CHAPTER 19

Deriving the Metric
A Project

Edmund Bertschinger & Edwin F. Taylor *

Perhaps I asked too large -
I take - no less than skies -
For Earths, grow thick as
Berries, in my native Town -

My basket holds - just - Firmaments -
Those - dangle easy - on my arm,
But smaller bundles - Cram.

—Emily Dickinson

I want to know how God created this world.
I am not interested in this or that phenomenon,
in the spectrum of this or that element.
I want to know his thoughts.
The rest are details.

—Albert Einstein

THIS IS A ROUGH VERSION OF THE FINAL CHAPTER. EFT COMMENTS IN CAPS.

19.1 INTRODUCTION

Paring down the field equations.

In their full glory, the Einstein field equations—which determine the metrics for any and all spacetimes—are a set of ten coupled nonlinear partial

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differential equations. They're so complicated that only a few dozen exact solutions are known, including those for black holes and cosmology used in this book. Fortunately, several of these solutions are closely related and relatively easy to derive. Here we explore the Einstein field equations for static spherical spacetimes (and shortly will define what this means). With these restrictions, the Einstein field equations become tractable, allowing us to derive the Schwarzschild metric and several other exact solutions.

Simplifying the field equations

The equatorial plane metric of a static spherical spacetime can be written

$$\begin{aligned} d\tau^2 &= A(r)dt^2 - B(r)dr^2 - r^2 d\phi^2 \\ &= e^{2\Phi(r)}dt^2 - \frac{dr^2}{1 - \dfrac{2M(r)}{r}} - r^2 d\phi^2 \end{aligned} \qquad (1)$$

Functions $A(r)$ and $B(r)$ in the first line of (1) represent general functions of map radius r, to be determined from the field equations. The more specialized functions $\Phi(r)$ and $M(r)$ in the second line make later derivations more convenient; their form is justified by compatibility with the field equations. The function $\Phi(r)$ means its value *at* r, while $M(r)$ means the mass *inside* r.

Meaning of static, spherical spacetime

Metric (1) is *static* because the coefficients are independent of t; it is *spherical* (or, more appropriately, circular in our usual equatorial plane) because the angular variable ϕ appears only in the differential $d\phi$, not as a coordinate itself in the metric. The increment $rd\phi$ is arc length along a circle of map radius r—our original definition of *reduced circumference* in Section 3 of Chapter 3, Curving.

Throughout this book, including in this final chapter, t and r are map coordinates. As always, we carry out every measurement and experimental test of the theory in a local inertial frame, for example a shell frame at the average radius $\bar{r}$ with local coordinates:

Every test in local coordinates

$$\Delta t_{\text{shell}} = e^{\Phi(\bar{r})}\Delta t \ , \quad \Delta y_{\text{shell}} = \frac{\Delta r}{\left[1 - \dfrac{2M(\bar{r})}{\bar{r}}\right]^{1/2}} \ , \quad \Delta x_{\text{shell}} = \bar{r}\Delta\phi \qquad (2)$$

Later we use shell coordinates to interpret various features of the metric.

2 ■ SOURCES OF CURVATURE

More than mass curves spacetime.

In general, spacetime curved by mass/energy density, plus . . .

Metric (1) is a generalization of the Schwarzschild metric. It remains indefinite until we specify the functions $\Phi(r)$ and $M(r)$. What should they depend on? Matter, energy and momentum cause spacetime to curve. The way we have written $M(r)$ suggests that the metric depends on mass and its spherically symmetric distribution in space, and therefore on mass density. This is true but is not the whole story. The relativistic equivalence of mass and energy

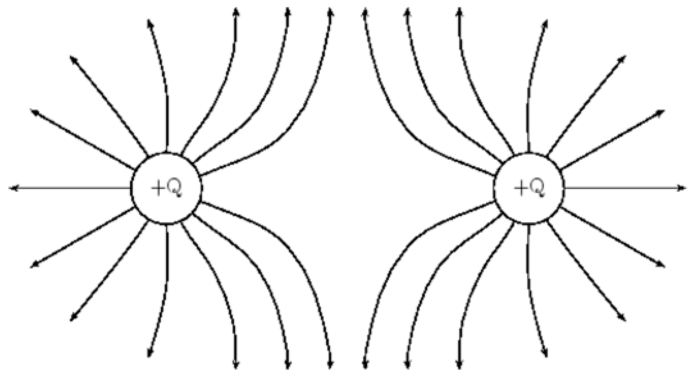

FIGURE 1 Electric field lines of two equal charges of the same sign, which leads to positive pressure of compressed electric field lines. MORE FIELD LINES IN BLANK REGION BETWEEM CHARGES TO SHOW LATERAL COMPRESSION OF LINES.

. . . momentum density, plus flows of energy and momentum.

implies that the metric must depend at least on the total mass/energy density $\rho(r)$. Einstein argued further—using the relativity principle—that the laws of physics must have the same form in every reference frame, that gravity must depend not only on energy density, but on momentum density and on the flows of energy and momentum. For a general situation these are described by ten quantities called stress-energy-momentum; fortunately life is simpler for static spherical spacetimes.

Ideal gas model: Static spherical spacetimes curved by energy density $\rho(r)$ plus pressure, $P(r)$.

One obvious source of spacetime curvature is the matter/energy density $\rho(r)$; for a static spherical spacetime the density must not depend on time t or angle ϕ. Is this all? No. Although the net momentum must vanish for a static spacetime, internal stresses (such as pressure) in the matter and energy sources have the capacity to generate motion, and these stresses play a role in generating curvature. One important special case is an ideal gas with energy density ρ and pressure P. Pressure differences generate momentum flows—a high-pressure gas will expand when released into a low-pressure medium. In general relativity, unlike Newtonian gravity, both $\rho(r)$ and $P(r)$ generate curvature. We will see later how this works when we write down the Einstein field equations for static spherical spacetimes.

Fields: Like charges compress field lines *perpendicular* to their direction.

An ideal gas is a theoretical model composed of randomly-moving point particles that interact only by collision. Most gases we meet every day behave as ideal gases. The ideal gas model works well for stars like the Sun, but fails to describe the energy and momentum carried by fields such as the electric field of a point charge. Electric fields have a more complex form of pressure. When the field is compressed in a direction perpendicular to the field lines—as happens, for example, when two like charges are brought together—the pressure is positive (Figure 1). The repulsive force between charges can be understood as the positive pressure of compressed electric field lines.

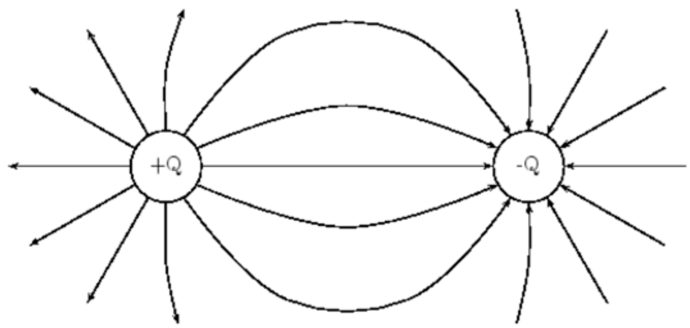

FIGURE 2 Electric field lines of two equal charges of opposite sign. The attractive force along a field line is *tension.* Positive tension is equivalent to negative pressure.

Opposite charges stretch field lines *along* their direction: tension.

When charges have opposite sign, the electric field lines join together (Figure 2) and behave like elastic bands under tension. Unlike charges attract rather than repel each other, signaling a change in sign of the pressure along the field lines. The attractive force along a line is called **tension**, given the symbol τ. Tension pulls rather than pushes, so negative tension is equivalent to positive pressure.

DOUBLE USE OF THE SYMBOL τ

We are sorry that in this chapter the symbol τ stands for both wristwatch time and tension. In the profession, tension is usually denoted by T, but that can be confused with temperature or global rain time. Most often in this chapter proper time appears as a differential $d\tau$, whereas tension is τ, not in a differential. An exception is equation (5). We will try to label the symbol whenever needed for clarity.

Positive tension = negative pressure.

Combining these ideas, the radial electric field of a point charge (Figure 3) has positive pressure P in the tangential directions (perpendicular to the field lines) and positive tension τ (with negative pressure $-\tau$) along the field lines. As we have already seen in Chapter 19, Cosmology and will show further below, negative pressure leads to surprising new effects of spacetime curvature.

In summary, for a static spherical spacetime the sources of curvature are energy density $\rho(r)$, tangential pressure $P(r)$, and radial tension $\tau(r)$. For an ideal gas, $\tau = -P$ everywhere. For fields, τ can differ from P and can be positive, which is equivalent to negative pressure.

3 ■ EINSTEIN FIELD EQUATIONS

The Einstein field equations are a mathematical fulfillment of the principle that mass-energy-momentum tells spacetime how to curve. For a static

FIeld equations for static, spherical spacetimes

spherical spacetime, three functions $\rho(r)$, $P(r)$ and $\tau(r)$ describe mass-energy-momentum while two functions $\Phi(r)$ and $M(r)$ govern spacetime geometry. The Einstein field equations for these quantities are the following three r-derivatives, the first one the radial derivative of mass:

$$\frac{dM}{dr} = 4\pi r^2 \rho \tag{3}$$

the second one the radial derivative of Φ,

$$\frac{d\Phi}{dr} = \frac{M - 4\pi r^3 \tau}{r(r-2M)} \tag{4}$$

and the third one the radial derivative of tension τ:

$$\frac{d\tau}{dr} = (\rho - \tau)\frac{d\Phi}{dr} - \frac{2}{r}(P + \tau) \tag{5}$$

First equation: Gravitating mass

Each of these equations has a simple interpretation. The first Einstein equation (3)—identical in general relativity and in the Newtonian limit—is called the mass continuity equation. Obtain the gravitating mass within radius r by integrating energy density over volume:

$$M(r) = M_0 + \int_0^r \rho(r)\, 4\pi r^2 dr\ . \tag{6}$$

where M_0 is a point mass at $r = 0$. For vacuum, $\rho = 0$ everywhere but at the center, and $M(r) = M_0$ is a constant. This is a spherical black hole: all the mass is concentrated at $r = 0$.

The Newtonian and general relativistic expressions for mass (6) are identical, but the interpretation is somewhat different because of the radial stretching of space (Chapter 3, Curving). The Newtonian volume of a shell is $4\pi r^2 \Delta r$ while the general relativistic volume is, from (2):

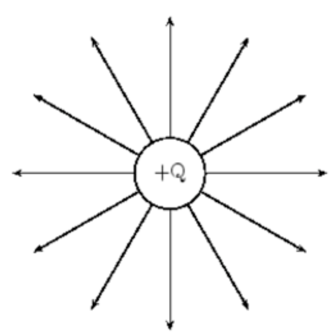

FIGURE 3 Electric field lines of one charge. Pressure is positive in directions perpendicular to the field lines and negative along the field lines.

$$4\pi\bar{r}^2\Delta y_{\text{shell}} \approx \left(1-\frac{2M(\bar{r})}{\bar{r}}\right)^{-1/2} 4\pi\bar{r}^2\Delta r \tag{7}$$

Therefore

$$M(r) = M_0 + \sum \Delta E \quad \text{where} \quad \Delta E \approx \left(1-\frac{2M(\bar{r})}{\bar{r}}\right)^{1/2} \Delta E_{\text{shell}}\ , \tag{8}$$

since the measured energy of the shell is

$$\Delta E_{\text{shell}} \approx \rho(\bar{r})4\pi\bar{r}^2\Delta y_{\text{shell}}\ . \tag{9}$$

The second of equations (8) is the same as equation (22) of Chapter 6, Plunging (for a point mass of a black hole and without the increments). Thus ΔE in (8) is the usual map energy of the stone. As we saw in Section 7 of Chapter 6, when mass is added to a black hole, the black hole mass increases not by the measured energy but instead by the map energy E. Here we see the same is true for a center of attraction: gravitational mass equals the map energy interior to a given shell.

Second equation: Gravitational acceleration

The second Einstein equation (4) gives the gravitational acceleration resulting from gravitational mass. In the Newtonian limit this is

$$g = -\frac{d\Phi}{dr} = -\frac{M}{r^2} \qquad \text{(Newton)} \tag{10}$$

while in curved spacetime, the acceleration of a stone dropped from rest by a shell observer is

$$\begin{aligned} g_{\text{shell}} &= -\lim_{\Delta y\to 0}\frac{\Delta\Phi}{\Delta y_{\text{shell}}} = -\left(1-\frac{2M(r)}{r}\right)^{1/2}\frac{d\Phi}{dr} \\ &= -\left(1-\frac{2M(r)}{r}\right)^{-1/2}\frac{[M(r)-4\pi r^3\tau]}{r^2}\ . \end{aligned} \tag{11}$$

When $\tau = 0$ and $M = M_0$, a point mass, this reproduces the initial local acceleration on the shell of a nonspinning black hole, equations (57) and (58) of Chapter 6, Plunging. When $\tau > 0$ (positive tension or negative pressure in the radial direction), the gravitational acceleration inward is decreased; for sufficiently large tension (sufficiently negative pressure) the acceleration can change sign and become outward-directed. We saw this previously in cosmology (Section 7 of Chapter 19), where negative pressure in the form of a cosmological constant can lead to an outward-directed gravitational acceleration. For constant density and pressure, one can easily show that g_{shell} is proportional to $\rho - 3\tau$. Interestingly, it is only the radial pressure $-\tau$ that enters here, not the tangential pressure P. Thus we guess that the gravitational field of an electric charge (with tension along its radial field lines) might be repulsive overall. This guess will be borne out below.

Given $\rho(r)$, we can find $M(r)$; that together with $\tau(r)$ allows us to integrate equation (4):

$$\Phi(r) = \int_0^r \frac{M(r) - 4\pi r^3 \tau(r)}{r[r - 2M(r)]}\, dr \ . \tag{12}$$

The first two Einstein equations suffice to determine the metric. The third one (5) is redundant with one of the equations of motion for the fluid. For an ideal fluid with $\tau = -P$,

$$\frac{1}{(\rho + P)} \frac{dP}{dr} = -\frac{d\Phi}{dr} \tag{13}$$

Third equation: Hydrostatic balance equation for ideal gas.

is called the hydrostatic balance equation. It says that the gravitational force on a fluid element at rest is balanced by the mechanical force due to pressure differences across the fluid element. In simple terms, a star will collapse under its own weight unless the pressure is high enough in the center to push outward and halt the collapse. The Newtonian version of (13) collapses $(\rho + P)$ to ρ. In Einstein's formulation the extra P is a contribution from special relativity.

For a non-ideal gas, or fields with radial tension, the hydrostatic balance equation takes a different form. It is interesting to consider the electric field of a point charge (make this into a query), for which τ is proportional to the square of the electric field, and therefore $\tau \propto r^{-4}$ (inverse square law of fields).

EB SAYS THAT FOR A POINT CHARGE (QUOTE) both τ and P are positive. There is a simple relation between $\rho(r)$, $P(r)$ and $\tau(r)$ for a point charge, but it cannot be obtained from the Einstein equations. It follows instead from the Maxwell equations. Certainly the exact expressions for these three quantities are of central importance to charged black holes. (UNQUOTE)

4 ■ SCHWARZSCHILD SOLUTIONS

At the end of 1915, Karl Schwarzschild wrote down equations (3) to (5) for an ideal gas with $\tau = -P$. He realized immediately that outside a star of radius R we have $\tau = P = 0$, so that equations (6) and (12) can be directly integrated.

QUERY 1. Schwarzschild metric for a black hole

A. Show that for the given conditions, $M =$ a constant and

$$\Phi = \int \frac{M\, dr}{r(r - 2M)} = \frac{1}{2} \ln \left(1 - \frac{2M}{r}\right) + \text{constant} \ . \tag{14}$$

B. Show that the integration constant in (14) must be zero so that at a great distance $\Phi \to 0$ and the metric (1) becomes that for flat spacetime. Show that, as a result,

$$e^{2\Phi(r)} = 1 - \frac{2M}{r} \tag{15}$$

C. From these results and equation (1) derive the Schwarzschild metric:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{1 - \dfrac{2M}{r}} - r^2 d\phi^2 \ . \qquad (16)$$

Schwarzschild metric for black hole

Schwarzschild intended that (16) apply outside the star of radius R. However, we can shrink this radius to zero while keeping the mass constant, yielding our Schwarzschild metric for a black hole, which Schwarzschild never envisioned.

Schwarzschild metric inside neutron star

Less well known is that Schwarzschild also gave the solution for the metric inside the star, in the case of a constant-density star. Regular stars support themselves against gravity by the pressure of hot gas, so they are much denser at the center than near the surface. The neutron star is in intermediate case: it has reached a density comparable to that of nuclear matter, which is extremely rigid against further compression. Neverthess, there is some variation of density under the crushing pressure near the center of a neutron star, so instead we examine the idea case of incompressible star.

QUERY 2. Conditions inside an incompressible star

A. Assume that a star is composed of an incompressible material and therefore has a uniform density ρ. Assume also an ideal gas model, so that $\tau = -P$ inside the star. Apply these assumptions and equation (6) to the second and third Einstein equations (4) and (5) to obtain:

$$-\frac{1}{(\rho + P)}\frac{dP}{dr} = \frac{4\pi r^2(\rho + 3P)}{3(r - 8\pi r^3 \rho/3)} \qquad (17)$$

B. Rewrite (17) as

$$\frac{dP}{(\rho + P)(\rho + 3P)} = -\frac{4\pi r\, dr}{3 - 8\pi\rho r^2} \qquad (18)$$

Carry out the differentials in the following equation to show that the result is (18).

$$d\left[\ln\left(\frac{\rho + 3P}{\rho + P}\right)\right] = d\left[\ln\left(1 - \frac{8\pi}{3}\rho r^2\right)^{1/2}\right] \qquad (19)$$

C. The arguments of the log functions on either side of equation (19) are equal, so we can write: #(intpressbal3)

$$\left(\frac{\rho + 3P}{\rho + P}\right)^2 = 1 - \frac{8\pi}{3}\rho r^2 + \text{constant} = 1 + \frac{8\pi}{3}\rho(r^2_{\text{surface}} - r^2) \equiv F^2(r) \qquad (20)$$

Where "constant" is a constant of integration. Verify the expression after the second equality in (20) by evaluating this equation just inside the surface of the star at r_{surface}, where ρ has the same constant value but $P \to 0$.

D. Solve equation (20) for P and show that the result is

$$P = \frac{\rho F(r)}{3 - F(r)} \tag{21}$$

where $F(r)$ is defined in (20).

E. Demand that the pressure be less than infinite at $r = 0$, the center of the black hole. Show that the resulting condition is

$$\rho < \frac{3}{\pi r_{\rm surface}^2} \tag{22}$$

F. Combine (22) and (6), with $M_0 = 0$, to show that:

$$M < 4 r_{\rm surface} \tag{23}$$

Neutron star just before collapse?

What is the maximum radius and mass of a neutron star just before collapse into a black hole? This question is currently at the forefront of nuclear astrophysics. Finding the answer is complicated by our ignorance about densities and phase changes in the interior of the "critical" neutron star; we do not know what is called its **equation of state**, the relation among pressure, density, and temperature for whatever material is present. Here are two of several possible alternatives: (a) The neutron star is nuclear matter of the kind we study on Earth. This nuclear matter is very "stiff," that is difficult to compress, but surely it is compressed some under the vicious pressure near the center of the neutron star. (b) Quarks are squeezed out of the protons and neutrons and create a **quark plasma** that can have several different phases. This would allow densities greater than that of nuclear matter.

QUERY 3. Collapse of a neutron star into a black hole

In this Query make the simplest possible assumption, namely that the about-to-collapse neutron star has uniform density equal to that of common nuclei. The nuclear density of all elements is approximately 10^{18} kilogram/meter^3. Use the constants and conversion factors inside the front cover plus results from (22) to carry out the following steps:

A. Convert the density of nuclear matter to units of meter^{-2}.

B. Find the maximum map radius $r_{\rm surface}$ of the neutron star. [Answer: 13 kilometers]

C. Find the maximum mass $M_{\rm max}$ of the neutron star. Express your result as a multiple of the mass of our Sun. [Answer: 4.4 times mass of the Sun]

D. Suppose that the radius of a neutron star with mass 4.4 times that of our Sun is definitively measured to be less than 13 kilometers. What alternative possibilities could account for this observation? [Answer: The density may be higher than we assumed, or the star is not an ideal gas, or general relativity is wrong.]

CAN WE NOW SET UP THE SCHWARZSCHILD INTERIOR METRIC INSIDE THE NEUTRON STAR? EB SAYS (QUOTE) Yes we can do that, and indeed it is quite interesting to derive the embedding diagram because it has a "cap". See Thorne BH&TW. (UNQUOTE). IN MY ATTEMPT TO FIND $\Phi(r)$ COMBINING (4) WITH (13) UNDER THE ASSUMPTION $\tau = -P$, I GET A DIFFERENTIAL EQUATION IN P.

5 ■ CHARGED BLACK HOLE

Consider a point charge of mass M and charge Q; the fields have $\rho = 0$, $P = \tau \propto Q^2 r^{-4}$ (give exact—SEE END OF SECTION 3). Integrate (12) to get $\Phi(r)$ for the charged black hole and obtain the Reissner-Nordstrom metric

$$d\tau^2 = \left(1 - \frac{2M}{r} + \frac{r_Q^2}{r^2}\right) dt^2 - \frac{dr^2}{1 - \frac{2M}{r} + \frac{r_Q^2}{r^2}} - r^2 d\phi^2 \qquad (24)$$

where

$$r_Q^2 \equiv \frac{Q^2 G}{4\pi\epsilon_0 c^4} \qquad (25)$$

in meters.

QUERY 4. Constant $r_{\rm Q}$ has the unit meter.

A. Show that the numerical constant $r_{\rm Q}$ has the unit meter, using the definition, $1/(4\pi\epsilon_0) \equiv 8.99 \times 10^9$ Newton-meter^2/coulomb^2 and constants inside the front cover.

B. What is the value of $r_{\rm Q}$ in meters for a one-coulomb electric charge?

To be completed; Outline: Analyze (24) and show that there is an event horizon only for $M > r_Q$. Show that there is also an inner Cauchy horizon. Also calculate $g_{\rm shell}$ and show that it changes sign inside the Cauchy horizon so that a freely-falling body is repelled away from the point charge. This repulsion is purely gravitational; it is NOT due to the electric force. Show the effective potential and compare with the Kerr black hole. Conclude that particles crossing the inner horizon are forced back out – not into the original universe but into a new one. Refer to the "bouncing" shells of Graves and Brill described in Kip Thorne's Black Holes and Time Warps. Show (query for students?) that there is an infinite blueshift at the Cauchy horizon and therefore a mass inflation singularity. Similarly to the Kerr solution, we expect that real charged black holes differ from the Reissner-Nordstrom solution inside the Cauchy horizon. Andrew Hamilton and others have written some papers giving possible geometries.

FOR QUERY IDEAS, SEE MISNER, THORNE, AND WHEELER, PAGE 841

6 ■ STATIC COSMOLOGICAL METRICS

For the case of a Universe containing matter and fields with constant map density and tension. The phrase *constant map density* means that density is everywhere uniform *in map coordinates*. Query 5 contrasts map density with directly-measured local inertial density. You show in Query 4 that for uniform map density and tension the general metric (1) takes the form.

$$d\tau^2 = \left(1 - \frac{8\pi}{3}\rho r^2\right)^{(3\tau-\rho)/2\rho} dt^2 - \frac{dr^2}{1-(8\pi/3)\rho r^2} - r^2 d\phi^2 \ . \qquad (26)$$

QUERY 5. Metric for a Universe with constant density and tension

A. Derive $M(r)$ for a static cosmology by integrating (6) with $M_0 = 0$ and map density $\rho =$ constant everywhere. Verify by substituting the resulting function $M(r)$ into the dr^2 term in the general metric (1) and showing that it leads to the corresponding term in (26).

B. Next find $\Phi(r)$ by substituting your result for Item A into (12) and using the indefinite integral

$$\int \frac{zdz}{a^2 - z^2} = -\frac{1}{2}\ln\left|a^2 - z^2\right| \qquad (27)$$

where ln is the natural logarithm. Show that the result is:

$$\Phi(r) = \frac{3\tau - \rho}{4\rho}\ln\left|1 - \frac{8}{2}\rho r^2\right| \qquad (28)$$

C. Verify metric (26) from the results of Items A and B .

D. Show that the expression ρr^2 in (28) is unitless and explain why that equation requires it. Show from (1) that $\Phi(r)$ must be unitless and verify that (28) meets this condition.

QUERY 6. Map density vs. measured density

Item A of Query 4 assumes uniform *map* density, that is uniform mass content in equal incremental volumes computed in map coordinates. But we know that map coordinates are arbitrary, so uniform map density does not tell us the magnitude of the local density measured by an observer in a local inertial frame.

A. From metric (26), set up local inertial shell coordinates.

B. Show that under the assumption of "uniform map density," shell observers at different average radii $\bar{r}$ will directly measure different local mass densities.

C. Demonstrate that for "uniform map density ρ," a shell observer at smaller average radius $\bar{r}$ measures *smaller* value of local density than a shell observer at larger average radius. Is this a sensible physical prediction?

QUERY 7. The deSitter metric

The deSitter Universe is static with no matter and no radiation, just the cosmological constant.

A. Derive the deSitter metric by setting $\tau = \rho = \Lambda/(8\pi)$ in static metric (26). HOW CAN THIS BE PURE COSMOLOGICAL CONSTANT IF RHO IS NOT EQUAL TO ZERO?

$$d\tau^2 = \left(1 - \frac{\Lambda}{3}r^2\right)dt^2 - \frac{dr^2}{1 - \Lambda r^2/3} - r^2 d\phi^2 \tag{29}$$

B. Show that the deSitter metric has an *outer* event horizon. Where is this event horizon? Analyze the redshift and blueshift of light moving radially in the deSitter Universe. CAN WE SAY SOMETHING INTERESTING AND/OR FUNDAMENTAL ABOUT THIS UNIVERSE? Does the deSitter Universe have a unique center at $r = 0$ that an observer can identify?

QUERY 8. Einstein's static Universe

A. Derive the metric for Einstein's static universe by setting $\rho = 3\tau = \Lambda/(8\pi)$ in (26):

$$d\tau^2 = dt^2 - \frac{dr^2}{1 - \Lambda r^2/3} - r^2 d\phi^2 \ . \tag{30}$$

Show that this result matches the Einstein static Universe of Chapter 19, Cosmology.

B. Show that Einstein's static Universe is spatially closed. HOW, BESIDES THE WAY ALREADY USED IN CHAPTER 19??

As a final query, show the coordinate transformation from (t, r) to $T, R)$ that converts it (WHICH METRIC?) into the exponentially expanding solution of inflationary cosmology,

$$d\tau^2 = dT^2 - e^{2HT}(dR^2 - R^2 d\phi^2) \tag{31}$$

where R is comoving radius and the scale factor is $\exp(HT)$ where $H^2 = \Lambda/3$. Compare with the inflationary universe model from Chapter 19, Cosmology. AT PRESENT THERE IS NO INFLATIONARY UNIVERSE MODEL IN CHAPTER 19.

7 ■ WORMHOLES

To be completed: Conclude with the search for wormholes: metrics similar to the Einstein static universe with no horizons but with curved space sections that join two flat spaces. Show that this requires $\tau > \rho$ and explain that no known matter or fields satisfy this condition. See Morris and Thorne (1988) for details. Show pretty pictures of what wormhole geometries would look like if an advanced civilization could create materials with $\tau > \rho$. Summarize some of the puzzles of Morris and Thorne.

Compare with the Kruskal wormhole for a Schwarzschild black hole, which should already have been discussed in the chapter Inside the BH.

8 ■ REFERENCES

Initial quotes:
Emily Dickinson: *The Poems of Emily Dickinson, Variorum Edition*, Edited by R. W. Franklin, 1998, Belknap Press of Harvard University Press, Cambridge, MA, Number 358, about 1862, page 382.
Einstein: *The New Quotable Einstein*, Alice Calaprice, page 194

[The field line figures 1-3 are taken from `http://en.wikibooks.org/wiki/FHSST_Physics/Electrostatics/Electric_Fields`. Redrawing them will avoid copyright infringement.]

For Einstein's equations for static spherical spacetimes (Section 3), see Morris and Thorne, American Journal of Physics, Volume 56, 1988, page 395.