CHAPTER

21

Cosmology
A Project

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Some say the world will end in fire,
Some say in ice.
From what I've tasted of desire
I hold with those who favor fire.
But if it had to perish twice,
I think I know enough of hate
To say that for destruction ice
Is also great
And would suffice.

—Robert Frost "Fire and Ice"

1 ■ CURRENT COSMOLOGY

Summary of current cosmology.

Will the Universe end? Will it end in fire: a high-temperature Big Crunch? Or will it end in ice: the relentless separation of galaxies that drift out of the view of our freezing descendents? The poet and the citizen are interested in these questions.

Looking back to the beginning

Cosmology is the study of the structure and content of the Universe and its development with time. We live in a golden age of astrophysics and cosmology: Observational satellites above Earth's atmosphere view the electromagnetic spectrum, from microwaves through gamma rays. These observations combine with ground-based observations in the visible and radio portions of the spectrum to yield a flood of images and data that arouse public interest and stimulate advances in theory. For the first time in human history, data and testable models inform our view of the Universe most of the way back to its beginning. Running these models forward in time allows us to evaluate alternative predictions of our distant future.

Summary of current view

Briefly, the current view is that some 13.7 billion years ago the Universe underwent a rapid expansion from a hot **big bang**, a process which created a

*Draft of Second Edition of *Exploring Black Holes: Introduction to General Relativity* Latest drafts at dropsite exploringblackholes.com with a request for comments.

variety of particles. Most of the particles decayed or annihilated with their antiparticles, leaving behind copious amounts of photons and neutrinos, a low density of **dark matter** particles of uncertain form, and finally trace amounts of the familiar electrons, protons and free neutrons. (Free neutrons decay with a half-life of 11 minutes.) During the first three minutes, collisions and cooling allowed some neutrons to bind to two protons to create stable helium nuclei and tiny amounts of deuterium (one proton and one neutron) and lithium (three protons plus four neutrons).

Microwave background radiation

About 380 000 years after the big bang, the temperature dropped to the point that electrons could bind to nuclei, creating neutral atoms. A gas of neutral atoms is transparent to optical light, to infrared light, and to microwaves—only when photon energy is large enough to ionize the atoms does the gas become opaque. So when electrons dropped into atoms, photons were freed to stream uninhibited through the Universe. The **cosmic microwave background radiation** we see arriving from all directions brings us photons from the moment at which electrons dropped into atoms, although they are vastly redshifted due to the subsequent expansion of the Universe. This background radiation is almost uniform, varying in different directions by only a few parts in 100 000. Gravitational attraction between slightly more dense regions of gas, amplified these fluctuations, which grew into the galaxies and clusters of galaxies that we now observe. The present chapter shows how these small fluctuations also tell us about the **spatial curvature** and the constituents of the Universe.

Dark matter and dark energy

Gravitational compression of matter into stars ignited nuclear fusion, combining lighter elements into heavier ones up to and including the most stable element, iron. Explosions of some dying stars spewed these elements into space, in the process creating nuclei heavier than iron. The initial expansion rate of the Universe slowed down due to the preponderant influence of matter attracting other matter. This attraction arose from both visible matter made of atoms and the **dark matter** whose presence is inferred from the stable rotation of galaxies and motion within galaxy clusters. In contrast, the present Universe appears to be increasing its rate of expansion due to a so-far mysterious **dark energy**. If the most popular current cosmological models are correct, the accelerating expansion could continue indefinitely. The present chapter describes more fully and further analyzes this apparently crazy prediction.

Study constituents of the Universe

In addition to nailing down more accurate numbers than those given by current observations, the goal of future observations is to explore the nature of dark matter, which evidently accounts for about 23 percent of the mass/energy in the Universe, and the nature of dark energy, which makes up about 73 percent. Baryonic matter, protons and neutrons in the form of atoms and their associated electrons—everything we see, touch, and are made of—accounts for only 4 percent of the contents of the Universe.

In this chapter we continue to apply Einstein's general relativity theory to cosmological models. It is possible that Einstein's theory fails at cosmological distances and over the immense lifetime of the Universe. If so, dark matter and dark energy may turn out to be the consequence of an outmoded theory. But so far Einstein's theory has never failed a clear test of its correctness. Therefore we continue to use it as the theoretical structure for our rapidly developing theory of the history, present state, and future of the Universe.

2 ■ FRIEDMANN-ROBERTSON-WALKER (FRW) MODEL OF THE UNIVERSE

Einstein's equations tell us how the Universe develops in time.

How does $R(t)$ vary with t?

Chapter 18 introduced the Robertson-Walker metric, expressed in co-moving coordinates χ and ϕ and the set of functions $S(\chi)$ that embody the curvature of spacetime (assumed to be uniform—on average—throughout the Universe). The Robertson-Walker metric contains the undetermined time-dependent radial function $R(t)$ and cannot provide a cosmological model until we know how $R(t)$ develops with time. Our task in the present chapter is to find an equation for $R(t)$ and to use it to describe the past history and predict the future fate of the Universe. In order to simplify the algebra that follows, we introduce a dimensionless **scale factor** $a(t)$ equal to the radial function $R(t)$ at any time divided by its value $R(t_0)$ at the present time t_0:

Answer with scale factor $a(t)$.

$$a(t) \equiv \frac{R(t)}{R(t_0)} \qquad (t_0 \equiv \text{NOW}) \tag{1}$$

Friedmann predicts how $a(t)$ varies with t.

In 1922 Alexander Alexandrovich Friedmann combined the Robertson-Walker metric with Einstein's field equations to obtain what we now call the **Friedmann equation**, which relates the time rate of change of the scale factor to the total mass-energy density ρ_{tot}, assumed to be uniform—on average—throughout the Universe. Even though uniform in space, the mass-energy density is a function of time, $\rho_{\text{tot}}(t)$. The resulting model of the Universe is called the **Friedmann-Robertson-Walker model** or simply the **FRW cosmology**. The Friedmann equation is:

$$H^2(t) \equiv \left[\frac{\dot{R}(t)}{R(t)}\right]^2 \equiv \frac{\dot{a}^2(t)}{a^2(t)} = \frac{8\pi\rho_{\text{tot}}(t)}{3} - \frac{K}{a^2(t)} \qquad \text{(Friedmann equation)} \tag{2}$$

where K is the constant parameter in the Robertson-Walker space metric of Chapter 18, with the values $K > 0$, $K = 0$, or $K < 0$ for a closed, flat, or open Universe respectively. A dot over a symbol indicates a derivative with respect to time t, in this case time read on the wristwatches of co-moving galaxies. In this chapter we describe the different constituents that add up to the total $\rho_{\text{tot}}(t)$.

Hubble parameter $H(t)$ varies with t.

The Friedmann equation (2) also contains a definition of the Hubble parameter $H(t)$, introduced in Chapter 18. The Hubble parameter changes as the scale factor $a(t)$ evolves with time. So remember: *When you see H, it is actually $H(t)$.* In this chapter we almost always use the value of H at the present time t_0 and give it the symbol H_0.

$H(t_0) \equiv H_0$ is its value NOW

$$H_0 \equiv H(t_0) \qquad \text{(Hubble parameter NOW)} \tag{3}$$

AN ASIDE ON UNITS: In the Friedmann equation (2) distance, time, and mass are all measured in meters; $a(t)$ is dimensionless, $\dot{a}(t)$ has the unit meter^{-1}, and density ρ_{tot} has the units of (meters of mass)/meter^3 = meter^{-2}. If you want to express everything in conventional units, such as mass in kilograms, then the Friedmann equation becomes (using conversion factors inside the front cover):

$$H^2(t) \equiv \frac{\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G}{3}\rho_{\text{tot}}(t) - \frac{Kc^2}{a^2(t)} \qquad \text{(Friedmann equation, conventional units)} \tag{4}$$

For simplicity we will use equation (2) in the following analysis.

Equation (2) can be written in a form that shows how expansion (H, that stretches space) fights with density (ρ_{tot}, whose gravitational attraction curves spacetime) to determine the value of K.

$$K = a^2(t)\left[\frac{8\pi}{3}\rho_{\text{tot}}(t) - H^2(t)\right] \tag{5}$$

Einstein links geometry with energy.

A large density ρ_{tot} in (5) tends to increase the value of K, giving the Universe positive curvature. In contrast, a large expansion rate H tends to lower the value of K. In all cases, $\rho_{\text{tot}}(t)$ and $H(t)$ vary together so as to make K independent of time. This remarkable fact reflects the conservation of energy: $(Ha)^2$ is proportional to the kinetic energy of an object in an expanding Universe, while the term proportional to density in equation (5) is proportional to minus the gravitational potential energy of that object. Thus the Einstein field equations link geometry and energy.

Critical density ρ_{crit} yields flat spacetime

We need a benchmark value for the density ρ_{tot}, something with which to compare observed values. A useful reference density is the **critical density** $\rho_{\text{crit}}(t)$, which is the total density for which spacetime is flat, a condition described by the value $K = 0$. For densities greater than the critical density ($\rho_{\text{tot}} > \rho_{\text{crit}}$) the Universe has a closed geometry ($K > 0$). For densities less than the critical density ($\rho_{\text{tot}} < \rho_{\text{crit}}$) the Universe has an open geometry ($K < 0$). The Friedmann equation (2) shows that the Hubble parameter H is a function of time. Therefore the critical density also changes with time. We define the **critical density at the present time** as $\rho_{\text{crit},\,0}$, determined by the Hubble constant H_0, the present value of the Hubble parameter. Substitute this value and $K = 0$ into the Friedmann equation (2) to obtain:

$$\rho_{\text{crit},\,0} \equiv \frac{3H_0^2}{8\pi} \qquad \text{(critical density NOW, flat spacetime)} \tag{6}$$

The ratio NOW of total density to critical density (for flat spacetime) is a parameter used widely in cosmology. This parameter is given the Greek symbol capital omega, Ω:

$$\Omega_{\text{tot},\,0} \equiv \frac{\rho_{\text{tot}}(t_0)}{\rho_{\text{crit},\,0}} \ . \qquad (7)$$

Throughout this chapter, we retain the subscript zero as a reminder that we mean the density measured NOW relative to the critical value NOW. Combining equations (5), (6), and (7) NOW (when $a(t_0) \equiv 1$) gives a simple relation between the curvature parameter K and density parameter $\Omega_{\text{tot},\,0}$:

$$K = H_0^2(\Omega_{\text{tot},\,0} - 1) \qquad \text{(NOW)} \qquad (8)$$

QUERY 1. Value of the critical density NOW

A. Estimate the numerical value of the critical density in equation (6) in units of (meters of mass)/meter^3 = meter^{-2}. For the value of H_0 see equation (28) and the following equations later in this chapter.
B. Express your estimate of the value of the critical density as a fraction of the density of water (one gram per cubic centimeter).
C. Express your estimate of the value of the critical density in units of hydrogen atoms (effectively, protons) per cubic meter.

Find time-variation of density components.

The Friedmann equation (2) relates the rate of change of the scale factor $a(t)$ to the contents of the Universe. Before we can solve this equation for $a(t)$, we need to list the contributions to the total density and determine the time dependence of each. We now examine some of the strange properties of the components of the total density ρ_{tot}. Section 3 catalogs the different components of the contents of the Universe and describes how each of them varies with scale factor $a(t)$. Section 7 details recent observations on which we base estimates of the amounts of these different components.

3 ■ CONTENTS OF THE UNIVERSE I: HOW DENSITY COMPONENTS VARY WITH THE SCALE FACTOR

Matter, radiation, and dark energy.

The Friedmann-Robertson-Walker model of the Universe has been widely accepted for 40 years, but observations during that time have significantly modified our picture of the contents of the Universe. Our knowledge has in fact grown significantly during just the last decade. Such is the excitement of being at the research edge of so large a subject.

Universe composed of matter, radiation, and dark energy.

We group the contents of the Universe into three broad categories: matter, radiation, and dark energy. Each category is chosen because of the way its contribution to the total density changes as the Universe expands. We describe these changes in terms of the scale factor $a(t)$, leaving until later (Section 6) the derivation of how this scale factor changes with time.

Matter

Matter: stars, gas, neutrinos, and dark matter.

The first category we refer to as **matter**. By matter we mean particles or nonrelativistic objects with mass much greater than the mass-equivalent of their kinetic energy. Objects in this category are:

- **STARS**, including white dwarfs, neutron stars, and black holes.
- **GAS**, mostly hydrogen, with a smattering of other elements and dust.
- **NEUTRINOS**, very light particles recently determined to have a small mass. Neutrinos are produced, among other ways, by the decay of free neutrons.
- **DARK MATTER**, the non-luminous stuff, as yet unidentified, that makes up most of the matter in the Universe.

Stars and gas: mostly protons & neutrons.

Stars, interstellar gas, and dust are made of atoms. Cosmologists sometimes call atomic matter **baryonic** matter because most of the mass is made of baryons—mostly protons and neutrons. The mass of an electron is negligible compared to the mass of an atomic nucleus, so even though the electron is not technically a baryon the distinction is unimportant when counting mass.

Luminous matter: 4% of Universe.

Current observations lead to the estimate that **luminous matter**, the stars we can see, make up about one percent of the density of the Universe, with stars and gas together totaling four percent. It seems incredible that all the stars, individually and in galaxies and groups of galaxies, taken together, have only a minor influence on the development of the Universe. Yet observations appear to force us to this conclusion.

Neutrino mass is negligible.

Cosmic background neutrinos have never been directly detected but their presence is inferred from our understanding of nuclear physics in the early Universe. They contribute at most a small fraction of one percent to all the mass in the Universe.

Dark matter holds galaxies together.

Dark matter is currently estimated to account for approximately 23 percent of the mass-energy of the Universe. What is dark matter? And how do we know that it contributes so large a fraction? We do not know what dark matter is, but from observations we infer its density and some of its properties. From the **rotation curves** of galaxies (the tangential velocities of gas as a function of distance from the center—Figure 5) we can derive the magnitude of gravitational forces needed to keep the galaxies from flying apart, and, by implication, the amount and distribution of matter in them. The results (Section 8) show that luminous matter in a galaxy, which of course is all that we can observe directly, typically provides only a few percent of the mass required to bind the galaxy together. Dark matter was originally postulated in the 1970s to complete the total needed to hold each galaxy together as it rotates. Observations on the dynamics of galaxy clusters—first made in the 1930s and greatly refined in the 1980s and 1990s—provide further evidence for the presence of dark matter.

MATTER density varies as $a(t)^{-3}$.

The energy density of a gas of particles (whether particles of baryonic matter or dark matter) is the number density n times the energy per particle: nE. For nonrelativistic matter the energy per particle is well approximated by its mass m, so the matter density becomes $\rho_{\rm mat} = nm$. The mass of the particle is a constant (independent of the expansion of the Universe). However, the number density n, the number of particles per unit volume, drops as the volume increases, varying with the scale factor as $a^{-3}(t)$, since volume is proportional to the cube of the linear dimension. By the definition in equation (1), the scale factor $a(t)$ has the value unity at the present age of the Universe t_0. Call $\rho_{\rm mat,\,0}$ the value of the matter density at the present time. Then at any time t we predict:

$$\rho_{\rm mat}(t) = \rho_{\rm mat,\,0}\, a^{-3}(t) \ . \tag{9}$$

Equation (9) tells us that if we know the matter density today and the scale factor $a(t)$ as a function of time, we can determine the value of the matter density at any other time, past or future. (Thus far we still have not found the time dependence of $a(t)$.)

Radiation

Radiation: mass much less than energy.

Particles whose mass is much less than their energy earn the name **radiation**. Today the category *radiation* consists almost exclusively of photons. At much earlier times, neutrinos were relativistic particles with kinetic energy much greater than their mass. At such times, they were a significant part of the radiation component.

Recombination: Universe becomes transparent.

At the present stage of the Universe, radiation is a whisper, but it used to be a shout. Shortly after the big bang radiation contributed the dominant fraction of the mass/energy density of the Universe. In the hot ionized plasma of the early Universe, radiation and matter were tightly coupled: photons continually scattered from free electrons, so photons could not move in straight lines. About 380 000 years after the big bang, however, the Universe cooled to a temperature of about 3 000 K, at which point electrons combined with protons to create hydrogen gas (with some helium and a trace amount of lithium). This period is called **REcombination**, even though the stable electron-nucleus combination was taking place for the first time. At recombination, the Universe became transparent to radiation, and photons were essentially decoupled from matter, free to stream across the Universe unimpeded. The cosmic microwave background radiation that we observe in all directions is a view of that early transition from opaque to transparent, with later expansion lowering the temperature to 2.73 degrees Kelvin. It is amazing to think that the low-energy photons that we detect as background radiation between the stars have been streaming freely for billions of years, not interacting with a single thing until they enter our detectors.

Universe expansion reduces photon energy as well as density . . .

The number of photons emitted by all the stars in the history of the Universe is tiny compared with the number of photons created in the hot big bang. In the early Universe these photons were continually being emitted, absorbed, and scattered, but the *number* of photons is approximately constant as the Universe expands. Therefore the *number of photons per unit volume* varies inversely as the scale factor cubed, or as $a^{-3}(t)$, just as the number of matter particles do. But there is an additional effect for photons. The equation $E = hf = hc/\lambda$ connects the energy E of a photon to the frequency f and wavelength λ of the corresponding electromagnetic wave. The symbol h stands for **Planck's constant**, with the value $h = 6.63 \times 10^{-34}$ kilogram-meter^2/second in conventional units. As this wave propagates through an expanding space, its wavelength increases in proportion to $a(t)$. This increased wavelength is observed as the redshift of light from distant galaxies. An increasing wavelength implies a *decrease* in the energy of each photon, an energy proportional to $a^{-1}(t)$. This leads to an extra (inverse) power of $a(t)$ compared with that for matter in equation (9) because of the drop in energy of each photon as the Universe expands. Let $\rho_{\mathrm{rad},\,0}$ represent the density of radiation at t_0, the present age of the Universe. Then we predict that the radiation density for other values of the scale factor $a(t)$ obeys the equation

. . . so radiation energy density varies as $a(t)^{-4}$.

$$\rho_{\mathrm{rad}}(t) = \rho_{\mathrm{rad},\,0}\, a^{-4}(t) \ . \qquad (10)$$

where $\rho_{\mathrm{rad},\,0}$ is the average radiation density right now.

Dark Energy

After matter and radiation, the remaining contribution to the contents of the Universe is rather bizarre stuff which we call **dark energy**. Dark energy is entirely unrelated to *dark matter*, the major component of *matter*. Dark energy is detected only indirectly, through its effects on cosmic expansion. Its composition is unknown. Dark energy is the component of the total density that accounts for the observed (and surprising) current increase in the rate of expansion of the Universe. Observations described in Sections 7 and 8 lead to the estimate that approximately 73 percent of the mass/energy of the Universe is in the form of dark energy.

Dark energy composition is unknown.

QUERY 2. Mass density of radiation

The cosmic microwave background radiation has a nearly perfect blackbody spectrum with current temperature $T_0 = 2.726$ K. The temperature decreases as the Universe expands

$$T = T_0\, a^{-1}(t) \tag{11}$$

The energy density (energy/volume) of blackbody radiation in conventional units is given by the equation

$$u_{\text{rad}} = \frac{\pi^2}{15}\frac{(k_{\text{B}}T)^4}{(c\hbar)^3} \equiv a_{\text{rad}} T^4 \ . \tag{12}$$

Here k_{B} is the Boltzmann constant, c is the speed of light, and $\hbar \equiv h/2\pi$ where h is the Planck constant. The quantity a_{rad} is called the **radiation constant**.

A. Show that equations (11) and (12) are consistent with equation (10).
B. Find the present value of the energy density corresponding to the cosmic background radiation, in kilograms per cubic meter. (We assume that the complete equivalence of energy and mass is by now second nature for you.)
C. Express your answer to part B as a fraction or multiple of the critical density, $\rho_{\text{crit},0}$.
D. Take the average energy of a photon in the gas of cosmic background radiation surrounding us to be $k_{\text{B}}T$. Estimate the present-day number of photons per cubic meter. Compare your result with the critical mass density expressed in the number of hydrogen atoms (effectively, protons) per cubic meter.
E. At what absolute temperature T would blackbody radiation energy density be equal to the value of the critical density $\rho_{\text{crit},0}$ at the present time?

Dark energy = vacuum energy?

Dark energy is a generic term which encompasses all of the various possibilities for its composition. One possibility is the so-called **vacuum energy**. We often think of the vacuum as "nothing," but that is not the picture offered by modern physics through quantum field theory, which defines the vacuum to be the state of lowest possible energy. As the Universe expands, this lowest possible vacuum energy density does not drop, but rather remains constant. Of what does vacuum energy consist? One can think of the vacuum as containing **virtual particles** that are continually being created and rapidly annihilated, according to quantum field theory. The presence of virtual particles is a well-known and well-tested consequence of the standard model of particle physics. For example, virtual particles in the surrounding vacuum have a small but detectable effect on the energy levels of the hydrogen atom. Virtual particles surely have gravitational effects, but it has proved very difficult to correctly estimate the magnitude of these effects.

Cosmological effects of vacuum energy are described using the **cosmological constant** symbolized by the capital Greek lambda, Λ. In 1917

TABLE 1 Contents of the Universe. (Subscript 0 means value NOW.)

Contents	Consisting of	Scale variation
Matter	stars, gas, dark matter, (neutrinos: negligible)	$\rho_{\text{matt},0}\, a^{-3}(t)$
Radiation	photons, (earlier: neutrinos)	$\rho_{\text{rad},0}\, a^{-4}(t)$
Dark energy	cosmological constant?	$\rho_\Lambda = \text{constant}$

Einstein: cosmological constant

Einstein added the cosmological constant to his original field equations in order to make the Universe static, that is to keep it from collapsing from what he assumed must be an everlasting constant state. Einstein later removed the cosmological constant from the field equations when Hubble showed in 1929 that the Universe is expanding, but the cosmological constant continues to pop up in different theories of cosmology, as it does here as a possible source of dark energy. The presence of the cosmological constant in modern theory does *not* require a static Universe. In the 1960s, Yakov Borisovich Zel'dovich and Erast B. Gliner showed that vacuum energy is equivalent to a cosmological constant.

Other more complicated candidates for dark energy could lead to a time-dependent energy density, but there is no current consensus about these possibilities. A full description of dark energy may have to await the development of a complete theory of quantum gravity, which does not yet exist. In this chapter we will not consider time-dependent dark energy.

Assume dark energy density is constant with time.

IF vacuum energy accounts for dark energy, THEN as the Universe expands the density of dark energy remains constant. We use the subscript Λ for dark energy to remind ourselves of our assumption that vacuum energy accounts for dark energy, and take ρ_Λ to be the symbol for dark energy density that is constant with time:

$$\rho_{\text{dark energy}}(t) \equiv \rho_\Lambda = \text{constant} \; . \tag{13}$$

DO WE NEED COMMENT ON "CONSERVATION OF TOTAL ENERGY OF THE UNIVERSE," WHICH (AS I UNDERSTAND IT) CANNOT BE DEFINED? JON GARRITY ASKS WHERE THE ADDED ENERGY COMES FROM TO KEEP DARK ENERGY DENSITY CONSTANT AS THE UNIVERSE EXPANDS.

Table 1 summarizes the contents of the Universe and the scale dependence of each component. The time-independent density of dark (vacuum) energy contrasts with the density of matter, proportional to $a^{-3}(t)$, and the energy density of radiation, proportional to $a^{-4}(t)$, both of which decrease as the Universe expands. As a result, dark energy influences the development of the Universe more and more as time passes.

Variation of the total density with the scale factor $a(t)$

Time variation of total density.

We can now write an expression for the time dependence of total density from equations (9), (10), and (13),

$$\rho_{\rm tot}(t) = \frac{\rho_{\rm mat,\,0}}{a^3(t)} + \frac{\rho_{\rm rad,\,0}}{a^4(t)} + \rho_\Lambda \ . \qquad (14)$$

Divide through by the critical density at the present time, equation (6), to express the result as fractions of the present critical density, as in equation (7):

$$\frac{\rho_{\rm tot}(t)}{\rho_{\rm crit,\,0}} = \frac{\rho_{\rm mat,\,0}}{\rho_{\rm crit,\,0}} a^{-3}(t) + \frac{\rho_{\rm rad,\,0}}{\rho_{\rm crit,\,0}} a^{-4}(t) + \frac{\rho_\Lambda}{\rho_{\rm crit,\,0}} \ . \qquad (15)$$

Fractional densities Ω

We want to plot equation (15) as a function of the scale factor $a(t)$. To do this we need numerical values for the three fractional densities in that equation. These fractional densities also define contributions to the total density parameter Ω defined in equation (7).

In Section 7 we describe current observations that yield the approximate values:

$$\Omega_{\rm mat,\,0} \equiv \frac{\rho_{\rm mat,\,0}}{\rho_{\rm crit,\,0}} = 0.27 \pm 0.03 \ . \qquad (16)$$

$$\Omega_{\Lambda,\,0} \equiv \frac{\rho_\Lambda}{\rho_{\rm crit,\,0}} = 0.73 \pm 0.03 \ . \qquad (17)$$

In QUERY 2 you showed that at the present time, the background radiation yields an energy density of approximately 5×10^{-5} times the critical density. Assuming that the neutrinos have zero mass and move with the speed of light would increase this by 68% implying

$$\Omega_{\rm rad,\,0} \equiv \frac{\rho_{\rm rad,\,0}}{\rho_{\rm crit,\,0}} \approx 8.4 \times 10^{-5} \ . \qquad (18)$$

Actually, the neutrinos are known today to be nonrelativistic, with mass, so this is not the correct value; nonetheless, their contribution to the density today is so small that the error made in equation (18) by assuming neutrinos are massless is negligible.

We live between matter domination and vacuum energy domination.

Figure 1 graphs equation (15) with numerical values given in equations (16), (17), and (18). Because each of the individual quantities is proportional to a power of $a(t)$, during times when one component dominates the total density, ρ versus $a(t)$ is a straight line on the log-log graph. Figure 1 shows that the radiation contribution has little effect at present, but was dominant at early stages because of the multiplier a^{-4} in equation (15). For a while after the radiation-dominated era, matter had the greatest influence on the evolution of the Universe. But the influence of matter is also fading by now because of the multiplier a^{-3}. The contribution of dark energy was negligible in the distant past but has an increasing effect at the present and later stages

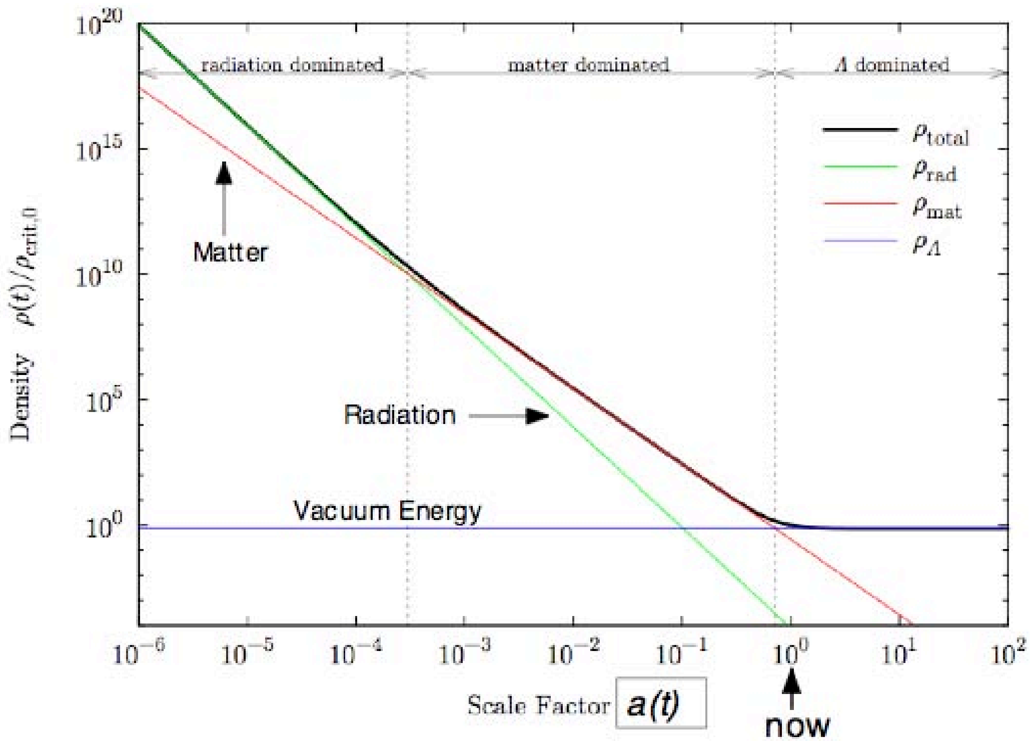

FIGURE 1 Total mass-energy density of the Universe (heavy line) in units of the present critical value plotted versus the expansion scale factor. The vertical dashed lines denote transitions between the radiation-dominated early phase, the matter-dominated middle era, and the vacuum-energy-dominated late stage of the Universe. (We assume here that dark energy is vacuum energy.)

of expansion, because its density remains constant as densities of matter and radiation decay away with the increase in $a(t)$. If the data and assumptions behind Figure 1 are correct, we are at the beginning of the era dominated by dark energy.

QUERY 3. Contributions to the Density

A. Use equation (15) to find the approximate values of $\rho_{\rm tot}(t)/\rho_{\rm crit,\,0}$ at the following times:

- at the end of the radiation-dominated era (that is, when radiation and matter make approximately equal contributions)
- at the end of the matter-dominated era (that is, when matter and dark energy make approximately equal contributions)
- NOW

- when $a(t) = 10^6$

B. What additional information do you need to answer the question: How many billions of years ago did the radiation-dominated era end?

It seems an odd coincidence that at the present moment—NOW in Figure 1—we are at the transition between the matter-dominated Universe and one shaped by vacuum energy. Is there a deep reason for this? Could life have developed on Earth at a different time on the curves of Figure 1?

Deep questions, indeed, which we encourage you to pursue. We do not see how to answer them with the limited range of skills developed in this book. Indeed, we do not see how to move past speculations to scientific verification, mainly because we have only one Universe in which this "experiment" is taking place.

4 ■ UNIVERSES WITH DIFFERENT CURVATURES

Effective potential for the Universe

Using the Friedmann equation (2), we can analyze the development of alternative model Universes with different assumptions for the curvature K. To put the Friedmann equation in a more useful form, divide it through by H_0^2 and substitute for the critical density from equation (6):

$$\left(\frac{H}{H_0}\right)^2 = \left(\frac{\dot{a}}{H_0 a}\right)^2 = \frac{\rho_{\text{tot}}}{\rho_{\text{crit},\,0}} - \frac{K}{H_0^2 a^2} \; . \qquad (19)$$

Time development of the Universe

where, remember, a dot over a symbol means its derivative with respect to time t. Re-express equation (19) in terms of the components of Ω_{tot} defined in equations (8), (16), (17), and (18):

$$\left(\frac{H}{H_0}\right)^2 \equiv \left(\frac{\dot{a}}{H_0\, a}\right)^2 = \Omega_{\text{mat},\,0}\, a^{-3} + \Omega_{\text{rad},\,0}\, a^{-4} + \Omega_{\Lambda,\,0} - \frac{K}{H_0^2\, a^2} \; . \qquad (20)$$

For the present time, t_0, when $a(t_0) = 1$, we can write equation (20) in the very simple form:

$$1 = \Omega_{\text{mat},\,0} + \Omega_{\text{rad},\,0} + \Omega_{\Lambda,\,0} - \frac{K}{H_0^2} \qquad \text{NOW} \; . \qquad (21)$$

This equation allows us to determine the curvature parameter K from measurements of $\Omega_{\text{mat},\,0}$, $\Omega_{\text{rad},\,0}$, and $\Omega_{\Lambda,\,0}$. Compare it with equation (8). Current observations lead to the conclusion that, within measurement uncertainties of about 2% in $\Omega_{\text{tot},0}$, the Universe is flat ($K = 0$), in agreement with equations (16), (17), and (18).

For any arbitrary time, equation (20) can be rearranged to read:

$$\dot{a}^2 - H_0^2 \left[\Omega_{\text{mat},0} a^{-1} + \Omega_{\text{rad},0} a^{-2} + \Omega_{\Lambda,0} a^2\right] = -K \ . \qquad (22)$$

Compare equation (22) with the corresponding Newtonian expression derived from the conservation of energy for a particle moving in the x-direction subject to a potential $V(x)$:

$$\dot{x}^2 + \frac{2V(x)}{m} = \frac{2E_{\text{total}}}{m} \qquad \text{Newtonian} \ . \qquad (23)$$

In the Newtonian case we can get a qualitative feel for the particle motion by plotting $V(x)$ as a function of position and drawing a straight line at the value of E_{total}. We use equation (22) for a similar purpose, to get a qualitative feel for the evolution of the Universe. Rewrite equation (22) as:

$$\dot{a}^2 + V_{\text{eff}}(a) = -K \ . \qquad (24)$$

Here the $-K$ on the right takes the place of total energy, and $V_{\text{eff}}(a)$ is an effective potential given by the equation

$$V_{\text{eff}}(a) \equiv -H_0^2 \left[\Omega_{\text{mat},0} a^{-1} + \Omega_{\text{rad},0} a^{-2} + \Omega_{\Lambda,0} a^2\right] \ . \qquad (25)$$

Isn't it remarkable that effective potentials appear when we analyze orbits of a stone (Chapter 9), trajectories of light (Chapter 12), and expansion of the Universe (in the present chapter)?

We summarize here the assumptions on which equations (22), (24), and (25) are based.

ASSUMPTIONS FOR THE DEPENDENCE OF $\dot{a}$ ON $a(t)$

Assumptions

1. The Universe is homogeneous (on average the same in all locations).
2. The Universe is isotropic (on average the same as viewed in all directions).
3. Dark energy is vacuum energy and therefore its density is constant, independent of $a(t)$.
4. *Background assumptions*: There are no other forms of mass-energy in the Universe; spacetime has four dimensions; general relativity is correct; the Standard Model of particle theory is correct, and so on.

"Effective potential" for the Universe

Figure 2 plots V_{eff}/H_0^2 as a function of $a(t)$, using the values of the densities given in equations (16), (17), and (18). For the range of $a(t)$ plotted, radiation has negligible effect. Figure 2 carries a lot of information about the history and alternative futures of the Universe according to different values of K. In the Newtonian analogy, an effective potential with a *positive* slope yields

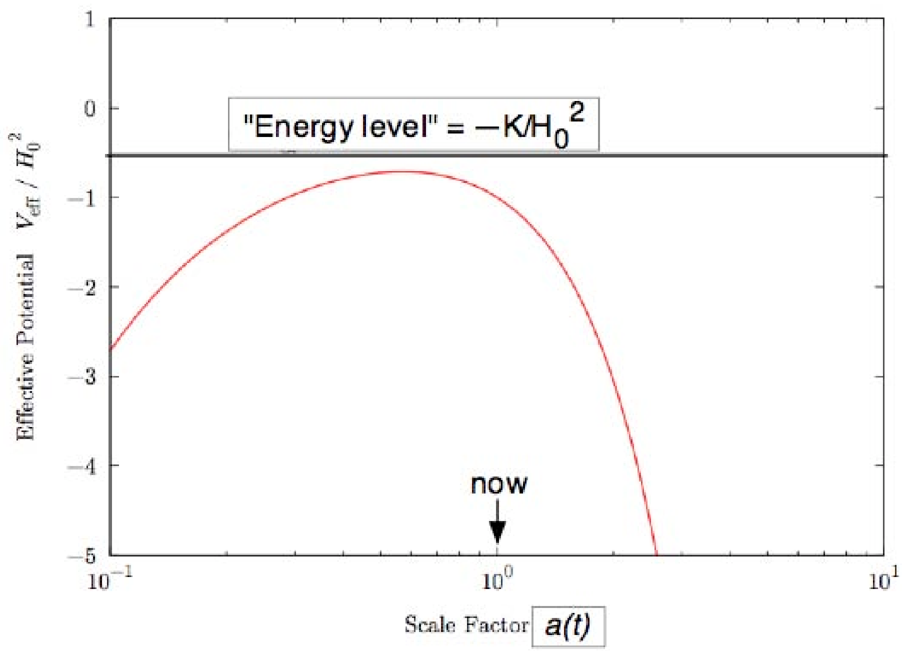

FIGURE 2 Effective potential governing the evolution of $a(t)$ according to equations (24) and (25). The"energy level" is set by $V_{\rm eff}/H_0^2 = -K/H_0^2$. The figure shows an example of a closed Universe that expands forever. Our Universe has $K = 0$ to a good approximation and will apparently expand forever.

a force tending to slow down positive motion along the horizontal axis, while the portion of the effective potential with a *negative* slope yields a force tending to speed up positive motion along the horizontal axis. These two conditions occur, respectively, to the left and the right of the peak at $a(t) \approx 0.57$. By analogy, then, $a(t)$ decelerates to the left of $a(t) \approx 0.57$ and accelerates to the right of $a(t) \approx 0.57$. This acceleration is due to dark energy. (CAUTION: Cosmological models described in older textbooks, written before dark energy was shown to be significant in the observed expansion of the Universe—say, before 1999—effectively assume that $\Omega_{\Lambda,0} = 0$ so that the expansion never accelerates.)

QUERY 4. The Friedmann-Robertson-Walker Universe

Figure 2 enables us to deduce many things about the history of the Universe. Answer the following questions about the predictions of this model under the assumption that the Universe begins with a big bang. Make a reasonable assumption about the qualitative influence of radiation on $V_{\rm eff}(a)$ for small $a(t)$.

1. True or false: The descending curve to the right of $a(t) \approx 0.57$ says that the Universe is contracting after $a(t)$ reaches this value.

2. Can the Universe be closed and expand forever?
3. Can the Universe be closed and recontract?
4. Can the Universe be open and expand forever?
5. Can the Universe be open and recontract?
6. Can the Universe be flat and expand forever?
7. Can the Universe be flat and recontract?
8. Describe qualitatively the evolution of a flat Universe ($K = 0$). Be specific about the evolution of $a(t)$ in the region to the right of the peak in the curve of V_{eff}/H_0^2.
9. What point on the graph of Figure 2 corresponds to a value of K that would lead to a static Universe? How could the Universe arrive at this configuration starting from a big bang? Is this static configuration stable or unstable, and what are the physical meanings of the terms *stable* and *unstable*?

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5 ■ SOLUTION FOR THE SCALE FACTOR

Integrating $\dot{a}(t)$

Our ignorance is stuffed into $a(t)$.

Thus far we have stuffed all our ignorance about the time development of the Universe into the scale factor $a(t)$, as given in equation (14) and plotted along the horizontal axes in Figures 1 and 2. We need to determine how $a(t)$ itself develops with time. To do this we integrate the Friedmann equation (2) as modified in equation (22). Using equations (8) and (21), rearrange (22) to read:

$$\frac{da}{dt} = H_0[\Omega_{\text{mat},\,0}(a^{-1}-1) + \Omega_{\text{rad},\,0}(a^{-2}-1) + \Omega_{\Lambda,\,0}(a^2-1) + 1]^{1/2} \qquad (26)$$

Integrate da/dt.

By eliminating the curvature we have shown that the components of Ω_0 completely determine the expansion of the Universe—they are *important!* Now invert this equation and derive an integral with the limits from NOW ($a(t_0) = 1$) to any other arbitrary $a(t)$:

$$t - t_0 = \frac{1}{H_0}\int_1^a \frac{da'}{[\Omega_{\text{mat},\,0}(a'^{-1}-1) + \Omega_{\text{rad},\,0}(a'^{-2}-1) + \Omega_{\Lambda,\,0}(a'^2-1) + 1]^{1/2}}\ . \qquad (27)$$

Here a' is the dummy variable of integration. We can integrate equation (27) numerically from the present time t_0 to either a future time t ($a > 1$) or to an earlier time ($a < 1$). The big bang occurred when $a = 0$.

In order to carry out the integration in (27), we need to put into the integral all of our time variations of the Ω functions. Before doing this, however, we express the constituents of (27) in convenient units. Recall that the scale factor $a(t)$ is unitless and is defined to have the value unity at present time, equation (1). If we choose to express time in years, then the time derivative $\dot{a}(t)$ will have the units years^{-1}. Then, the current value of the

Hubble constant H_0 will also be expressed in the unit of years^{-1}. This is a different unit than those conventional in the field. Recent observations yield the following approximate value for H_0 in conventional units:

$$H_0 = 72 \pm 3\ \frac{\text{kilometers/second}}{\text{Megaparsec}} \ . \tag{28}$$

QUERY 5. H_0 in years^{-1}

Use conversion factors inside the front cover to convert the units of (28) to years^{-1}. Verify the resulting value is:

$$H_0 \approx 7.37 \times 10^{-11}\,\text{year}^{-1} \ . \tag{29}$$

Approximate age of the Universe: $t_0 \approx H_0^{-1}$

It is not a coincidence that the coefficient $H_0^{-1} = 1.36 \times 10^{10}$ years in equation (29) approximates the estimated age of the Universe: $t_0 \approx 13.7$ billion years. If $a(t)$ represented a linear expansion, then $a = At$ for some constant A, and because $a = a(t_0) = 1$ today, the age of the Universe would be $t_0 = A^{-1}$. The Hubble constant is $H_0 \equiv \dot{a}(t_0)/a(t_0) = A$. So, for the case of linear expansion, $t_0 = H_0^{-1}$. Although the actual solution $a(t)$ is not linear in our Universe, $a(t_0)/t_0$ is close to $\dot{a}(t_0) = H_0$ because the Universe has recently made the transition from deceleration to acceleration. Therefore the age of the Universe approximately equals the **Hubble time** H_0^{-1}.

QUERY 6. Various kinds of Universes

Integrate equation (27) in three simplifying cases, under the assumption that spacetime is flat ($K = 0$).

A. Assume the Universe contains only matter and that $\Omega_{\text{mat},0} = 1$. Find an expression for $a(t)$ and the exact value of $H_0 t_0$.

B. Assume the Universe contains only radiation and that $\Omega_{\text{rad},0} = 1$. Find an expression for $a(t)$ and the exact value of $H_0 t_0$.

C. Assume that the Universe contains only dark energy and that $\Omega_{\Lambda,0} = 1$. Find an expression for $a(t)$.

D. *Optional.* Discuss the validity of your results for parts A, B, and C for $t < t_0$ and in particular for $t = 0$.

Integrating equation (27) requires that we know the values of the components of the total density. Remember that the total density parameter Ω_{tot} determines the curvature parameter according to equation (21). Therefore (27) has been integrated numerically for several cases as shown in Figure 3. The model with dark energy present clearly undergoes accelerated expansion at late times.

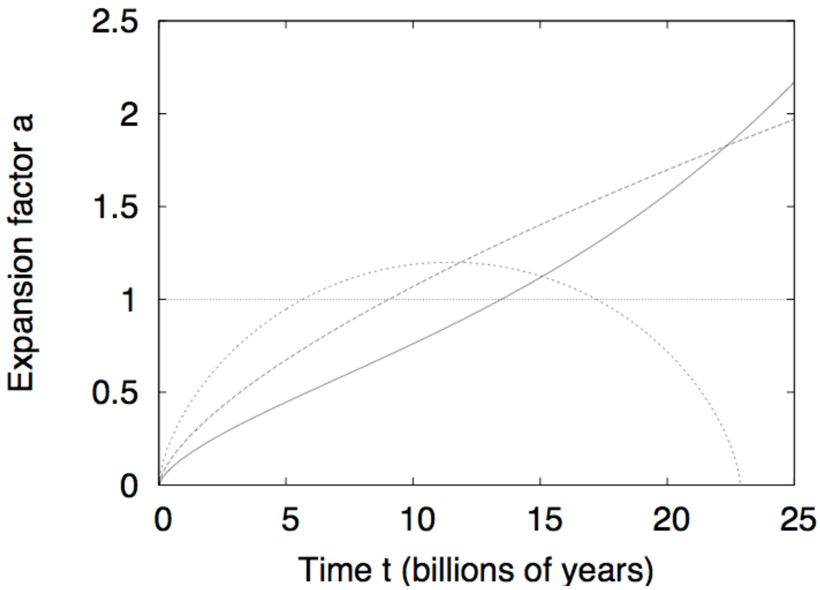

FIGURE 3 Expansion scale factor $a(t)$ versus time for three different models. The solid curve is the favored model with $\Omega_{\mathrm{mat,}\,0} = 0.27$ and $\Omega_{\Lambda,0} = 0.73$. The two dotted curves show alternative models with no dark energy, $\Omega_{\mathrm{mat,}\,0} = 1$ and $\Omega_{\mathrm{mat,}\,0} = 1$. Can you tell which is which? The curves all have the same slope where they cross $a = 1$, because that slope is the measured current value H_0 of the Hubble constant, equations (28) and (29).

6 ■ LOOK-BACK DISTANCE AS A FUNCTION OF RED SHIFT

Where are earlier emitters NOW?

Box 4 of Chapter 18 showed that the calculated physical distance now to an object that emitted light at time t and is observed by us now is (when expressed using the scale factor)

$$d_0(t) = \int_t^{t_0} \frac{dt'}{a(t')} \tag{30}$$

where t' is a dummy variable. We call d_0 the **look-back distance**. In Box 4 of Chapter 18 we approximated $a(t) \approx H_0 t$ to deduce that $d_0 = 40+$ billion light years for $t = 0.7$ billion years after the big bang as the time of emission. We can now improve on this estimate, using our new understanding of $a(t)$.

? *Wait! How do we verify with observations the current distance of 40+ billion light years to an object that emitted light 0.7 billion years after the big bang?*

! We cannot verify this current distance with observation! The speed of light is finite. Right now we see the emitting object as it was 0.7 years after the big bang. We have no direct information about its condition since then. The 40+ billion light year present distance is our projection under a set of assumptions about the motion of this emitter in the approximately 13 billion years since it emitted the light we see now.

QUERY 7. Look-back distance d_0 in terms of red shift z.

Because astronomers measure redshift z, not time t, we rewrite (30) using the relation between redshift and expansion, equation (28) of Chapter 18, which now becomes

$$1 + z(t) = \frac{1}{a(t)} \tag{31}$$

A. Differentiate both sides of (31) and use equation (2) to write H as a function of z:

$$H(z) = -a(t)\frac{dz}{dt} \tag{32}$$

B. Substitute the result into equation (30) and show that

$$d_0(z) = \int_0^z \frac{dz'}{H(z')} \tag{33}$$

where z' is a dummy variable of integration.

Look-back d_0 *vs* z

Now we can numerically integrate equation (33) using the best-fit FRW model. Figure 4 shows the calculated "look-back" (present) distance to a galaxy with observed redshift z.

7 ■ WHY IS THE RATE OF EXPANSION OF THE UNIVERSE INCREASING?

Negative pressure pushes!

Acceleration $\ddot{a}(t)$ of scale factor $a(t)$

Figure 3 displays changes in the scale factor $a(t)$ of the Universe as a function of time. The slope of the curve at any point is the rate of expansion $\dot{a}(t)$ at that time. Changes in the slope correspond to changes in this expansion rate. We can call the rate of change of the expansion rate the *acceleration of the scale factor, symbolized by a double dot:* $\ddot{a}(t)$. Why does the Universe change its rate of expansion?

Matter-dominated era: expansion slows down.

For the matter-dominated era, one can understand that matter mutually attracts and "holds back" or "slows down" the expansion, as shown in the left-hand portion of Figure 2. But the expansion in the dark-energy-dominated era clearly violates this explanation, since the rate of expansion increases

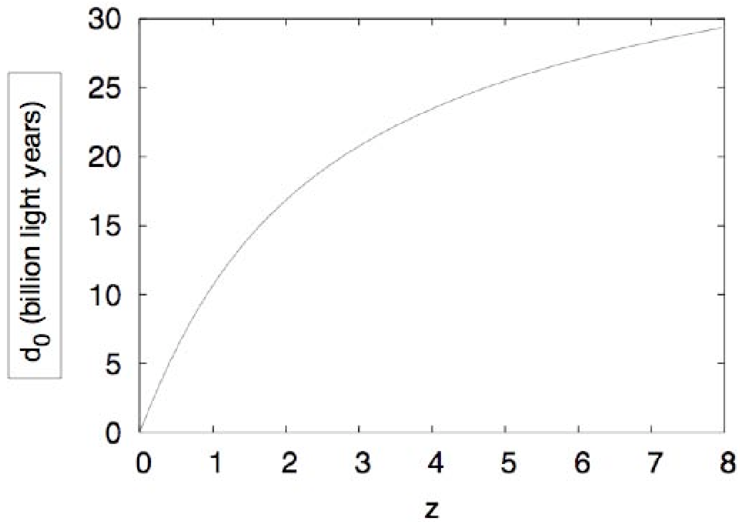

FIGURE 4 The present-day "look-back distance" d_0 to objects at redshift z. As explained in Box 4 of Chapter 18, in an expanding Universe an object that we see now (at its earlier position) is at present much farther away from us in light-years than the age of the Universe in years.

there. What is the physical reason for this increased expansion rate? This question is the subject of the present section.

Thermo background

Begin with some basic thermodynamics. The first law of thermodynamics says that as the volume of a box of gas increases by dV, the energy of the gas inside it decreases by an amount PdV where P is the pressure of the gas, as long as no heat flows into or out of the box. The energy change PdV goes into the work done by the gas due to its pressure acting on the outward-moving wall of the box. The energy of the gas is simply the volume that it occupies times its energy density. However, we measure energy in units of mass, so the energy density is just the mass density $\rho_{\rm tot}$. Therefore we have

$$d(\rho_{\rm tot}V) = -P_{\rm tot}dV \ . \tag{34}$$

Expanding gas cools.

It turns out that this relation holds whenever the volume of a gas changes, regardless of the shape of the box. It even holds when there are no walls at all! It implies a general result: an expanding gas cools.

$\ddot{a}(t)$ depends on pressure.

In the following QUERY 9 you show that the second time derivative, the acceleration $\ddot{a}(t)$ of the scale factor, depends not only on the density $\rho_{\rm tot}$ but also on pressure. Pressure, along with total density, appears in Einstein's field equations. In special relativity, pressure and energy density transform into each other under Lorentz transformations in a way analogous to (but not the same as) electric and magnetic fields. Energy density in one inertial frame implies pressure in another. [Box on this?] Since the Einstein field equations are written to be valid in any frame, pressure must make a contribution to gravity (spacetime curvature). Positive pressure has an attractive gravitational effect similar to positive energy density.

Positive pressure slows expansion.

The gravitational effect of pressure may seem paradoxical: the *greater* the positive pressure, the *more negative* the value of $\ddot{a}$, the acceleration of the scale factor. We are used to watching pressure expand things like a bicycle tire. The stretching surface of an expanding balloon is often used as an analogy to the expanding space of our Universe. These images can carry the incorrect implication that positive pressure is what makes the Universe expand. A balloon is expanded by pressure *differences*: the pressure inside the balloon is higher than the pressure outside combined with the balloon surface tension. Pressure differences produce mechanical forces. By contrast, we are considering a homogeneous pressure, the same everywhere—there is no "outside" of the Universe for it to expand into. There is no mechanical force of pressure in this case, only a gravitational force.

QUERY 8. Acceleration of the Scale Factor

A. Divide the energy conservation equation (34) through by dt (in other words, consider the differential energy change in an increment of time dt) and apply it to a local volume V that has the current value V_0 and expands (or possibly contracts) with the Universe according to the equation $V = V_0\, a^3(t)$. Show that

$$\dot{\rho}_{\rm tot} = -3\frac{\dot{a}}{a}\left(\rho_{\rm tot} + P_{\rm tot}\right) \ . \tag{35}$$

B. Rewrite the Friedmann equation (2) as

$$\dot{a}^2 = \frac{8\pi}{3}\rho_{\rm tot}a^2 - K \ . \tag{36}$$

Take the time derivative of both sides of (36) and substitute equation (35) to obtain the equation for the acceleration of the cosmic scale factor:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}\left(\rho_{\rm tot} + 3P_{\rm tot}\right) \tag{37}$$

This equation predicts that for a positive total density and positive total pressure, the scale factor will decelerate with time. The surprise comes when the pressure term is calculated to be negative for dark energy, so that $\ddot{a}/a$ can become positive—this is what is usually meant by the phrase "the expansion of the Universe is accelerating".

Negative pressure speeds up expansion.

Here comes the big surprise. In QUERY 10 you show that dark energy leads to *negative* pressure. In contrast to positive pressure, negative pressure tends to *increase* the rate of expansion of the Universe. And recent observations bring evidence that we live in a Universe whose rate of expansion is increasing, not decreasing as our model would predict if only matter and radiation were present. Now for the details.

QUERY 9. Pressure from Different Sources

A. Solve equation (35) for $P_{\rm tot}$ and show that the result is:

$$P_{\rm tot} = -\frac{a}{3\dot{a}}\dot{\rho}_{\rm tot} - \rho_{\rm tot} \ . \tag{38}$$

Equation (38) is linear in $\dot{\rho}_{\rm tot}$ and $\rho_{\rm tot}$. Therefore we can apply it separately to the different components of which $\rho_{\rm tot}$ and $P_{\rm tot}$ are composed. In parts B through D below, apply equation (38) to each component of the density to find the individual pressures due to matter, dark energy, and radiation. (Hint: In each case, express the first term on the right of (38) using the corresponding fractional density.)

B. Apply equation (38) to nonrelativistic matter for which $\rho_{\rm mat}(t) = \rho_{\rm mat,\,0}\,a^{-3}(t)$. What is the pressure $P_{\rm mat}(t)$?

C. Apply equation (38) to dark energy for which $\rho_\Lambda(t) = \text{constant} = \rho_{\Lambda,\,0}$. What is the pressure $P_\Lambda(t)$? This surprising result has a large effect on the fate of the Universe.

D. Finally, apply equation (38) to a gas of photons. Though we can neglect $\rho_{\rm rad}$ in describing how the Universe behaves today, Figure 1 shows that in the early Universe $\rho_{\rm rad}$ was in fact larger than the corresponding matter term $\rho_{\rm mat}$ and could not be neglected. For radiation, $\rho_{\rm rad}(t) = \rho_{\rm rad,\,0}\,a^{-4}(t)$. What is the pressure of radiation $P_{\rm rad}(t)$?

E. Substitute your results of parts B through D into equation (37) to find an expression for $\ddot{a}$ as a function of $a(t)$:

$$\ddot{a} = -\frac{4\pi}{3}\left[\rho_{\rm mat,\,0}\,a^{-2} + 2\rho_{\rm rad,\,0}\,a^{-3} - 2\rho_{\Lambda,\,0}\,a\right] . \tag{39}$$

F. Assuming that $\rho_{\rm rad,\,0}$ is negligible, show that the condition for acceleration today ($a = 1$) is

$$\Omega_{\Lambda,0} > \frac{1}{2}\Omega_{\rm mat,0} \ . \tag{40}$$

Negative pressure? OK. Negative mass? No.

The result of part C of QUERY 10 tells us that the pressure of the vacuum is negative, a result unfamiliar in elementary thermodynamics. However, it is perfectly physical—neither the energy density nor the pressure of the vacuum arise from physical particles. The vacuum has constant energy density produced by quantum fluctuations. Conservation of energy—represented by equation (35)—then implies that the pressure must be negative. Negative pressure—but not negative mass density—is physically allowed.

History of changes in expansion

Equation (39) gives us a history of the changes in expansion rate since the big bang. Early in the expansion, when the dimensionless scale factor $a(t)$ was very small, the dominant term on the right side of (39) was due to radiation, because a^{-3} was large. As $a(t)$ increased, the matter term, proportional to a^{-2}, came to dominate. These radiation and matter terms in (39) resulted in negative acceleration of $a(t)$, that is a *decrease* in the expansion rate $\dot{a}$. More recently, as $a(t)$ approached its current value one, the negative dark energy term, proportional to a,has become more and more important. At the present age of the Universe, the net result is a positive value of the acceleration $\ddot{a}(t)$, that is an *increase* in the expansion rate $\dot{a}(t)$.

MATTER: positive mass and zero pressure

RADIATION: positive energy density and positive pressure.

DARK ENERGY: positive mass, but *negative* pressure.

What is the *physical reason* for these changes in acceleration of the dimensionless scale factor $a(t)$? Simply that matter has mass and zero pressure, while radiation energy density and pressure are both positive. Both mass and positive pressure contribute to a deceleration of $a(t)$, a decrease of $\dot{a}(t)$, as seen in (39). In contrast, dark energy contributes positive mass but *negative* pressure. The same equation shows us that negative pressure of dark energy contributes pressure to an acceleration of $a(t)$, that is an increase in $\dot{a}(t)$, an effect that dominates as $a(t)$ becomes large.

QUERY 10. Einstein Static Universe

Einstein introduced the cosmological constant Λ to make the Universe static according to general relativity. This constant Λ is related to ρ_Λ by

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} \qquad \text{(conventional units)} \tag{41}$$

To change to metric units, use the usual shortcut, setting $G = 1$. Then

$$\rho_\Lambda = \frac{\Lambda}{8\pi} \qquad \text{(metric units)} \tag{42}$$

Einstein's model included only matter ρ_{mat} and the cosmological constant Λ.

A. From (36), show that $\dot{a} = 0$ and $a = 1$ (Universe always has the same scale factor as now) imply

$$K = \frac{8\pi}{3}\left(\rho_{\text{mat}} + \rho_\Lambda\right) \qquad (\dot{a} = 0) \tag{43}$$

B. From (39), show that $\ddot{a} = 0$ implies

$$\rho_{\text{mat}} - 2\rho_\Lambda = 0 \qquad (\ddot{a} = 0) \tag{44}$$

C. Combine these to deduce that Einstein's static Universe is closed, with spatial curvature

$$K = \Lambda = 8\pi\rho_\Lambda = 4\pi\rho_{\text{mat}} \qquad (\text{Einstein's static Universe}) \tag{45}$$

D. From Figure 2 deduce whether Einstein's model is stable or unstable. What do you think Einstein's real blunder was? (I PRESUME THAT THESE CONDITIONS LEAD TO AN ENERGY PERCHED AT THE TOP OF THE FUNCTION V_{eff}/H_0^2 IN FIGURE 2, BUT THE MANIPULATION IS COMPLICATED AND NOT OBVIOUS. WE MAY NEED INTERMEDIATE STEPS. THE LAST QUESTION IS TOO MUCH LIKE "WHAT AM I THINKING.")

E. Suppose $\Lambda < 0$. Is a static Universe possible then?

Now that we have a model for the time development of the Universe, we need to validate the assumptions that went into it, namely the values of $\Omega_{\mathrm{mat},0}$ and $\Omega_{\Lambda,0}$ given in equations (16) and (17) along with the value of $\Omega_{\mathrm{rad},0}$ given in equation (18). For that validation we turn to observations.

8 ■ CONTENTS OF THE UNIVERSE II: OBSERVATIONS

Galaxy rotation and cosmic background radiation

In this section we examine the observational evidence for the quantitative amounts of the different components of our Universe: matter (visible baryonic plus dark matter), dark energy, radiation. This will allow us, in Section 10, to draw numerical conclusions about our Universe now and to use our present model to project these results into the past and future.

Galaxy Rotation: Evidence for Dark Matter

Evidence for dark matter.

How do we know that dark matter exists around and within galaxies? The most direct evidence comes from observing the orbits of stars or gas around a galaxy. Spiral galaxies are perfect for this exercise—their rotating disks contain neutral hydrogen gas that emits radiation with a rest wavelength of 21 cm. If we see the galaxy edge on, then as gas orbits the galaxy it moves directly away from us on one side of the galaxy and directly towards us on the other side. We then use the Doppler effect to measure the speed of the gas as a function of its distance from the center of the galaxy. The result is a **rotation curve**.

Rotation curve.

Figure 5 shows the rotation curve of a nearby edge-on spiral galaxy. It is quite different from a graph of the orbital speeds of planets in the Solar System, which decrease with increasing distance from the sun according to Kepler's Third Law. Spiral galaxies by contrast almost always have nearly-constant rotation curves at radii outside of their dense centers.

Surface luminosity density

The evidence for dark matter comes when we ask what one would *expect* the rotation curve to be if the gravitating mass were composed of only the observed stars and gas. Now think of the galaxy facing us like a dinner plate held at arm's length, with stars rotating in circular paths a distance R from the center of the disk. Optical measurements of spiral galaxies show that the **surface luminosity density**, $\Sigma(R)$, varies exponentially from the center to the edge to a very good approximation:

$$\Sigma(R) = \Sigma_0 \exp(-R/h) \ . \qquad (46)$$

The surface luminosity density is defined as the total luminosity emitted along a column perpendicular to the galactic disk, taken to be the direction toward us. In this equation Greek sigma Σ in the function $\Sigma(R)$ simply means "surface" and is not a summation sign. The constant Σ_0 is surface luminosity

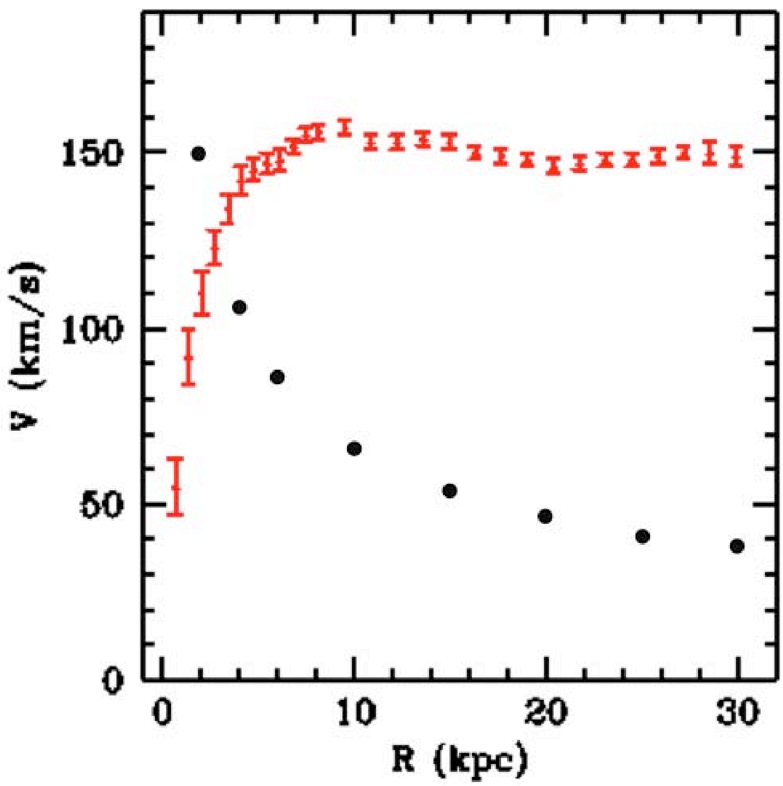

FIGURE 5 Upper plot: Rotation curve for spiral galaxy NGC 3198, from Begeman 1989, *Astronomy and Astrophysics*, 223, 47. Filled dots: Points showing the shape of a rotation curve if the attractive mass were concentrated at the center, for example in our solar system. The vertical position of the filled-dot curve depends on the value of the central mass, but the shape of the curve does not.

density at the center of the galaxy. We assume that the galaxy is sparse enough so that light from the stars across the thickness of the disk simply adds in the direction toward us. Surface luminosity density has units of luminosity (typically watts or solar luminosities, $L_{\rm Sun}$) per unit area (typically square meters or square parsecs).

The form of equation (46) has two constants: Σ_0, the central surface luminosity density, and h, the disk's scale length. For NGC 3198 the approximate values for these parameters are

$$\Sigma_0 = 100\ L_{\rm Sun}/\text{parsec}^2\ , \quad h = 2.7\ \text{kiloparsec}\ . \tag{47}$$

One solar luminosity ($L_{\rm Sun}$) is the amount of power emitted by the sun in optical light. To get the luminosity dL emitted between radii R and $R + dR$ of the galactic disk, multiply by the area of the annulus: $dL = \Sigma(R)\,2\pi R\,dR$. The total light emitted out to radius R follows immediately by integration.

Mass *vs* lumnosity

To predict the rotation curve arising from luminous matter we need to know how much *mass* there is, not how much *light* the stars emit. If the luminous matter in galaxies is mainly stars like the sun, then the light in solar luminosities, $L_{\rm Sun}$, equals approximately the mass in solar masses, $M_{\rm Sun}$. In other words, if the total light emitted from the center out to radius R is $L(R)$, then the total luminous mass (stars and gas) is $M(R) = \Upsilon L(R)$ where capital Greek upsilon Υ is a factor called the **mass-to-light ratio** and whose units are solar mass per solar luminosity, that is $M_{\rm Sun}/L_{\rm Sun}$. If all stars in the galaxy were identical to our sun, then Υ would have the value unity. However, not all stars have the same mass-to-light ratio. A reasonable range for spiral galaxies is $0.5 < \Upsilon < 5$.

In Query 11 you apply these ingredients to show that NGC 3198 contains substantial amounts of dark matter. Make the following assumptions:

1. To describe motion of stars, assume mass density of the galaxy is spherically symmetric, but a function of radius R. (The tangential speed of stars in the disk has approximately the same value regardless of whether the mass is distributed in a thin disk or in a more spherical halo.)
2. Motion of stars in a galaxy can be described using Newtonian mechanics, including Newton's result that total mass inside a spherically symmetric distribution leads to a gravitational force equivalent to the force due to that total mass concentrated at the center of the sphere.
3. Stars in the galaxy move in circular orbits at a speed V that is a function of radius R.
4. The surface mass density follows the same function as the surface luminosity density, implying that the mass enclosed in a sphere of radius R is

$$M(R) = \Upsilon \int_0^R \Sigma_0 e^{-r/h}\, 2\pi r\, dr \ . \qquad (48)$$

In Query 11 you show that assumption 4 is incorrect.

QUERY 11. Dark Matter from a Rotation Curve

Combine Figure 5 with the surface luminosity density of equation (46), to show that the galaxy contains far more mass than can be accounted for by the stars.

A. Set up the Newtonian equation of motion and use it to find an expression for the circular speed V as a function of radius R, in terms of the enclosed mass $M(R)$
B. Carry out the integration in equation (48) and use it to obtain a prediction for $V(R)$. Qualitatively describe the predicted $V(R)$. Does it have a maximum value? Does it approach a nonzero constant as $R \to \infty$? If not, how does it behave for $R \gg h$? Also, how does it behave for $R \ll h$?

C. The observed rotation curve will exceed the predicted one if there is dark matter present, which is not accounted for by equation (48). Use Figure 5 and assume that the luminous matter predominates for $R < 5$ kpc, what is the maximum mass-to-light ratio Υ for the luminous matter in NGC 3198?

D. From the results of the previous parts together with Figure 5, determine the ratio of total mass to luminous mass contained within 30 kpc from the center of NGC 3198.

Using galaxy rotation curve measurements, and a related technique in clusters of galaxies that uses X-ray emitting gas, for about 20 years astronomers have estimated $0.1 < \Omega_{\mathrm{mat},0} < 0.4$. The uncertainty was relatively large (a factor of 4!) because measurements based on luminous "tracers" (such as gas in galaxies or clusters) are insensitive to dark matter far away from galaxies. However, over the last decade increasingly sophisticated measurements of dark matter in and around galaxies have led to a consensus range $0.2 < \Omega_{\mathrm{mat},0} < 0.35$.

Cosmic Microwave Background Radiation

The Universe is filled with a nearly uniform glow of microwaves called the cosmic microwave background (CMB) radiation. This radiation has a **blackbody** spectrum, whose intensity as a function of frequency f is given by the Planck law, discovered in 1900 by Max Planck:

$$I(f) = \frac{2hf^3}{c^3}\,\frac{1}{e^{hf/k_{\mathrm{B}}T} - 1}\ . \qquad (49)$$

Radiation that has this spectrum (this dependence on frequency) is produced by an opaque medium with temperature T. The microwave background radiation fits the Planck law stunningly well—the *CO*smic *B*ackground *E*xplorer (COBE) satellite measured the spectrum to match the Planck Law to about 1 part in 10^4 in the early 1990s. Figure 6 shows the measured spectrum; the estimate of the best-fit temperature has increased by 0.001 K to $T_0 = 2.726$ K since this figure was made in 1998.

Why blackbody spectrum?

At first glance, the microwave background radiation is absurd—the Universe is not opaque, and the matter that emitted the radiation was much hotter than 3 degrees above absolute zero. However, the microwave background radiation is a messenger from the early Universe, and it has aged and become stretched out along the trip. Remarkably, the form of the Planck law—the shape of the function (49) for different temperatures—is preserved by the cosmic redshift. As the Universe expands, the frequency of every light wave and the temperature of the radiation decrease in proportion to $1/a(t)$. In other words, at redshift z the radiation temperature was higher. Using equations (11) and (31), we find:

$$T(z) = (1 + z)T_0\ . \qquad (50)$$

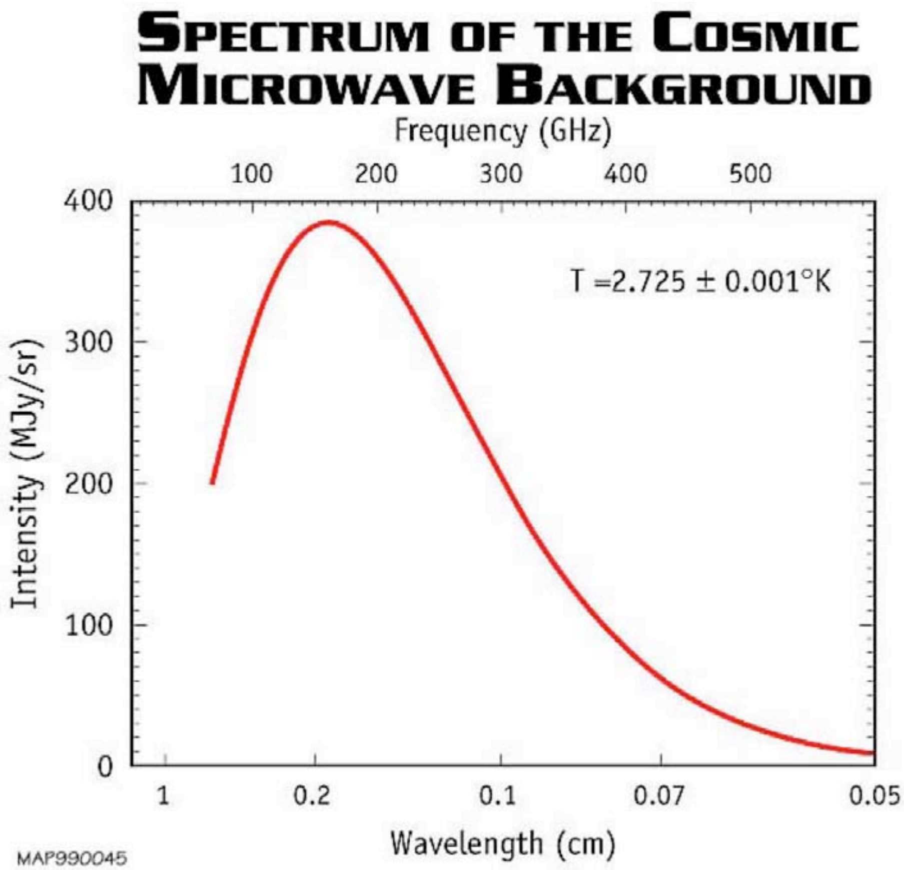

FIGURE 6 Spectrum of the cosmic microwave background radiation measured in the 1990s by the FIRAS instrument aboard the COBE satellite. (From the WMAP website.)

This is an example of how cosmologists use redshift as a proxy for time since the big bang.

Redshift at recombination.

Most of the gas filling the Universe is hydrogen. Neutral atomic hydrogen gas is transparent to microwaves, to infrared light, and to optical light—only when the photon energy becomes large enough to ionize hydrogen does the gas become opaque. For the conditions prevailing in the Universe, hydrogen gas ionizes at a temperature comparable to that of the surface layer of cool stars, $T \approx 3000$ K. Conclusion: the microwave background radiation was produced at a redshift $z \approx 3000/2.726 = 1100$. We call the time when this occurred the *recombination time*

Earliest view of the Universe.

The age of the Universe at the time when hydrogen became transparent, $t_{\rm CMB}$, follows from $a(t_{\rm CMB}) \approx 2.726/3000$. A rather complicated argument leads to the value $t_{\rm CMB} \approx 380\,000$ years. The CMB radiation gives us a picture

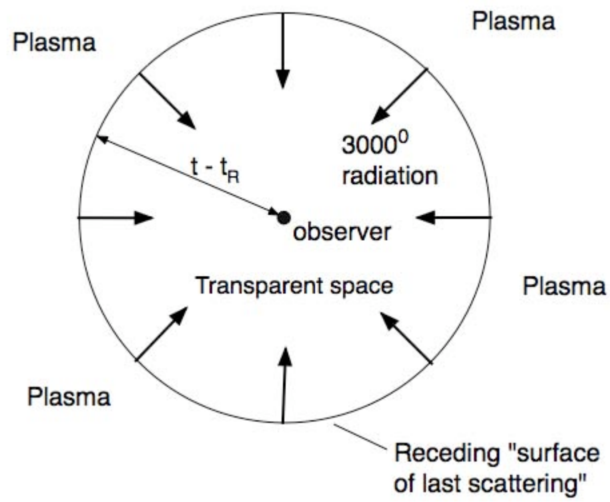

FIGURE 7 Observer's view of a *static* model Universe after recombination time $t_{\rm R}$, at which entire Universe becomes transparent to radiation. However, looking outward, observer cannot receive signal from entire Universe, but will see just-released radiation from receding "surface of last scattering" a distance $t - t_{\rm R}$ away. In a static Universe, this radiation would be at the recombination temperature of $\approx 3000^0$ Kelvin, about that of the surface of our Sun. However, in our expanding Universe, this radiation has been down-shifted to a temperature of 2.726^0 Kelvin, forming the cosmic microwave background radiation.

of the Universe nearly 13.7 billion years ago. Currently this is our earliest view of the Universe; only neutrinos and gravitational waves could have penetrated the primordial plasma, bringing us information from farther back toward the time of the big bang.

?

This is hard to visualize. From where is the cosmic microwave background originating? From the direction of the center of the Universe? What direction is that?

!

There is no unique center of the Universe; every observer has the impression of being at the center, as explained in Chpater 18. Looking outward in every direction, we see radiation from the receding **surface of last scattering** that has been down-shifted to a temperature of 2.783^0 Kelvin, as illustrated in Figure 7.

COBE satellite

What do we see when we look at microwave radiation from the early Universe? The spectrum tells only part of the story. To see the rest, we can look at images of the sky in microwaves. The first sensitive all-sky maps of the microwave background radiation were made in the early 1990s by the COBE satellite. In 2001 a new microwave telescope called the *W*ilkinson *M*icrowave

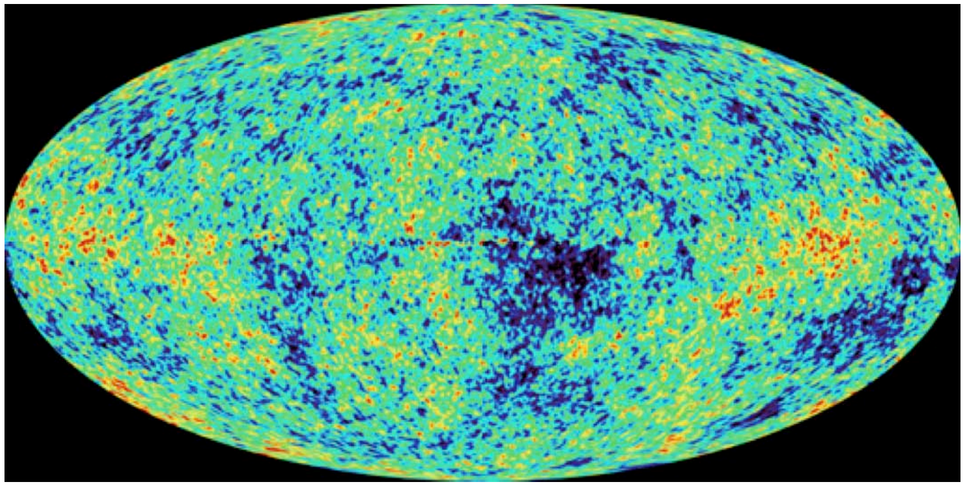

FIGURE 8 An all-sky map of the cosmic microwave background radiation at high contrast made by the WMAP satellite, with radiation from the nearby milky way stars removed. The oval is a projection of the entire sky onto the page. The colors in the original are "false colors" that indicates the temperature of the radiation ranging from $T_0 - 2 \times 10^{-4}$ K (black) to $T_0 + 2 \times 10^{-4}$ K (red) where T_0 is the average temperature. The early Universe had slight temperature variations. (Image courtesy of the WMAP Science Team, from the WMAP website.)

*A*nisotropy *P*robe (WMAP) was launched into orbit. It has greatly refined our picture of the early Universe.

WMAP satellite

Figure 7 shows an image of the microwave brightness around the sky made by WMAP. The Planck law is an excellent fit to the spectrum in a fixed direction of the sky; however, the temperature is not the same in all directions. The temperature varies by a few parts in 10^5 from place to place in the early Universe. These fluctuations are, we believe, the seeds from which galaxies, stars, and all cosmic structure formed during the past 13 billion years.

Map of fluctuations: fingerprint of early Universe

In this chapter we focus on the average properties of the Universe rather than the fluctuations. However, the map of fluctuations is also a treasure trove of information about the cosmic parameters. This is because the pattern of fluctuations in the sky provides a kind of fingerprint of the early Universe. For example, Figure 7 shows that the fluctuations have a characteristic angular size of about 1 degree.

Fluctuations due to sound waves.

The 1 degree scale has a direct physical significance and can be used to measure the curvature of the Universe. The fluctuations in temperature are due to sound waves in the hot gas of the early Universe: the Universe was filled with a super low frequency static created in the aftermath of the big bang. Sound waves compressed and rarefied the gas, changing its temperature. Sound waves oscillated in time but they also oscillated in space at a given moment of time. The temperature fluctuations we see in the microwave background give a snapshot of the spatial variation of these sound waves 380 000 years after the big bang!

Sound waves: "standard ruler."

The 1 degree scale corresponds to the distance that a sound wave could travel from the time the sound waves were created in the big bang until 380 000 years later, when they were revealed to us as fluctuations in the cosmic microwave background radiation. This distance of travel constitutes a *standard ruler*. If we know the size of this standard ruler in meters and the distance the released radiation has since travelled to reach our telescopes — *and* we know the spatial geometry (open, closed, or flat)—then we can predict the angular size of the fluctuations. In practice, we measure the angular size and other quantities enabling us to accurately determine the standard ruler size and the distance travelled. This method is called "baryon acoustic oscillations" (BAO). See Figure 8. The details are beyond the level of this book, but the result is not: The angular size measurement implies that the cosmic spatial curvature K is very small and consistent with zero. The spatial geometry of the Universe appears to be the simplest one possible: flat space. On the other hand, dark matter and dark energy curve space*time* in such a way that the cosmic expansion accelerates. What a strange Universe we live in!

9 ■ EXPANSION HISTORY FROM STANDARD CANDLES

Finding t from red shift z

Astronomers do not directly measure $a(t)$. As discussed in Chapter 18, they measure redshift z and luminosity distance $d_{\mathrm{L}}(z)$. The observable redshift is used as a proxy for the unobservable cosmic time t via equation (31). The goal here is to determine time t from redshift z. From equations (31) and (32)

$$\frac{dz}{dt} = -(1+z)H(z) \tag{51}$$

where H is the Hubble expansion rate at time t related to redshift z by equation (31). In an expanding Universe, $(1+z)H > 0$, so redshift increases looking backwards in time. If astronomers could measure $H(z)$ directly, we could integrate it to get $t(z)$:

$$t_0 - t(z) = \int_0^z \frac{dz}{(1+z)H(z)} \tag{52}$$

Unfortunately, $H(z)$ is very difficult to measure directly. The luminosity distance d_{L} is much easier, especially since the refinement of Type Ia supernovae as standard candles (Chapter 18). The relation between d_{L} and z can be found starting from results of Chapter 18. Along a light ray $d\tau = 0$ coming from a distant supernova to our telescope, equation (17) of Chapter 18 gives

$$dt = -R(t)d\chi \tag{53}$$

which implies

$$R(t_0)\chi = R(t_0)\int_t^{t_0} \frac{dt'}{R(t')} \tag{54}$$
$$= \int_t^{t_0} \frac{dt'}{a(t')}$$

Equation (53) of Chapter 18 gives

$$\frac{d_{\mathrm{L}}(z)}{1+z} = R(t_0)S(\chi) \tag{55}$$

where $S(\chi)$ is given by equations (18)-(20) of Chapter 18. Therefore, in a flat Universe ($K = 0$, implying $S = \chi$),

$$\frac{d_{\mathrm{L}}(z)}{1+z} = \int_t^{t_0} \frac{dt'}{a(t')} \qquad \text{(flat Universe)} \tag{56}$$

Thus, if $d_{\mathrm{L}}(z)$ is measured at many different redshifts, one can determine $t(z)$ by differentiating (56) and re-integrating it again. Differentiating:

$$\frac{d}{dz}\left[\frac{d_{\mathrm{L}}(z)}{1+z}\right] = -\frac{dt}{a(t)} = -(1+z)dt \qquad \text{(flat Universe)} \tag{57}$$

then reintegrating:

$$t_0 - t(z) = \int_0^z \left\{\frac{d}{dz}\left[\frac{d_{\mathrm{L}}(z)}{1+z}\right]\right\} \frac{dz}{1+z} \qquad \text{(flat Universe)} \tag{58}$$

Alternative Universe models

which must be integrated numerically. More complicated formulae are required if $K \neq 0$, but the idea is similar. In practice, measurements are too imprecise to determine $d_{\mathrm{L}}(z)$ with enough accuracy so that equation (58) can be used directly. Instead, astronomers construct different model Friedmann-Robertson-Walker universes by adopting choices for parameters $\Omega_{\mathrm{mat},\,0}$ and $\Omega_{\Lambda,\,0}$. They integrate equation (26) to get $a(t)$, then substitute into (56) (or its generalization for a non-flat Universe) to predict $d_{\mathrm{L}}(z)$.

10 ■ THE UNIVERSE NOW: THE OMEGA DIAGRAM

Squeeze the Universe model from all sides.

Supernovae and the microwave background radiation constrain the values of $\Omega_{\mathrm{mat},0}$ and $\Omega_{\Lambda,0}$. We have already seen that radiation contributes very little to the critical density today. The major contributors are thus matter (dark matter plus baryons) and dark energy, which we model as a cosmological constant.

During the last five years, our knowledge of the density parameter values has gone from shadowy outline to measurements of 10% accuracy. Figure 8 illustrates our current knowledge about the key parameters based on

Squeezing the parameters

observations of Type Ia supernovae (SNe), the cosmic microwave background radiation (CMB), and the Baryon Acoustic Oscillations (BAO). The microwave background data clearly show that the Universe is close to flat, if not exactly so. They also imply a nonzero dark energy contribution, especially when combined with the baryon acoustic oscillations. The latter measurement is most sensitive to $\Omega_{\text{mat},0}$ and indicates that there is too little matter to close the Universe. Microwave background and BAO data independently support the radical claim made by the supernova observers in 1998 that the Universe is accelerating. We found out earlier that the expansion accelerates if $\Omega_{\Lambda,0} > \frac{1}{2}\Omega_{\text{mat},0}$.

Figure 8 does not include all of the constraints on the Omegas. When they are applied, the result is equations (16) and (17). Future satellite missions should shrink the uncertainties in the Omegas to less than 0.01. Once they do, we may still be left with two outstanding mysteries: What are dark matter and dark energy?

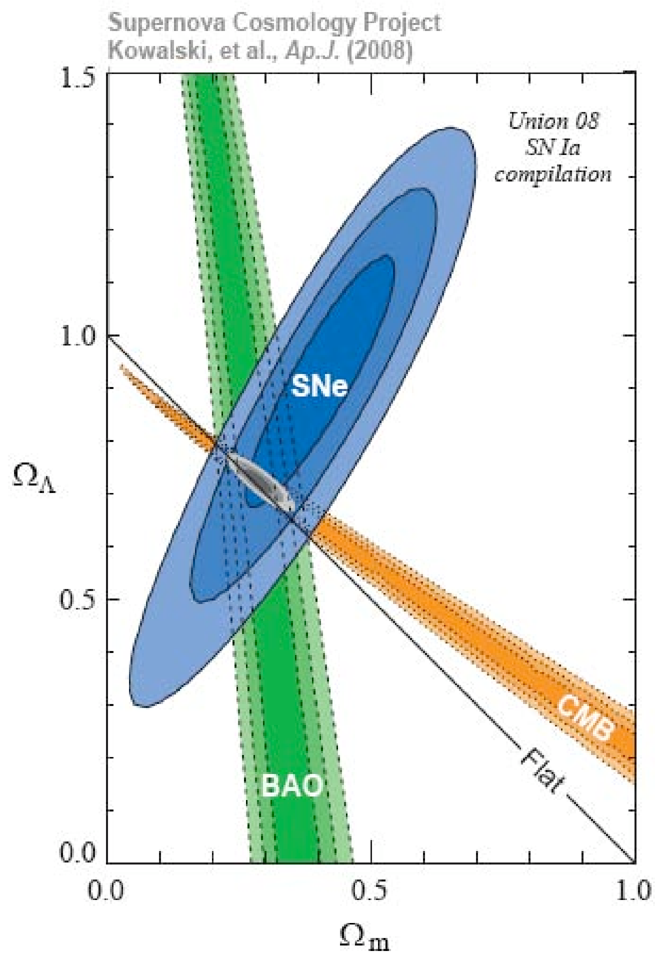

FIGURE 9 The Omega Diagram. Parameters $\Omega_{\rm M}$ and Ω_Λ are called $\Omega_{\rm mat,\,0}$ and $\Omega_{\Lambda,\,0}$ in this chapter. Relative amounts of matter and vacuum energy in the universe at present corresponds to the relatively tiny region of intersection of three sets of measurements: Type Ia supernovae (SNe), the cosmic microwave background radiation (CMB), and "baryon acoustic oscillations" (BAO). Darkest regions represent 99.78% statistically confidence level and the lighter two represent 95% and 68% confidence levels, respectively. The straight line represents conditions for a flat Universe.

QUERY 12. No Big Bang?

Are all points on the Omega diagram allowable? Some can be excluded because they never have a hot dense phase. In other words some regions correspond to "No big bang."

A. Consider a FRW Universe with $\Omega_{\mathrm{mat},0} = 1$ and $\Omega_{\Lambda,0} = 3$. Neglect radiation. What are $V_{\mathrm{eff}}(a)/H_0^2$ and $-K/H_0^2$ for this case?
B. Sketch $V_{\mathrm{eff}}(a)/H_0^2$ similar to Figure 2 for the parameters of part A. Show that the Universe has a turning point in the past, so that it was impossible to start from $a = 0$ (the big bang) and get to $a = 1$ (today) in this model.
C. Consider models with $\Omega_{\mathrm{mat},0} = 0$and only dark energy with $\Omega_{\Lambda,0} > 0$. Show that these models also have a turning point at $a > 0$.
D. Show that a given model *cannot* have a big bang if there exists a solution $a = a_{\mathrm{min}}$ of the equation:

$$V_{\mathrm{eff}}(a) + K = 0 \qquad \text{where } 0 < a_{\mathrm{min}} < 1 \tag{59}$$

E. Show that the Universe will recollapse if there exists a solution $a = a_{\mathrm{max}}$ of (59) with $a_{\mathrm{max}} > 1$.

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11 ■ FIRE OR ICE?

You choose the end of the Universe.

Will the Universe end in fire or ice? You choose the answer to this question:

Fire or ice?
You predict.

ANSWER: FIRE if the temperature $T \to \infty$ for large times. This requires $a(t) \to 0$ for large times in equation (11). This happened, in effect, at the big bang. It will happen again if the expansion reverses, leading to a big crunch, that is $a \to 0$ in the future (Part E of Query 10).

ANSWER: ICE if the temperature $T \to 0$ for large times, or $a \to \infty$ as $t \to \infty$. What does Figure 2 imply for this case?

With the background of this chapter, you are an informed cosmologist.

DECIDE: Will the Universe end in fire or ice?

12 ■ REFERENCE

The publisher's (future) website for this book will contain current web references to topics in this chapter.