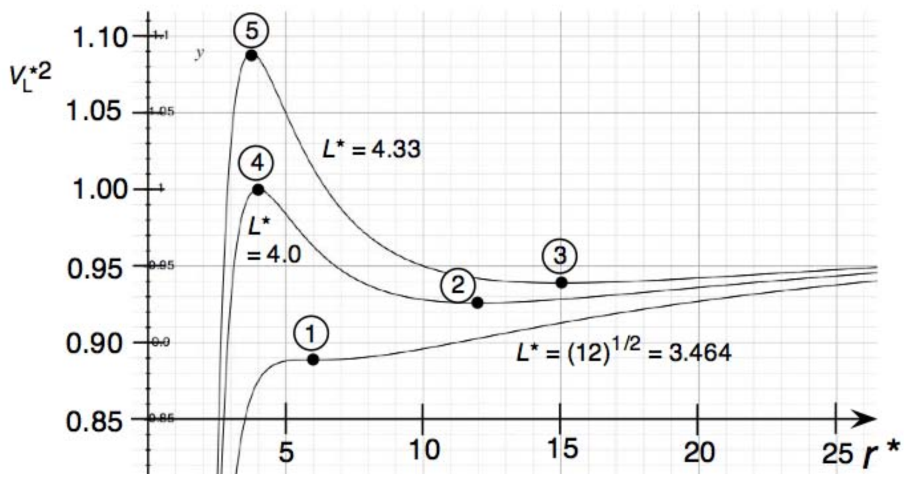

FIGURE 9.4 Radii of stable and knife-edge circular orbits around a black hole for a stone or unpowered spaceship with various values of map angular momentum L^*. The radius of each *stable* circular orbit (circled numbers 1, 2, 3) lies at an effective potential *minimum*. The stable circular orbit of smallest radius lies at $r^* = 6$ (circled number 1). The stone cannot be in a stable circular orbit for $r^* < 6$. The radius of each *knife-edge* circular orbit (circled numbers 4 and 5) lies at a *maximum* of the squared effective potential. The unstable orbit of smallest radius—a limiting case—is at $r^* = 3$.

(#CircularOrbits)

A. Multiply out the right side of (9.21) and divide through by 2 to yield (#NewtRad)

$$\frac{1}{2}\left(\frac{dr^*}{d\tau^*}\right)^2 = \frac{1}{2}\left(E^{*2} - 1\right) - \left(-\frac{1}{r^*} + \frac{L^{*2}}{2r^{*2}} - \frac{L^{*2}}{r^{*3}}\right) \qquad \text{(Schwarzschild)} \qquad (9.24)$$

B. Here is Newton's expression for angular momentum in unitless coordinates, with Newton's universal time t^*: (#NewtL)

$$L^* \equiv r^{*2}\frac{d\phi}{dt^*} \qquad \text{(Newton, universal time } t^*\text{)} \qquad (9.25)$$

Show that Newton's expression for the square of the velocity of the stone is, in unitless coordinates: (#Newtv)

$$v^2 = \left(\frac{dr^*}{dt^*}\right)^2 + r^{*2}\left(\frac{d\phi}{dt^*}\right)^2 = \left(\frac{dr^*}{dt^*}\right)^2 + \left(\frac{L^*}{r^*}\right)^2 \qquad \text{(Newton)} \qquad (9.26)$$

C. Now Newton's expression for gravitational potential energy per unit mass—chosen to go to zero far from the center of attraction—is $U^*(r^*) \equiv U(r^*)/m = -1/r^*$. Write down Newton's conservation of energy equation and solve it for the radial velocity. Show that the result is: (#Newtdrdt)

$$\frac{1}{2}\left(\frac{dr^*}{dt^*}\right)^2 = E^* - \left[-\frac{1}{r^*} + \frac{1}{2}\left(\frac{L^*}{r^*}\right)^2\right] = E^* - V^*_{\text{Newt}}(r^*) \qquad \text{(Newton)} \qquad (9.27)$$

where **Newton's effective potential** is $V^*_{\rm Newt}(r^*) = V_{\rm Newt}(r^*)/m$.

D. Sketch for the Newtonian case a diagram like that of Figure 9.3 with a plot of $V^*_{\rm Newt}(r^*)$ along with horizontal lines for different values of E^*. Describe the resulting orbits and contrast them to those for motion in Schwarzschild spacetime.

Of course general relativity expression (9.24) is not just another version of Newton's equation (9.27)—general relativity is *not* the same as Newton's mechanics! However, look at the basic similarity of the right-hand sides of these two equations: a constant term from which we subtract a function of radius—the "effective potential"—that varies with the value of angular momentum L^*.

Conclusion of this analysis: It is the negative third term in the effective potential on the right side of (9.24), with r^{*3} in its denominator, that drives the effective potential downward as r^* approaches the event horizon—thereby creating the pit in the potential. This third term is the child of curved spacetime described by Schwarzschild.

QUERY 9.5. Difference of squares?

The right-hand side of Newton's equation (9.27) is the difference between E/m and $V_{\rm L}/m$, whereas the Schwarzschild equation (9.23) is the difference between the corresponding *squares* E^{*2} and $V^{*2}_{\rm L}$. Why this difference? Investigate using the following outline or some other method:

A. Recast (9.22) in non-unitless form. Show that the result is (#eq:32A)

$$\frac{V_{\rm L}^2(r)}{m^2} \equiv \left(1-\frac{2M}{r}\right)\left[1+\frac{(L/m)^2}{r^2}\right] \qquad \text{(Schwarzschild)} \tag{9.28}$$

B. Evaluate this expression in the limit that the black hole fades away: $M \to 0$, so that spacetime becomes flat. You can still calculate map angular momentum with respect to any point you choose, for example the former center of the black hole. Show that map energy will have a different numerical value in flat spacetime, but is still a constant of motion. Show that relativistic equation (9.24) becomes, for $M \to 0$, (#SchwarzMequalsZero)

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}\left(\frac{E}{m}\right)^2 - \frac{1}{2} - \frac{1}{2}\left(\frac{L}{mr}\right)^2 \qquad \text{(Schwarzschild, } M \to 0) \tag{9.29}$$

Surprise: In flat spacetime the difference of squares in (9.23) remains. This shows that the difference in form between equations (9.27) and (9.23) results from special relativity rather than from general relativity.

C. Show that, for $M \to 0$, Newton's expression (9.27) becomes: (#NewtdrdtA)

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \frac{E}{m} - \frac{1}{2}\left(\frac{L}{mr}\right)^2 \qquad \text{(Newton, } M \to 0) \tag{9.30}$$

D. Special relativistic equation (9.29) is still not the same as Newton's equation (9.30). However, if we let the speed of the stone become small compared with the speed of light, then, first of all,

$d\tau \rightarrow dt$ on the left of (9.29), where dt is the differential of Newton's universal time. Show that the right sides of the two equations also become equal in the limit of low velocity.

E. *Optional:* From the above analysis, show that it is purely a matter of convention whether we define the Schwarzschild effective potential in (9.22) to be the first power or the second power of V_{L}/m. Equation (9.23) would look a little screwy if the right-hand side contained E/m to the second power and V_{L}/m to the first power, but the content would be exactly the same, given the corresponding power of V_{L}/m in its definition (9.22).

Shape of effective potential curve

The effective potential has this wonderful feature: its radial dependence is determined only by the angular momentum of the satellite, not at all by its map energy. Map energy is constant throughout the motion of the satellite, so energies E^* of different values plot as horizontal lines on the effective potential graph, as in Figure 9.3. We use the resulting combined plot to predict the qualitative shape of various orbits in a given potential—that is, for a given map angular momentum—as analyzed in the caption to that figure.

Turning points

Note that $dr^*/d\tau^*$ in equation (9.23) is real only where E^{*2} has a value greater than $V_{\mathrm{L}}^{*2}(r^*)$. Therefore both ends of the horizontal map energy line labeled E_2^{*2} in Figure 9.3 terminate where it meets the $V_L^{*2}(r^*)$ curve. At these points, called **turning points**, $E^{*2} = V_{\mathrm{L}}^{*2}$, so $dr^*/d\tau^* = 0$ in (9.23). At a turning point the radial component of motion stops for an instant (while tangential motion continues), then the radial motion changes sign. In Figure 9.3 the satellite's radius oscillates back and forth between turning points labeled A and B. Earth and each solar planet oscillates radially back and forth in a radial motion similar to that labeled E_2^{*2} in Figure 9.3, each around a minimum of its own solar effective potential that depends on its map angular momentum. (Planetary motions are almost exactly those predicted by Newton, but not quite: see Chapter 12.)

Stable orbit at potential minimum

In this chapter we focus on the circular orbit, illustrated by the radius of the open circle labeled E_1^{*2} in Figure 9.3. In that circular orbit the stone's map energy rests at the minimum of the effective potential curve and the stone rides round and round the black hole without changing radius. Several such stable radial positions for the black hole case are numbered 1 through 3 in Figure 9.4. A circular orbit at the radius of a minimum of the effective potential is called a **stable circular orbit**, because a slight radial displacement puts the stone into a region where the slope of the potential urges its radius back toward the equilibrium value.

Knife-edge orbit at potential maximum

Einstein opens up a second set of radial locations where the effective potential also has zero slope, illustrated by points labeled 4 and 5 in Figure 9.4. Each of these is a *maximum* of the effective potential curve; *exactly* at this radius the stone experiences no tendency to move either to larger radius or to smaller radius, so will stay put radially, riding round and round the black hole at constant radius. We call these **knife-edge circular orbits**, because slight departure from the knife-edge radius leads to decisive motion either radially outward, or else—horrors!—inward toward a dark fate. "Why, oh why," our captain cries, "didn't I carry along a booster rocket! A tiny rocket boost pushing us outward could have reversed our initially slow inward motion, allowing us to escape. But now it's too late!"

Comment 9.3. Circle forever on a knife edge?
Suppose that our spaceship is in a knife-edge orbit. Technically a *knife-edge orbit* is also an *unstable orbit*. Slight cosmic wind, firing of a projectile, or ejection of the day's trash may give our spaceship a tiny radial motion. Once displacement from the peak of effective potential occurs, the slope of the effective potential urges the spaceship *farther away* from the point of zero slope, either outward toward great distance, or else inward toward the singularity.

Classical prediction: Circulate on knife-edge orbit forever.

In this book we assume that a stone placed in a knife-edge orbit circulates at that radius forever. Is this true in the real world? No. Classical physics often fails in the real world. General relativity is a classical (non-quantum) theory, which predicts the infinite-time circulation of a stone on the peak of the effective potential. Classical physics also predicts that a pencil balanced exactly on its point on your desk will stay balanced indefinitely. In contrast, quantum mechanics warns that you cannot—even in principle—set your pencil both exactly vertical and with exactly zero motion (code phrase: *Heisenberg's Uncertainty Principle*). Of course your pencil falls over. Quantum mechanics makes a similar prediction that the stone falls off the knife-edge orbit, either returning outward to a great distance or diving into the black hole. But in this book we restrict ourselves to classical general relativity and its prediction that a stone circulates on the peak of the effective potential forever; we do not listen to quantum mechanics, even though in this case quantum mechanics makes the correct prediction.

9.5 ■ PROPERTIES OF CIRCULAR ORBITS

Details! Details!

A series of Queries helps you to explore some properties of circular orbits.

QUERY 9.6. Map radii of circular orbits

A. Circular orbits are possible at radii where the effective potential has zero slope. Zero-slope of the effective potential occurs at the same radius as zero-slope of the *square* of the effective potential. Take the radial derivative of both sides of (9.22), set this derivative equal to zero, and show the result is a quadratic equation in r^*: (#eq:41)

$$r^{*2} - L^{*2} r^* + 3L^{*2} = 0 \qquad \text{(circular orbit)} \tag{9.31}$$

B. Solve quadratic equation (9.31) for the radii r^* of circular orbits:

$$r^* = \frac{L^{*2}}{2}\left[1 \pm \left(1 - \frac{12}{L^{*2}}\right)^{1/2}\right] \qquad \text{(circular orbit)} \tag{9.32}$$

Refer to Figure 9.4. The plus sign in (9.32) yields the radius at the minimum of the effective potential and therefore yields the radii of stable circular orbits. The minus sign yields the radius at the maximum of effective potential and thus the radii of knife-edge circular orbits of smaller radius.

C. Show that there are no circular orbits of any kind for angular momentum given by the inequality $|L^*| < (12)^{1/2}$. Show that for the minimum angular momentum, the radius of the circular orbit is $r^* = 6$. This is the *stable* circular orbit of smallest radius. See the case with the circled number 1 in Figure 9.4.

D. *Optional:* Verify the difference between minima and maxima of the effective potential $V_{\rm L}^{*2}$ by taking its second derivative with respect to r^* to determine whether this second derivative is positive (local minimum of the potential) or negative (local maximum in the potential) at specified values of r^*.

QUERY 9.7. Map angular momentum of satellite in circular orbit

Solve (9.31) to find $(L^*)^2$ for a circular orbit at map radius r^*: (#Lstar)

$$L^{*2} = \frac{r^{*2}}{r^* - 3} \qquad \text{(circular orbit)} \tag{9.33}$$

Note that this expression is valid for both stable and knife-edge circular orbits and is invalid for $r^* < 3$, where circular orbits do not exist. (Infinite angular momentum at $r^* = 3$ is a limiting case.)

QUERY 9.8. Satellite shell speed in a circular orbit

Compute the shell speed of the satellite in a circular orbit.

A. Consider two ticks of the satellite clock, separated by wristwatch time $\Delta\tau^*$ and by zero distance in the satellite frame, but separated by time $\Delta t^*_{\rm shell}$ and by distance $\Delta x^*_{\rm shell} = \bar{r}^*\Delta\phi$ in the shell frame. The relation between $\Delta t^*_{\rm shell}$ and $\Delta\tau^*$ is just the special-relativity expression (#eq:44)

$$\Delta t^*_{\rm shell} = \gamma_{\rm shell}\Delta\tau^* = (1 - v^2_{\rm shell})^{-1/2}\Delta\tau^* \tag{9.34}$$

where $\gamma_{\rm shell}$ has an obvious definition. Knowing map angular momentum, we can now use (9.34) to reckon shell speed: (#eq:45)

$$\begin{aligned} v_{\rm shell} &= \lim_{\Delta t^*_{\rm shell}\to 0}\left(\frac{\bar{r}^*\Delta\phi}{\Delta t^*_{\rm shell}}\right) = (1 - v^2_{\rm shell})^{1/2}\,\frac{r^{*\,2}d\phi}{r^*d\tau^*} \\ &= (1 - v^2_{\rm shell})^{1/2}\,\frac{L^*}{r^*} \end{aligned} \tag{9.35}$$

From this equation, show that (#eq:46)

$$v^2_{\rm shell} = \left[1 + \left(\frac{r^*}{L^*}\right)^2\right]^{-1} \qquad \text{(circular orbit)} \tag{9.36}$$

From equation (9.33) show that (#eq:47)

$$\left(\frac{r^*}{L^*}\right)^2 = r^* - 3 \qquad \text{(circular orbit)} \tag{9.37}$$

Substitute this into equation (9.36) to find (#eq:48)

$$v_{\text{shell}}^2 = \frac{1}{r^* - 2} \qquad \text{(circular orbit, } r^* > 3\text{)} \tag{9.38}$$

Equation (9.38) is valid for both stable and knife-edge circular orbits.

B. What is the value of the shell speed v_{shell} in the stable orbit of smallest radius, $r^* = 6$ (part D of Query 9.9)?

C. Verify that the minimum map radius of the knife-edge circular orbit is $r^* = 3$. (*Hint:* What is the upper limit of the shell speed of a stone?)

D. From (9.33) show that, as a limiting case, the map angular momentum L^* is infinite for the knife-edge circular orbit of minimum radius.

Comment 9.4. Infinite map angular momentum?

How can the map angular momentum possibly go to infinity(Item D)? It does so only as a limiting case. According to (9.11), the map angular momentum is equal to $L^* = r^{*2} d\phi/d\tau^*$. The relation between wristwatch time $d\tau^*$ and shell time dt^*_{shell} is given by (9.34), the usual time-stretch formula of special relativity. As the satellite speed approaches the speed of light, the advance of wristwatch time becomes smaller and smaller compared with the advance of shell time. Then it takes nearly zero wristwatch time for the satellite to circulate once around the black hole. Because $d\tau^*$ is in the denominator of the expression for angular momentum, the map angular momentum L^* grows large without limit. The speed of light is the limiting speed of a stone, so this is a limiting case, never quite reached in practice. This analysis has consequences for time travel using knife-edge circular orbits around a black hole; see Exercise 9.6 at the end of this chapter.

QUERY 9.9. Shell and map energies in a circular orbit

A. Using the definition of E_{shell} for the locally-flat shell spacetime and (9.38), show that (#EshellCirc)

$$E^*_{\text{shell}} \equiv \frac{1}{\left(1 - v^2_{\text{shell}}\right)^{1/2}} = \left(\frac{r^* - 2}{r^* - 3}\right)^{1/2} \qquad \text{(circular orbit)} \tag{9.39}$$

B. Show that the general equation connecting shell and map energies from equation (6.22), in unitless coordinates, is: (#shellvsmapE)

$$E^*_{\text{shell}} = \left(1 - \frac{2}{r^*}\right)^{-1/2} E^* \tag{9.40}$$

C. Using (9.39) and (9.40), show that for a circular orbit: (#EmapCirc)

$$E^* = \frac{r^* - 2}{r^{*1/2}(r^* - 3)^{1/2}} \qquad \text{(circular orbit)} \tag{9.41}$$

All of these equations apply to both stable and knife-edge circular orbits.

D. In (9.41) set $r^* - n = r^*(1 - n/r^*)$, then use our standard approximation (inside the front cover) to show that to first order at large radius: (#largerradiusE)

$$E^* \approx 1 - \frac{1}{2r^*} \qquad \text{(circular orbit, } r^* \gg 1) \tag{9.42}$$

Apply the same approximation to (9.40) and combine the result with (9.42) to obtain: (#largeradiusEshell)

$$E^*_{\text{shell}} \approx \left(1 + \frac{1}{r^*}\right) E^* \approx 1 + \frac{1}{2r^*} \qquad \text{(circular orbit, } r^* \gg 1) \tag{9.43}$$

E. Find the numerical value of both E^* and E^*_{shell} at the following values of r^*:
(a) as $r^* \to \infty$. Why does this result make obvious sense?
(b) at the minimum radius for a stable circular orbit,
(c) at the minimum radius for a knife-edge circular orbit. Justify your result.

QUERY 9.10. Map speed of a satellite in a circular orbit

A satellite in orbit has tangential map speed equal to the tangential map distance covered per unit map time. We *defined* map radius r as reduced circumference, that is, the measured circumference of a circle concentric to the black hole divided by 2π. From this definition, the tangential differential map distance is $rd\phi$. So the tangential map velocity is (using unitless coordinates):(#deftangspeed)

$$v_\phi \equiv \frac{r^* d\phi}{dt^*} \qquad \text{(tangential map velocity)} \tag{9.44}$$

A. Find an expression for the map speed in a circular orbit as follows: Use expression (9.11) for the map angular momentum to replace $d\tau$ in the Schwarzschild metric (9.15) by an expression in $d\phi$. From definition (9.44) show that the square of the map speed in circular orbit is (#eq:49)

$$v_\phi^2 = \frac{1 - 2/r^*}{1 + (r^*/L^*)^2} = \frac{1}{r^*} \qquad \text{(circular orbit)} \tag{9.45}$$

The final step in (9.45) uses equation (9.37). Equation (9.45) shows that a circular orbit of smaller radius has higher tangential map speed.

B. What is the value of the map speed of a satellite in the stable circular orbit of smallest radius?

C. Is equation (9.45) valid for *knife-edge* circular orbits? If so, what is the value of the map speed in the knife-edge circular orbit of minimum radius?

D. In a circular orbit of given radius r^*, is map satellite speed greater or less than shell satellite speed?

QUERY 9.11. Map, shell, and orbiter wristwatch times for one circular orbit (#myquery6) From the results of previous Queries, verify the following expressions for the time period of one circular orbit. (#deftangspeed)

A. Show that Schwarzschild map time lapse for one circular orbit is (#eq:99)

$$t^* = 2\pi r^{*3/2} \qquad \text{(one circular orbit)} \tag{9.46}$$

B. Show that the local shell time for one circular orbit is (#eq:98)

$$\Delta t^*_{\text{shell}} \approx (r^* - 2)^{1/2} 2\pi r^* \qquad \text{(one circular orbit)} \tag{9.47}$$

For the minimum (knife-edge) orbit radius $r^* = 3$, explain why the shell period is equal to the circumference of the orbit.

C. Use (9.34) to show that the orbiter wristwatch time lapse for one circular orbit has the value (#eq:97)

$$\Delta\tau^* \approx (r^* - 3)^{1/2} 2\pi r^* \qquad \text{(one circular orbit)} \tag{9.48}$$

Why is $\Delta\tau^*$ for one orbit smaller than $\Delta t^*_{\text{shell}}$ for one orbit? For the minimum (knife-edge) orbit $r^* = 3$, explain why the wristwatch time goes to zero.

D. For a circular orbit of very large radius, explain why Schwarzschild map time, shell time, and orbiter wristwatch time all have the same value, namely $2\pi r^{*3/2}$.

9.6 ■ TOY MODEL OF A QUASAR

Beacon of the heavens

Quasar

A **quasar** ("quasi-stellar object") is a distant astronomical object that pours out radiation at a prodigious rate. A quasar is a steady source of light that we can see at a great distance. Almost certainly a quasar is a spinning black hole (Chapters 17 and 18), but here we take a first stab at modeling a quasar using a non-rotating black hole. This sort of quick preliminary analysis is called a **toy model**.

Accretion disk

A likely model of quasar emission postulates an **accretion disk**, a gas cloud in the form of a plane that swirls around the black hole. Friction among neighboring atoms and ions heats the accretion disk to high temperature. We observe the resulting electromagnetic emission at great distance from the

Radiating away orbit energy

quasar. This radiated energy comes from the change in orbital map energy of each atom as it moves successively to orbits of lower and lower map energy. (Significant energy change takes place over many orbits, so one can still speak of the energy of the current circular orbit.)

Friction moves map angular momentum outward.

What is the mechanism of this orbit change? Equation (9.32) tells us that map angular momentum, not map energy, determines the radius of a circular orbit. Figure 9.4 shows that "our atom" decreases its angular momentum as it moves from a higher circular orbit to a lower circular orbit. How is this angular momentum extracted? A crude model notes that atoms in an adjacent slightly higher orbit circulate more slowly than "our atom." Friction between atoms in these two orbits tends to increase velocity—and therefore angular momentum—of atoms in the higher orbit. In turn, atoms moving faster in the adjacent smaller orbit transfer map angular momentum to "our atom." This mechanism, if correct, moves map angular momentum outward through the accretion disk. The vertical axis in Figure 9.4 also indicates the change in map energy that accompanies this decrease in orbit radius, map energy that is radiated away for us to see at a great distance.

Minimum stable orbit at $r^* = 6$

Sooner or later our atom reaches radius $r^* = 6$, the minimum radius of a stable circular orbit, labeled one in Figure 9.4. At this point our atom continues to lose map angular momentum to atoms in the next-higher orbit, but in this case no next-smaller stable circular orbit exists. Any further loss of map angular momentum drops our atom out of orbit. Once it crosses the horizon, no further radiation can reach any external observer.

QUERY 9.12. Map energy given up by "our atom."

A. Start with an atom in a circular orbit at a great distance, so its initial map energy is approximately equal to its mass, $E^* \approx 1$. Now think of its map energy later, as the atom moves in the stable circular orbit of minimum radius, $r^* = 6$. Using (9.41), find the map energy E^* of the atom in this minimum-radius circular orbit. How much map energy has the atom given up during the process of dropping gradually from a large distance to the minimum-radius stable circular orbit?

B. Suppose that the atom emits as electromagnetic radiation all the map energy it gives up (from Item A) as it spirals down to the circular orbit at $r^* = 6$. To one significant digit, the total radiated map energy is $E^* = 0.06$. Find this value to three significant digits. What percentage is your result of the initial map energy of the atom?

Comment 9.5. How much emitted energy?

No nuclear reaction on Earth releases as much as one percent of the rest energy of its constituents. Chapter 17 shows that for a black hole of maximum spin, the fraction of initial mass radiated away by a stone that spirals down from a great distance to a similar minimum-radius stable circular orbit approaches half of its rest mass. No wonder quasars are such bright beacons in the heavens!

?

Wait a minute! Map energy is not directly measurable. Yet in this model of a quasar we equate the energy of radiation detected at a distance with the drop in Schwarzschild map energy.

!

Nice point. Far from the black hole, however, shell energy has the same value as map energy: equation (6.15). The observer remote from the black hole receives energy equal to the change in map energy E, which is independent of radius. This map energy is smaller than the shell energy of radiation emitted near the black hole, as you show in the following Query. Gravitational red shift decreases the energy of this radiation as it climbs out of the gravitational well.

QUERY 9.13. Radiation climbs the ladder

A. Use (9.39) to calculate the *shell* energy E^*_{shell} of the atom (a) at a great distance from the black hole, and (b) in the circular orbit at $r^* = 6$. Answer of (b) to two significant digits is $E^*_{\text{shell}} = 1.2$. Calculate this answer to three significant digits.

B. We want to calculate the radiated energy of the atom measured at a great distance. Is this radiated energy equal to the shell energy of the atom at a great distance *minus* its shell energy at $r^* = 6$? Explain your answer.

9.7 ■ REFERENCES

Emily Dickinson poem at the beginning of the chapter is from R. W. Franklin, *The Poems of Emily Dickinson, Variorum Edition* 1998, The Belknap Press of Harvard University. This poem is variation E of the poem with Franklin number 1570, written about 1882. Reprinted and modified with permission of Harvard University.

GRorbits software program that displays orbits of a stone and light flash available at `http://stuleja.org/grorbits/`

9.8 ■ EXERCISES

Use unitless quantities in the following exercises, but practice converting results to everyday units such as meters and seconds.

9.1. Three Views of a Circular Orbit

A shell observer on the shell of average map radius $\bar{r}$ compares measurements with an observer in a circular orbit moving tangentially past him with speed v_{shell}. The orbiter is in a local inertial frame, and so, by definition, is the shell observer.

Therefore, special relativity applies, and we can use Lorentz transformations between the two frames, calling the shell frame "laboratory," the orbiting frame "rocket," and the measured relative speed v_{shell}. Choose the positive tangential x-axis to be the direction of motion of the orbiter "rocket" frame with respect to the shell "laboratory" frame.

A. The orbiter does one circuit and returns to the same shell clock. What time lapse $\Delta t^*_{\text{shell}}$ does this shell clock record for one circuit, in terms of v_{shell}?

B. During one complete circuit of the shell, the orbiting clock "runs slow" by the usual stretch factor $\gamma = (1 - v_{\text{rel}}^2)^{-1/2}$ of special relativity when compared with the shell clock to which it returns. What time lapse $\Delta t^*_{\text{orbiter}}$ does the orbiter's clock record between sequential passes over the recording shell clock, in terms of v_{shell}?

C. What is the map angular momentum L^* of the orbiter in terms of v_{shell}? (The answer is *not* $r^* v_{\text{shell}}$.)

D. What time lapse t^* does the Schwarzschild mapmaker record for one circuit of the orbiter, in terms of v_{shell}?

E. Part D of Query 9.4 shows that the smallest radius for a stable circular orbit is $r^* = 6$; equation (9.38) determines that in this orbit the orbiter's shell speed $v_{\text{shell}} = 0.5$, half the speed of light. Assume the central attractor to be Black Hole Alpha, with $M = 5000$ meters. Following, to one significant digit, are L/m and the times of one orbit for the shell observer, orbiter, and mapmaker. Find the value of L/m and these times to three significant digits. (#OneOrbitTimes)

$$\begin{aligned} L/m &\approx 2 \times 10^4 \text{ meter} && && (9.49)\\ \Delta t_{\text{shell}} &\approx 4 \times 10^5 \text{ meter} && \text{(one orbit)}\\ \Delta t_{\text{orbiter}} &\approx 3 \times 10^5 \text{ meter} && \text{(one orbit)}\\ \Delta t &\approx 5 \times 10^5 \text{ meter} && \text{(one orbit)} \end{aligned}$$

F. *Optional:* Notice that the orbiter of Part E completes one circuit of the black hole in approximately one millisecond on her wristwatch. If you ignore tidal effects, does this extremely fast rotation produce extreme *physical discomfort* for the orbiter? If she closes her eyes, does her orbiting make her dizzy?

G. How do you respond to the objection that a complete orbit vastly exceeds the dimension of an inertial frame and, at least for Parts A through E of this exercise, one orbit can take longer than the shell time during which the shell frame remains inertial?

9.2. When are Newton's Circular Orbits Almost Correct?

Your analysis of the Global Positioning System (GPS) in Chapter 4 calculated values of radius and orbital speed of a GPS satellite in circular orbit using

Newton's mechanics, with the prediction that the general relativistic analysis gives essentially the same values of radius and speed for this application. Under what circumstances are circular orbits predicted by Newton indistinguishable from circular orbits predicted by Einstein? Answer this question using the following outline or some other method.

A. Find Newton's expression similar to equation (9.32) for the radius of a stable circular orbit, starting with equation (9.27).

B. Recast equation (9.32) for the general-relativistic prediction of r^* for stable orbits in the form (#eq:100)

$$r^* = r^*_{\mathrm{Newt}}(1-\epsilon) \tag{9.50}$$

where r^*_{Newt} is the radius of the orbit predicted by Newton and ϵ is the small fractional deviation of the actual orbit from Newton's prediction. This expression neglects differences between the Newtonian and relativistic values of L^* when expressed in the same units. Use the approximation inside the front cover to derive a simple algebraic expression for ϵ as a function of r^*_{Newt}.

C. Set your expression for ϵ equal to 0.01 as a criterion for good-enough equality of the radius according to both Newton and Einstein. Find an expression for r^*_{min}, the smallest value of the radius for which this approximation is valid.

D. Find a numerical value for r^*_{min} in meters for our Sun. Compare the value of $r*_{\mathrm{min}}$ with the radius of the Sun.

E. What is the value of ϵ for the radius of the orbit of the planet Mercury, whose orbit has an average radius 0.387 times that of Earth?

9.3. Kepler's Laws of Planetary Motion

Johannes Kepler (1571-1630) provided a milestone in the history of mechanics with his **Three Laws of Planetary Motion**, deduced from a huge stack of planetary observations made by Tycho Brahe.

1. A planet orbits around the Sun in an elliptical orbit with the Sun at one focus of the ellipse.
2. The radius vector from the Sun to the planet sweeps out equal areas in equal times.
3. The square of the period of the planet is proportional to the cube of the planet's mean distance from the Sun.

A. Show by a simple symmetry argument that Kepler's Second Law is true for circular orbits around a black hole.

B. From equation (9.46) show that for *circular* orbits Kepler's Third Law is also valid for circular orbits around a black hole (when expressed in Schwarzschild map coordinates).

C. Kepler's Third Law is sometimes called **The 1-2-3 Law** from the exponents in the following equation. Show that for circular orbits, in our regular notation using meters, (#eq:55)

$$M \equiv M^1 = \omega^2 r^3 \tag{9.51}$$

where $\omega \equiv 2\pi/t$, with t the map time for one orbit.

D. *Preview:* Is Kepler's First Law true for noncircular orbits near a black hole? We shall see that the planet Mercury departs slightly from this law (Chapter 11). Look at Figure 10.9. State your conclusion about the validity of Kepler's First Law for orbits about a black hole.

9.4. Time Travel Using Black Hole *Stable* Circular Orbits

You are on a panel of experts called together to evaluate a proposal from the Space Administration to travel forward in time using the difference in rates between a clock in a stable circular orbit around a black hole and our clocks remote from the black hole. Give your advice about the feasibility of the scheme, based on the following analysis or on your own.

A. Consider two sequential ticks of the clock of a satellite in a stable circular orbit around a black hole. We want to find the ratio $d\tau^*/dt^*$. The numerator in this fraction is equal to the wristwatch time $d\tau$ between the ticks in the frame of the satellite; the denominator is the map time lapse dt^*, which is also the wristwatch time lapse of a distant observer at rest. Use the expression for map angular momentum to eliminate $d\phi$ from the Schwarzschild metric in this case to obtain (#eq:58)

$$\left(\frac{d\tau}{dt}\right)^2 = \left(\frac{d\tau^*}{dt^*}\right)^2 = \frac{1-(2/r^*)}{1+(L^*/r^*)^2} = 1 - \frac{3}{r^*} \tag{9.52}$$

where the final step uses equation (9.37).

B. What is the value of the ratio $d\tau^*/dt^*$ for the stable circular orbit of smallest radius, $r^* = 6$?

C. What rocket speed in flat spacetime gives the same ratio of rocket clock time to "laboratory" time as the stable circular orbit of smallest radius?

D. Does the proposed time travel method require rocket fuel?

E. Based on this analysis, do you recommend in favor of—or against—the Space Administration's proposal for time travel using stable circular orbits around a black hole?

9.5. Time Travel Using the Black Hole *knife-edge* Circular Orbits

Whatever your own vote on the time travel proposal of Exercise 9.4, the majority on your panel rejects the proposal because it requires extra rocket

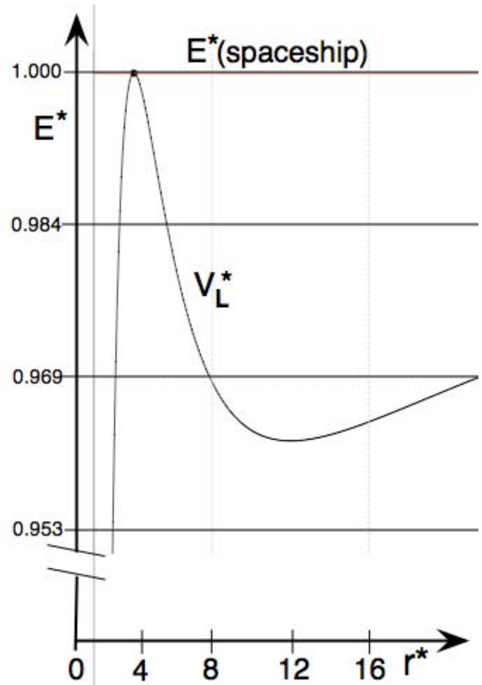

FIGURE 9.5 Insertion into a knife-edge orbit at radius $r^* = 4$ with map energy $E^* \approx 1$, equal to that of a spaceship moving slowly at large radius in a direction chosen to give it the value of L^* required to establish the peak value for $V_{\rm L}^*(r^*)$.
(#UnstableCircle)

power for insertion into and extraction from the circular orbit at $r^* = 6$. The Space Administration comes back with a new proposal that uses a knife-edge circular orbit, assuming that an automatic device can fire small rockets to maintain the satellite safely on the radial knife-edge of the effective potential. The Space Administration notes that such an orbit can be set up to require *zero* rocket burns, either for insertion or extraction from a knife-edge circular orbit. As an example, they present Figure 9.5 for the case of nonrelativistic distant velocity, so that the map energy of the satellite is $E^* \approx 1$. The direction of the remote velocity of the satellite is adjusted to achieve the value of L^* required so that $V_{\rm L}^* = E^* = 1$ at the peak, as shown in Figure 9.5. They boast that the time stretch factor is increased enormously by high satellite speed in the knife-edge orbit without the need for rocket burns to achieve that speed.

A. The condition shown in Figure 9.5 means that $V_{\rm L}^*(r^*) = 1$ at the peak shown in equation (9.22). The resulting equation plus equation (9.33) are two equations for the two unknowns r^* and L^*. Solve them to find the result: $r^* = 4$ and $L^* = 4$. *Optional:* Describe in words how the commander of the spaceship sets the desired value of L^* while still at a great distance, without changing the remote nonrelativistic speed v_{far}.

B. What is the factor $d\tau^*/dt^*$ for the spaceship in this orbit? What speed in flat spacetime gives the same time-stretch ratio?

C. Does the spaceship require a significant rocket burn to leave its knife-edge circular orbit and return remote position? If so, what will be its speed at that distant location?

D. After its long interstellar trip, the spaceship approaches the black hole at relativistic speed, that is $E^* > 1$. The crew does not want to use a rocket burn to change spaceship speed, but rather only its direction of motion to enter a knife-edge circular orbit with the same map energy it already has. Draw a figure similar to Figure 9.5 for this case. Can the astronauts find a knife-edge circular orbit on which to perch, no matter how large is the initial speed of the spaceship at a remote distance?

The small size of the ratio $d\tau^*/dt^*$ for the case shown in Figure 9.5 and analyzed in Item C leads the review panel to reject the proposal to use a knife-edge circular orbit for the purpose of time travel. However, the review panel approves the use of a knife-edge circular orbit as an essentially zero-cost parking orbit from which to take data on a black hole over an extended time. Little rocket power is required to put the approaching spaceship into that orbit, only the thrust to change approach direction in order to give the spaceship the correct value of L^*. After they finish collecting data, the astronauts can choose the time of radially-outward push-off so that they return toward home base at the same speed at which they approached, even if this speed is relativistic (Item E). In summary, once launched toward a black hole the explorers need little rocket power to go into circular orbit, to study the black hole and to return. Further details in Chapter 10.

9.6. Newtonian Orbits OK for Global Positioning System Analysis.

Your analysis of the Global Positioning System (GPS) in Chapter 4 calculated values of radius and orbital speed of a GPS satellite in circular orbit using Newtonian mechanics, with the prediction that the general relativistic analysis gives essentially the same values of radius and speed. Make this comparison for the twelve-hour orbits of GPS satellites.

WHAT IS THE MEANING OF A COMPARISON OF NEWTONIAN RADIUS OF A CIRCULAR ORBIT WITH ITS SCHWARZSCHILD MAP RADIUS?