CHAPTER 20

Expanding

Edmund Bertschinger & Edwin F. Taylor [*]

> *I want to know how God created this world. I am not interested in this or that phenomenon, in the spectrum of this or that element. I want to know his thoughts. The rest are details.*
>
> *What really interests me is whether God could have created the world any differently; in other words, whether the requirement of logical simplicity admits a margin of freedom.*
>
> —Albert Einstein

1 ■ DESCRIBING THE UNIVERSE AS A WHOLE

Finding words that correctly describe the unbounded

Where are we?

Is the Universe a giant black hole? How can we tell? Recall John Wheeler's one-sentence summary of general relativity: *Spacetime tells mass how to move; mass tells spacetime how to curve.* In the case of a black hole, the mass/energy that curves spacetime is all concentrated on a singularity.

In fact, our Universe is completely different from a black hole: Hundreds of billions of galaxies, each containing about one hundred billion stars, are scattered more or less uniformly, like grains of sand on a beach (Figure 1). We inhabit one of these grains and gaze on the beach, wondering about its size and shape.

We are in no special place.

Assume the Universe is uniform in space.

How can we describe the Universe as a whole? Start by describing the part we can see, and—in the absence of evidence to the contrary—extrapolate using the hypothesis that the place we live is not special but common and ordinary. We see, on average, a uniform distribution of galaxies going out to distances of billions of light years from Earth. As a first—and it turns out, accurate—approximation, look for metrics describing the curvature caused by a uniform distribution of mass. Make no hypothesis about how far this distribution extends. Instead, examine all possibilities consistent with general relativity, compute their predictions, and let astronomical observations select the viable model or models—according to whether or not observations fulfill the predictions.

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[*] Draft of Second Edition of *Exploring Black Holes: Introduction to General Relativity* Latest drafts at dropsite exploringblackholes.com, with request for comments.

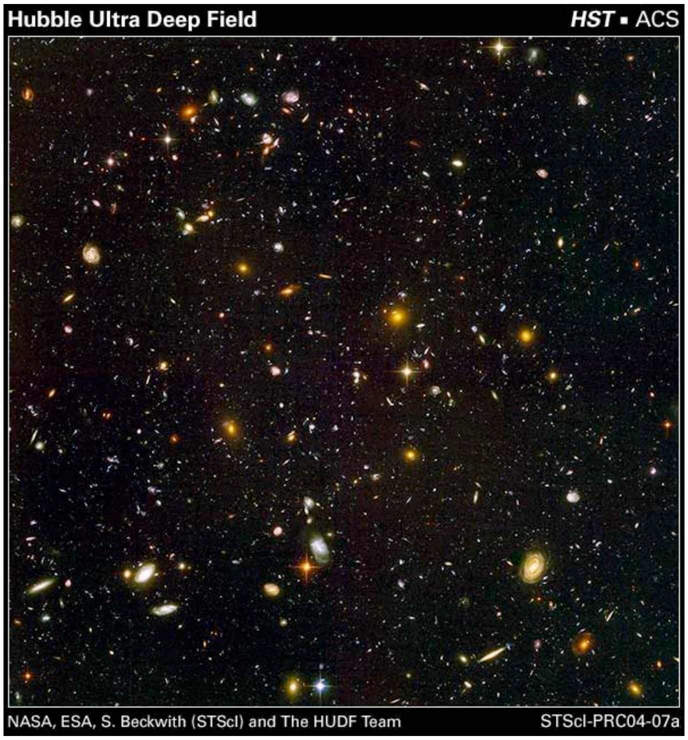


FIGURE 1 “Ultra deep field” image from the Hubble Space Telescope, which was named after astronomer Edwin P. Hubble, who showed that the so-called “spiral nebulae” are distant galaxies moving away from us at high speeds. Every dot and every smear in this image is a galaxy, with the exception of only a few stars in our galaxy. (Can you tell which ones they are?)

Cosmological constant comes, goes . . . then comes back again!

Restrict attention to metrics that are uniform in space? Why not also uniform in time—a Universe that remains unchanged as the eons roll? In the absence of evidence to the contrary this would be the simplest hypothesis. Indeed, in the very first cosmological model inspired by general relativity, Einstein in 1917 looked for metrics describing a static Universe filled smoothly with mass. He found that no static metric was compatible with his newly-invented field equations unless he introduced a new term into those equations, what he called the **cosmological constant** denoted by the Greek capital letter lambda, Λ. Later, after acknowledging Hubble’s discovery that galaxies are flying apart from one another, Einstein regretted the addition of Λ to his field equations. Astonishingly, today we know that there is something very similar, if not identical, to Λ at work in the Universe, as described in Project 10, Cosmology.

Brief history of the Universe

Today we know far more about the Universe than Einstein did almost a century ago. We know that the Universe is not static but evolving. We know that galaxies have not always existed but were formed starting about 13 billion

Box 1. Is this the only Universe?

Are there multiple universes, parallel universes, or baby universes? General relativity theorists write about all these and more. In this book we investigate the simplest model Universe—just one simply connected space—consistent with observations.

Cosmologists often distinguish "the observable universe" from all that there is or might be, citing plausible arguments that space could be very different trillions of light years away. Here we restrict discussion to the simplest generalization of the observable universe, one—*the* Universe—that is everywhere similar to what we see in our vicinity.

years ago. We know that almost 14 billion years ago all that we see was concentrated as a hot gas far denser than a neutron star. We know that this gas expanded and thinned, starting from a moment we call the big bang.

How do we know?

How do we know these things? And how do we describe an evolving, expanding Universe? The present chapter starts to assemble this description beginning with the metric of a spatially uniform, static Universe, then generalizes the metric to include general features of time development. However, a detailed prediction of time development requires a knowledge of the constituents of the Universe and how these constituents change with time. Project 10, Cosmology, adds a description of what we know about the constituents of the Universe, then harnesses the structure assembled in the present chapter to analyze the past and predict alternative futures for our Universe.

2 ■ SPACE METRICS FOR A STATIC UNIVERSE

Describing a uniform space

Space metric for uniform space curvature

A Universe filled uniformly with mass and energy—on average—has *uniform space curvature* everywhere. In this book we deal mainly with two space dimensions plus time. In one popular global map coordinate system, the most general constant-curvature *space* metric has the following form on the r, ϕ plane:

$$ds^2 = \frac{dr^2}{1 - Kr^2} + r^2 d\phi^2 \tag{1}$$

Flat, closed, and open spaces

The value of the parameter K determines the shape of the space, which in turn determines the range of r:

$$\text{for } K = 0, \qquad 0 \le r < \infty \qquad \text{flat space} \tag{2}$$

$$\text{for } K > 0, \qquad 0 \le r \le \frac{1}{K^{1/2}} \qquad \text{closed space} \tag{3}$$

$$\text{for } K < 0, \qquad 0 \le r < \infty \qquad \text{open space} \tag{4}$$

We are comfortable with the shape of a flat space, equation (2). We will see that the closed space has the shape of a sphere and the open space the shape of a saddle.

Variable χ automatically satisfies limits.

The limit (3) on closed space can be automatically satisfied with a coordinate transformation. Let

$$r \equiv \frac{1}{K^{1/2}} \sin\chi \qquad (K > 0 \text{ and } 0 \leq \chi \leq \pi) \tag{5}$$

Here χ is the Greek lowercase letter "chi" (pronounced "k/eye," like "high") which corresponds to the letter x in the Roman alphabet. The sine function automatically limits the range of r to that given in (3). However, there is a problem with the coordinate r: it has the same value in the two hemispheres of the sphere (Figure 2). The coordinate χ does not have this problem; it is single-valued.

The differential dr is

$$dr = \frac{1}{K^{1/2}} \cos\chi d\chi \qquad (K > 0 \text{ and } 0 \leq \chi \leq \pi) \tag{6}$$

With these transformations the metric for the closed, constant-curvature space (1) and (3) becomes

$$ds^2 = \frac{1}{K}\left(d\chi^2 + \sin^2\chi\, d\phi^2\right) \qquad (K > 0 \text{ and } 0 \leq \chi \leq \pi) \tag{7}$$

Equation (7) is identical to the space metric for the surface of Earth derived in Interlude 1, SR to GR—page 6, equation (3)—which reads:

$$ds^2 = R^2(d\lambda^2 + \cos^2\lambda\, d\phi^2) \qquad (\textit{space metric}: \text{Earth's surface}) \tag{8}$$

The expressions in the parentheses on the right sides of both (8) and (7) refer to the unit sphere. In Interlude 1 we use the latitude λ rather than the colatitude χ. The two are related by the following equation, illustrated in Figure 2:

$$\chi \equiv \frac{\pi}{2} - \lambda \tag{9}$$

Transformation (9) replaces sine in (7) with cosine in (8).

Describing closed space

Thus for $K > 0$ the shape of constant-curvature space is that of a spherical surface with a radius R whose square is equal to $1/K$. The space represented by the surface of the sphere is homogeneous and isotropic: the same everywhere and in all directions. Same shape means same physical experience: If you start walking "straight in the χ-direction" in this closed space, sooner or later you will return to your starting point.

Using R instead of K, equation (7) becomes

$$ds^2 = R^2\left(d\chi^2 + \sin^2\chi\, d\phi^2\right) \qquad (\text{closed space},\ \ 0 \leq \chi \leq \pi) \tag{10}$$

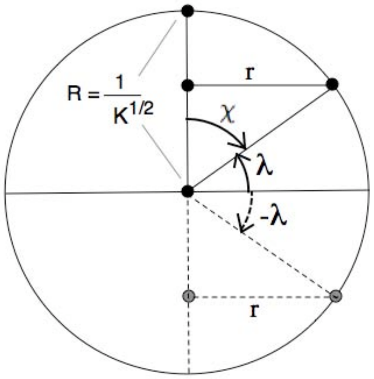


FIGURE 2 Relation between latitude λ and colatitude χ to determine the north-south coordinate on the sphere of radius $R = 1/K^{1/2}$. Latitude λ ranges over the values $-\pi/2 \leq \lambda \leq +\pi/2$, whereas colatitude χ ranges over $0 \leq \chi \leq \pi$. Equation (9) gives the relation between χ and λ, while (5) gives the relation between χ and r. This figure also shows that the radius r is a "bad" coordinate, since it is double-valued, failing to distinguish between northern and southern latitude. In contrast, χ is single-valued between $\chi = 0$ (north pole) and $\chi = \pi$ (south pole).

where the expression in the parenthesis on the right also embodies the shape of the unit sphere.

Describing open space

The values $K < 0$ in metric (1) lead to an *open* space, as shown by the alternative transformation:

$$r \equiv R \sinh \chi \qquad (\text{open space},\ 0 \leq \chi < \infty) \tag{11}$$

where $R^2 = -1/K$ and sinh is the hyperbolic sine. The hyperbolic sine and cosine are defined by the equations

$$\sinh \chi \equiv \frac{e^{\chi} - e^{-\chi}}{2} \qquad \text{and} \qquad \cosh \chi \equiv \frac{e^{\chi} + e^{-\chi}}{2} \tag{12}$$

Equation (11) shows r to be a monotonically increasing function of χ, so there is no worry about a single value of χ representing more than one location. The differential dr is

$$dr = R \cosh \chi \, d\chi \qquad (\text{open space},\ 0 \leq \chi < \infty) \tag{13}$$

and the corresponding space metric is

$$ds^2 = R^2 \left(d\chi^2 + \sinh^2 \chi d\phi^2 \right) \qquad (\text{open space},\ 0 \leq \chi < \infty) \tag{14}$$

The expression in the parentheses on the right side of this equation embodies an open space that has a uniform negative curvature, therefore is uniform. The saddle surface shown in Figure 3 has a single central point whose curvature is

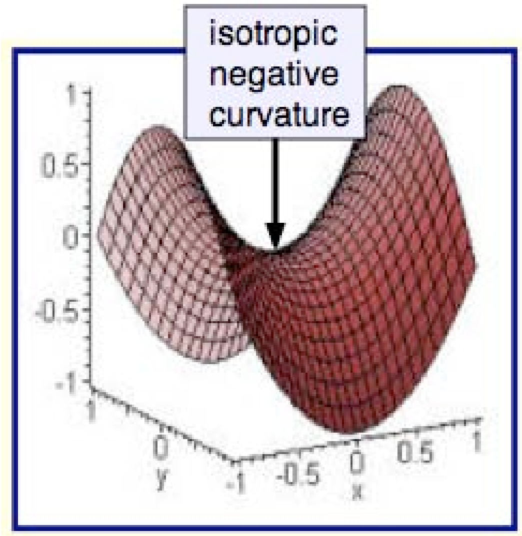


FIGURE 3 The saddle shape has intrinsic negative curvature, but it has only a single (central) point in whose close neighborhood the negative curvature is the same in all directions. Elsewhere on the surface the curvature is negative but varies from place to place and is different in different directions. It is not possible to embed in three spatial dimensions a two-dimensional surface that has uniform negative curvature everywhere.

negative and the same in all directions. Unfortunately it is not possible to embed in three spatial dimensions a two-dimensional surface that has uniform negative curvature everywhere. The best we can do is the saddle shape, Figure 3, which has isotropic negative curvature at only one point.

Describing
flat space

For flat spacetime, equation (2) tells us that $K = 0$ in (1), leading directly to the transformation:

$$r = R\chi \qquad dr = Rd\chi \qquad (\text{flat space},\ \ 0 \leq \chi < \infty) \tag{15}$$

so that the metric for flat space is

$$ds^2 = R^2\left(d\chi^2 + \chi^2 d\phi^2\right) \qquad (\text{flat space},\ \ 0 \leq \chi < \infty) \tag{16}$$

If you start walking "straight in the χ-direction" in an open or flat space, you will *not* return to your starting point.

3 ■ ROBERTSON-WALKER SPACETIME METRIC

A Universe that expands

"Expands" means
$R(\text{constant}) \rightarrow R(t)$

Observation tells us that the Universe "expands with time." What does that mean? Space metric (10) describes the surface of Earth, with R equal to

Box 2. What is the Universe expanding into?

A common misconception is that the Universe expands in the same way that a balloon expands or a firecracker explodes: into an existing three-dimensional space. That is wrong: Both time and space come into existence with the big bang and space itself expands with time. In general relativity there is nothing—no extra dimensions—into which the Universe expands.

If you stick with the image of the expanding balloon for the closed Universe, then you must assume that the surface of the balloon is all that exists. Galaxies are scattered across its surface and human observers are surface creatures who view nothing but what lies on the surface. At the beginning of expansion the surface evolves from a point-event that is also the beginning of time. During the subsequent expansion, every surface creature sees other points on the balloon move away from him. In general relativity there is no extra dimension into which the model balloon is expanding.

Earth's radius. If we were to inflate the Earth like a balloon, its radius R would increase with time but its property of uniform space curvature would remain. By analogy, we can describe a Universe that expands while keeping the same shape by replacing the static radial function R in equations (10), (14), and (16) with a radial function $R(t)$ that increases with time. In the 1930s, Howard Percy Robertson and Arthur Geoffrey Walker proved that the *only* spacetime metrics that describe an evolving, spatially uniform Universe take the form:

$$\boxed{d\tau^2 = dt^2 - R^2(t)\left[d\chi^2 + S^2(\chi)d\phi^2\right] \qquad \text{(Robertson-Walker metric)} \qquad (17)}$$

Robertson-Walker metric

The function $S(\chi)$ is defined for different shapes of the Universe by generalizing equations (15), (5), and (11) respectively:

$$S(\chi) = \chi \qquad (\text{flat Universe}, \quad 0 \leq \chi < \infty) \qquad (18)$$

$$S(\chi) = \sin\chi \qquad (\text{closed Universe}, \quad 0 \leq \chi \leq \pi) \qquad (19)$$

$$S(\chi) = \sinh\chi \qquad (\text{open Universe}, \quad 0 \leq \chi < \infty) \qquad (20)$$

Comoving coordinates

Coordinates χ and ϕ are called **comoving coordinates** because a galaxy with fixed χ and ϕ simply "rides along" as the radial function $R(t)$ increases.

For a closed Universe, $R(t)$ can be interpreted loosely as the "radius of the Universe." However, for flat or open Universes $R(t)$ has no such interpretation. We will call it simply the *radial function of the Universe.*

THE CENTER OF THE UNIVERSE CAN BE ANYWHERE. For all three models of the Universe described by (18) through (20), the location $\chi = 0$ appears to be a favored point, for example the north pole for the closed Universe or the center of the saddle for the open Universe or the origin of the infinite flat sheet for the flat Universe. Because the Universe is assumed to be completely uniform, however, any point can be chosen as $\chi = 0$ (and as the origin of ϕ). That arbitrary point then becomes the north pole or the center of the saddle or the origin on the flat sheet. The mathematical model allows every observer to assume that s/he is at the center of the Universe. (Talk about ego!)

Box 3. Is a static, uniform Universe possible?

The Robertson-Walker metric (17) is more general than general relativity. Whether or not the Robertson-Walker metric satisfies Einstein's field equations depends on the variation of the radial function $R(t)$ with time. At any time, $R(t)$ depends on what the Universe is made of and how much of each constituent is present at that time and was present at earlier times. Project 10, Cosmology, examines the presence and density of the constituents of the Universe at different times, displays the resulting functions $R(t)$ that satisfy Einstein's equations, and traces the consequences for our current model of the time development of the Universe. In the present chapter we simply assume that $R(t)$ started with value zero at the big bang and thereafter increases monotonically with time.

In 1917 Einstein thought that the Universe was not only uniform in space, but also unchanging in time. Such a spacetime has the spacetime metric (17) with R a constant. Is this a valid metric for the Universe?

Einstein showed that metric (17) with R = constant does *not* satisfy his field equations for a Universe uniformly filled with matter. However, by adding the cosmological constant Λ to his field equations, he obtained a unique solution for a closed Universe, the case described by (19). The effect of Λ is to create a cosmic repulsion that keeps galaxies from being drawn together by gravity. In Project 10, Cosmology, we will see that something very like Λ—now called *dark energy*—repels galaxies so much that at the present stage of the Universe distant galaxies fly away from our own with increasing speed.

Map time read on wristwatch of comoving observer

The squared time differential dt^2 in (17) has no coefficient; in Robertson-Walker map coordinates, time has no warpage. Indeed, for $d\chi = d\phi = 0$, passage of coordinate time t tracks the passage of wristwatch time τ. The interpretation is simple: coordinate time is that recorded on comoving clocks, those that ride along "at rest" with respect to the space coordinates of the expanding Universe. The same was true for the gravitational wave metric in Project 9, Gravitational Radiation.

Space and time exist only for $t > 0$.

We should also give a range for time t in order to complete the definition of the spacetime region described by equations (17) through (20). However we cannot specify a range of time until we know details of the radial function $R(t)$. For big bang models of the Universe—time development from an initial singularity—the radial function $R(t) = 0$ at $t = 0$. In this book we examine big bang models, for which spacetime exists only for $t > 0$.

4 ■ REDSHIFT

Light we receive from a great distance increases in wavelength in an expanding Universe.

Choose our location at the center of the Universe and our time (now) to be t_0.

We can *choose* to place ourselves at the center of the Universe, that is at $\chi = 0$ and assume that we stay at the center permanently. Then every current observation *that we make* is an event that takes place at global map location $\chi = 0$ and *now*, which we will call map time t_0.

$$\text{Observation NOW on Earth has map coordinates } t \equiv t_0,\ \chi \equiv 0 \qquad (21)$$

Suppose that a distant star is fixed in comoving coordinates χ and ϕ, so it rides along as the radial function $R(t)$ increases. Let the star emit a light flash at $(t_{\text{emit}}, \chi_{\text{emit}})$, which we observe on Earth at $(t_0, 0)$.

For light, $d\tau = 0$ and for radial motion $d\phi = 0$ in (17). Write the resulting metric with time and space terms on opposite sides of the equation, take the square root of both sides, and integrate each one:

$$\int_{t_{\rm emit}}^{t_0} \frac{dt}{R(t)} = \int_0^{\chi_{\rm emit}} d\chi = \chi_{\rm emit} \qquad \text{(light, d}\phi = 0\text{)} \tag{22}$$

Emit and detect two light flashes.

Now think of a second light flash emitted from the same star at event $(t_{\rm emit} + \Delta t_{\rm emit}, \chi_{\rm emit})$ and observed by us at $(t_0 + \Delta t_0, 0)$. The two flashes can represent two sequential positive peaks in a continuous wave. We assume that the emitter is located at constant χ, so the second flash travels the same distance in χ as the first. Hence the right-hand integral has the same value for both flashes. Therefore

$$\int_{t_{\rm emit}+\Delta t_{\rm emit}}^{t_0+\Delta t_0} \frac{dt}{R(t)} = \chi_{\rm emit} \qquad \text{(light)} \tag{23}$$

Compare the time limits of the integrals on the left sides of (22) and (23). The integration in (23) starts later by $\Delta t_{\rm emit}$ and ends later by Δt_0. In consequence, when we subtract the two sides of equation (22) from the corresponding sides of equation (23), the result is.

$$\int_{t_0}^{t_0+\Delta t_0} \frac{dt}{R(t)} - \int_{t_{\rm emit}}^{t_{\rm emit}+\Delta t_{\rm emit}} \frac{dt}{R(t)} = 0 \qquad \text{(light)} \tag{24}$$

Approximate this equation to first order in $\Delta t_{\rm emit}$ and Δt_0, leading to

$$\frac{\Delta t_0}{R(t_0)} \approx \frac{\Delta t_{\rm emit}}{R(t_{\rm emit})} \qquad \text{(light)} \tag{25}$$

Let the two flashes represent two sequential peaks in a continuous wave. Then the time between flashes in meters equals the wavelength in meters.

$$\frac{\Delta t_0}{\Delta t_{\rm emit}} = \frac{\lambda_0}{\lambda_{\rm emit}} = \frac{R(t_0)}{R(t_{\rm emit})} \qquad \text{(light)} \tag{26}$$

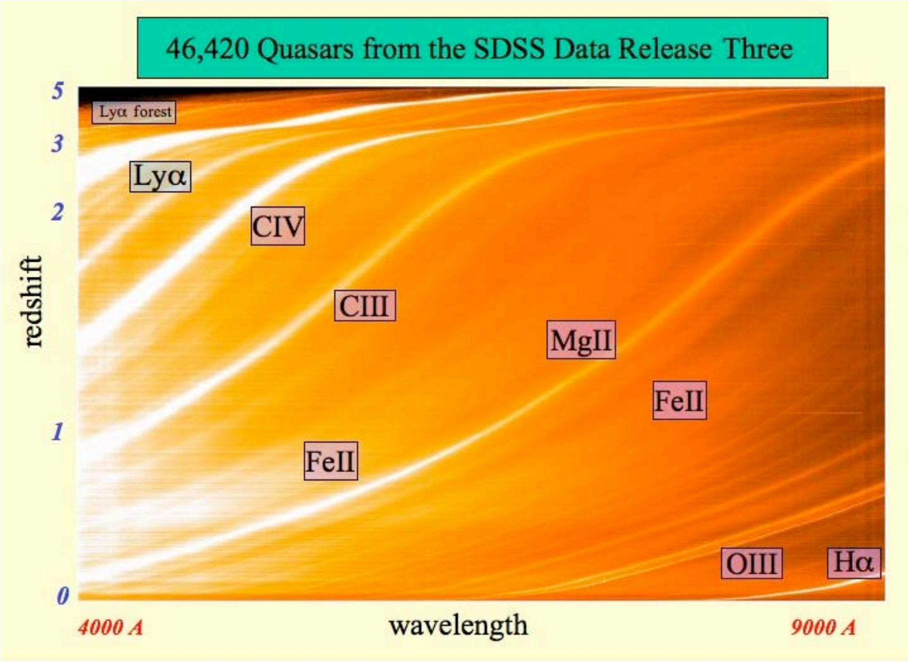
Redshift z

In this equation an equality sign replaces the approximately equal sign in (25) because one wavelength of light λ is truly infinitesimal compared with the radial function $R(t)$ of the Universe. It is customary to measure the fractional change in wavelength using a dimensionless parameter z, called the **redshift**, defined by the equation

$$\lambda_0 \equiv (1+z)\lambda_{\rm emit} \qquad \text{(light)} \tag{27}$$

where we call $1+z$ the **stretch factor**. Then equation (26) can be written

$$1 + z \equiv \frac{\lambda_0}{\lambda_{\rm emit}} = \frac{R(t_0)}{R(t_{\rm emit})} \qquad \text{(stretch factor)} \tag{28}$$



In addition to images, the SDSS has measured the spectra of light from more than a million celestial sources. The spectrum of an object shows the intensity of its light as a function of wavelength. This picture shows the spectra of more than 46,000 quasars from the SDSS 3rd data release; each spectrum has been converted to a single horizontal line, and they are stacked one above the other with the closest quasars at the bottom and the most distant quasars at the top. Bright bands show the emission produced by specific ions of hydrogen, carbon, oxygen, magnesium, and iron. For more distant quasars, these emission lines are shifted to longer wavelengths by the expansion of the universe. This redshift of spectral lines is what the SDSS measures to determine the distances to quasars and galaxies.

Credit: X. Fan and the Sloan Digital Sky Survey.

FIGURE 4 A remarkable plot of the redshifts z of the spectra from more thatn 46 thousand quasars taken by the Sloan Digital Sky Survey (SDSS). The spectrum of each quasar lies along a single horizontal line at a vertical position corresponding to its redshift z. Some prominent spectral lines from different atoms are labeled: Lyα is the Lyman alpha line of hydrogen. Roman numeral I following an element is the neutral atom; Roman numeral II is the singly ionized atom, and so forth. Thus MgII is singly ionized magnesium and CIV is triply ionized carbon. The observed wavelength λ_0 increases with increasing z. (The redshift scale is nonlinear so the bands are not straight lines.)

In other words, when we train our telescopes on a source with redshift z, we are observing light emitted at a time when the Universe was a factor $(1 + z)$ smaller than it is today.

Cosmological redshift

The change in wavelength described by equation (28) is called the **cosmological redshift**. The time of observation t_0 is greater than the time of emission t_{emit}, and for an expanding universe $R(t_0) > R(t_{\mathrm{emit}})$. Therefore the observed light has a longer wavelength than the emitted light; the color of light visible to our eyes shifts toward the red end of the spectrum, hence the term “redshift.” The same fractional increase in wavelength occurs for electromagnetic radiation of any frequency, so the term *redshift* applies to microwaves, infrared, ultraviolet, x-rays, and gamma rays.

Redshift a Doppler shift?

Equation (27) appears not to describe a Doppler shift in the special relativity sense. Both emitter and observer are *at rest* in their comoving coordinate χ; nevertheless, they observe the light to have different wavelengths. In a sense the expansion of the Universe “stretches out” the wavelength of the light as it propagates. In another sense, however, the cosmological redshift is a cumulative redshift, because a star at fixed χ is at a physical distance $R(t)\chi$ that grows with time. In other words, it moves away from us. For $z \ll 1$, we will see in Section 6 that the cosmological redshift *is* a Doppler shift.

When we see light of a given frequency that has been emitted from a distant galaxy, how do we know that it has been redshifted? With what do we compare it? From laboratory experiments on Earth, we know the discrete pattern of radiation frequencies emitted by a particular atom or molecule. Then an identical set of *ratios* of frequencies in light from a distant star tells us what element or molecule we are observing. Then the value of the shift in any one frequency is the redshift. Figure 4 shows redshifted spectral lines (bright: emission lines; dark: absorption lines) from many abundant atoms in the light from distant quasars.

Astronomers use z for t_{emit}.

Because it is easy to measure a galaxy’s redshift z, astronomers use z as a proxy for t_{emit} in equation (26). See Figure 5. Whenever you read a news article about a galaxy formed during the first billion years of the Universe, remember that astronomers do not measure time; they measure redshift. The distant galaxies in the news have $z > 6$: in the process of traveling to us, the wavelength of their light has been stretched by a factor more than 7! Optical light is redshifted to the infrared. This is why the James Webb Space Telescope—the successor to the Hubble Space Telescope—is looking in the infrared region of the spectrum for light from the most distant galaxies, those that appeared earliest in the development of the Universe.

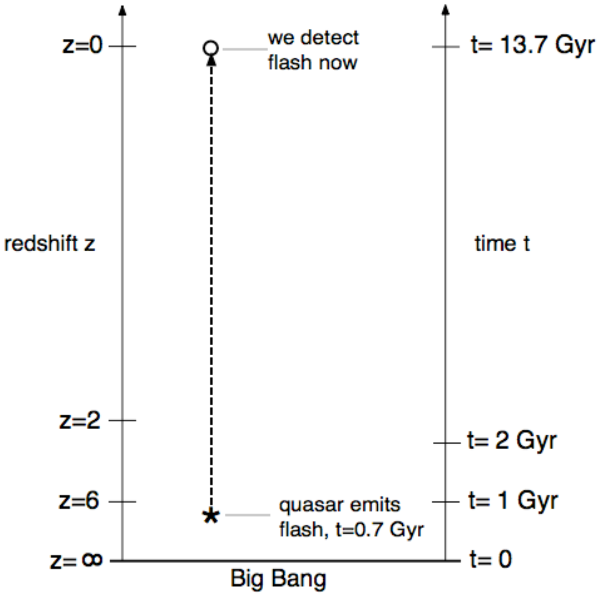


FIGURE 5 Schematic diagram comparing redshift z with cosmic time t. Calibration of the right hand scale depends on the time development of the Universe, through $R(t)$, based on our current model. Astronomers use redshift as a proxy for time, both because it is directly measurable and also because it does not change as we revise our cosmic time scale. The flash emission and detection is the case analyzed in Box 4.

5 ■ HOW DO GALAXIES MOVE?

Apply the Principle of Maximal Aging to the motion of a galaxy.

Transverse galaxy motion is difficult to detect.

Limit attention to radial motion.

We have a disability in viewing the distant Universe from what amounts to a single point: the Earth and its solar system. The redshift of light from distant galaxies gives us a handle on their radial recession. However, transverse motion of a remote source is too small to detect directly in a human lifetime. (See the exercises.) We can sometimes infer transverse motion, for example when the redshift pattern of sources close to one another implies that they are flying apart as the result of an explosion or that they are orbiting a common center of attraction. In this and following sections, however, we limit attention to sources that move radially away from us.

Galaxy motion from Principle of Maximal Aging

How do galaxies move in the global coordinate system of metric (17)? As usual, the metric tells us about structure of spacetime but does not specifiy the motion of a stone—or a galaxy. For that we need the Principle of Maximal Aging, which requires that total wristwatch time be a maximum along a worldline that crosses adjoining patches.

For radial motion, the metric (17) becomes:

Box 4. How far away (now) is the most distant galaxy that we see (now)?

We see *now* the most distant galaxies as they *were* when they emitted the light: at, say, $t_{\rm emit} = 0.7$ billion years after the big bang (Figure 5). The current age of the Universe is $t_0 \approx 13.7$ billion years, so $t_0 - t_{\rm emit} \approx 13$ billion years. Naively, then, we might expect that these galaxies lie at a distance of about 13 billion light years from us. However, this is false; they actually lie much further away in the present day. Why? Because these galaxies have moved farther away from us during the 13 billion years that it took for their light to reach us. How much farther? What is the true distance *now* to a galaxy formed at time $t_{\rm emit} = 0.7$ billion years ago?

Use the Robertson-Walker metric (17) with $d\tau = 0$ to obtain the map distance between the emitting galaxy (at $\chi = \chi_{\rm emit}$) and Earth (at $\chi = 0$) at any particular time t. This distance is given simply by $R(t)\chi_{\rm emit}$, since the emitter continually "rides along" at the constant comoving coordinate $\chi_{\rm emit}$. The present separation $d_0 \equiv \sigma_0$ is then just $R(t_0)\chi_{\rm emit}$ with $\chi_{\rm emit}$ given by (22).

$$d_0 = R(t_0)\chi_{\rm emit} = R(t_0)\int_{t_{\rm emit}}^{t_0} \frac{dt}{R(t)} \qquad (29)$$

We cannot complete this calculation until we know how the radial function $R(t)$ increases with time t. That is the task of Project 10, Cosmology. For a rough estimate of the present distance d_0, assume that the radial function increases uniformly with time: $R(t)/R(t_0) = t/t_0$. Then the integral in (29) can be carried out using $t_{\rm emit} = 0.7$ billion years and the present time $t_0 = 13.7$ billion years:

$$\begin{aligned} d_0 &= t_0 \int_{t_{\rm emit}}^{t_0} \frac{dt}{t} = t_0 \ln \frac{t_0}{t_{\rm emit}} \qquad (30) \\ &= t_0 \ln \frac{13.7}{0.7} = 13.7 \times 2.97 = 40.7 \end{aligned}$$

in billions of light-years. We call d_0 the **look-back distance**. According to this rough model, look-back distances of galaxies that emitted light 13 billion years ago are something like $d_0 = 40$ billion light years. This is their calculated distance away from us now. We can refine this estimate by using a more accurate radial function $R(t)$, but the present distance to these remote galaxies is almost certainly larger than 40 billion light years.

$$d\tau^2 = dt^2 - R^2(t)d\chi^2 \qquad (d\phi = 0) \qquad (31)$$

This metric is valid for any function $S(\chi)$ in (17), whether for a flat, closed, or open model Universe. By just looking at this metric, can we anticipate constants of motion? One metric coefficient depends explicitly on t through the function $R(t)$. In all our earlier derivations, energy as a constant of motion required that no metric coefficient be an explicit function of time. Therefore metric (17) tells us that energy will *not* be conserved in the motion of galaxies. However, for radial motion ($d\phi = 0$) the metric coefficients do not depend explicitly on χ, so there will be a conserved quantity related to motion in χ, a kind of radial momentum.

The stone crosses two adjoining patches (Figure 6). Label A and B the segments of its path across the respective patches. Consider three events: Two at the opposite edges of the patches and one where they join. To find momentum as a constant of motion, we fix the times of all three events and fix the locations of the two events at the outer ends of the two segments. Then we vary the position of the connecting event (and the boundary between patches) in order to maximize total wristwatch time.

Over one patch, $R(t)$ is treated as being constant, so each patch is flat. Define

$$R_{\rm A} \equiv R(\bar{t}_{\rm A}) \qquad (32)$$

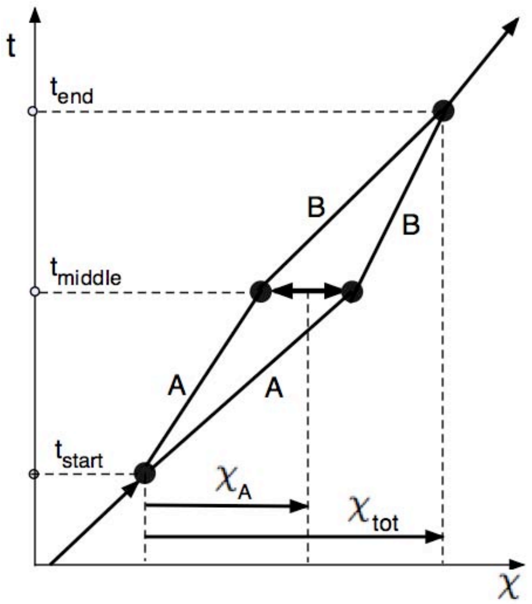


FIGURE 6 Greatly magnified picture of alternative worldlines across incremental segments A and B used in the derivation of the constant of motion (38). We vary the position χ_{A} of the middle event between segments A and B and demand that the total wristwatch time across both segments be maximum. The origin of this diagram is NOT necessarily at the zero of either time or radial position.

$$R_{\mathrm{B}} \equiv R(\bar{t}_{\mathrm{B}})$$

where $\bar{t}_{\mathrm{A}}$ and $\bar{t}_{\mathrm{B}}$ are the average times when the galaxy crosses patch A and B, respectively. Define the time for the galaxy to cross each patch as:

$$t_{\mathrm{A}} \equiv t_{\text{middle}} - t_{\text{start}} \tag{33}$$

$$t_{\mathrm{B}} \equiv t_{\text{end}} - t_{\text{middle}}$$

Let χ_{A} be the *change* in coordinate χ across segment A and χ_{B} be the corresponding change across segment B. Then $R_{\mathrm{A}}\chi_{\mathrm{A}}$ is the radial distance across segment A and $R_{\mathrm{B}}(\chi_{\text{tot}} - \chi_{\mathrm{A}})$ the radial distance across segment B, with χ_{A} variable. Then the metric (31) across the two patches becomes:

$$\tau_{\mathrm{A}} = \left[t_{\mathrm{A}}^2 - R_{\mathrm{A}}^2\,\chi_{\mathrm{A}}^2\right]^{1/2} \tag{34}$$

and

$$\tau_{\mathrm{B}} = \left[t_{\mathrm{B}}^2 - R_{\mathrm{B}}^2\,(\chi_{\text{tot}} - \chi_{\mathrm{A}})^2\right]^{1/2} \tag{35}$$

Fix the times t_{start}, t_{middle}, and t_{end} at the edges of the two segments. This fixes the values of t_{A}, t_{B}, R_{A}, and R_{B} through equations (33) and (32).

Now vary χ_{A} to maximize the total wristwatch time $\tau_{\mathrm{tot}} = \tau_{\mathrm{A}} + \tau_{\mathrm{B}}$ across both segments:

$$\begin{aligned}\frac{d\tau_{\mathrm{tot}}}{d\chi_{\mathrm{A}}} &= \frac{d\tau_{\mathrm{A}}}{d\chi_{\mathrm{A}}} + \frac{d\tau_{\mathrm{B}}}{d\chi_{\mathrm{A}}} \\ &= -\frac{R_{\mathrm{A}}^2\,\chi_{\mathrm{A}}}{\tau_{\mathrm{A}}} + \frac{R_{\mathrm{B}}^2\,(\chi_{\mathrm{tot}} - \chi_{\mathrm{A}})}{\tau_{\mathrm{B}}} \\ &= -\frac{R_{\mathrm{A}}^2\,\chi_{\mathrm{A}}}{\tau_{\mathrm{A}}} + \frac{R_{\mathrm{B}}^2\,\chi_{\mathrm{B}}}{\tau_{\mathrm{B}}} = 0\end{aligned} \tag{36}$$

or

$$\frac{R_{\mathrm{B}}^2\,\chi_{\mathrm{B}}}{\tau_{\mathrm{B}}} = \frac{R_{\mathrm{A}}^2\,\chi_{\mathrm{A}}}{\tau_{\mathrm{A}}} \tag{37}$$

Constant of radial motion Q_{r}

Now the usual argument: The left side of (37) refers to segment B alone, the right side to segment A alone. We have found a quantity that has the same value for each segment—that is, a constant of motion. Restore differentials and define a constant of motion Q_{r}.

$$Q_{\mathrm{r}} \equiv mR^2\frac{d\chi}{d\tau} = mR\left(\frac{Rd\chi}{d\tau}\right) \equiv Rp_{\mathrm{r}} \quad \text{is a constant of motion} \tag{38}$$

where (38) provides a definition of local radial momentum p_{r} because $Rd\chi$ is a measured distance, from (17). Here m is the mass of a stone—or of a galaxy! Let the motion be radial only, so $p_{\mathrm{r}} = p$. Then (38) is still valid as $m \to 0$ for a photon, with $p = E$. In other words $R(t)E$ is constant for light, which means that as $R(t)$ increases, the energy E of photons decreases—another example of cosmological redshift.

Two possible radial motions

We can distinguish two possible radial motions of a galaxy that leave Q_{r} constant. In the first χ remains constant as t increases, so $d\chi/d\tau = 0$ and $Q_{\mathrm{r}} = p_{\mathrm{r}} = 0$. Each such "comoving" galaxy rides outward with $R(t)$; two galaxies at different values of χ move apart as $R(t)$ increases with time. For flat spacetime ($S = \chi$) one can think of a set of concentric rings of galaxies fixed in the comoving coordinate χ. As time progresses the radius of each ring increases with $R(t)$. Figure 7 shows that radial distances $R(t)\chi$ and tangential distances $R(t)\chi\phi$ both increase proportionally to $R(t)$. This is true for every observer. There is no unique center; every observer can plot the expansion of the Universe in global coordinates with himself at the center.

In the second possible radial motion that leaves Q_{r} constant, a galaxy moves radially with respect to comoving coordinate χ. (Most galaxies have at least a slightly non-zero Q_{r} because of local gravity from spatial inhomogeneities.) Or one can think of a stone thrown radially out of a comoving galaxy. For such motion one can rewrite (38) as:

$$p_{\mathrm{r}} = \frac{Q_{\mathrm{r}}}{R(t)} \tag{39}$$

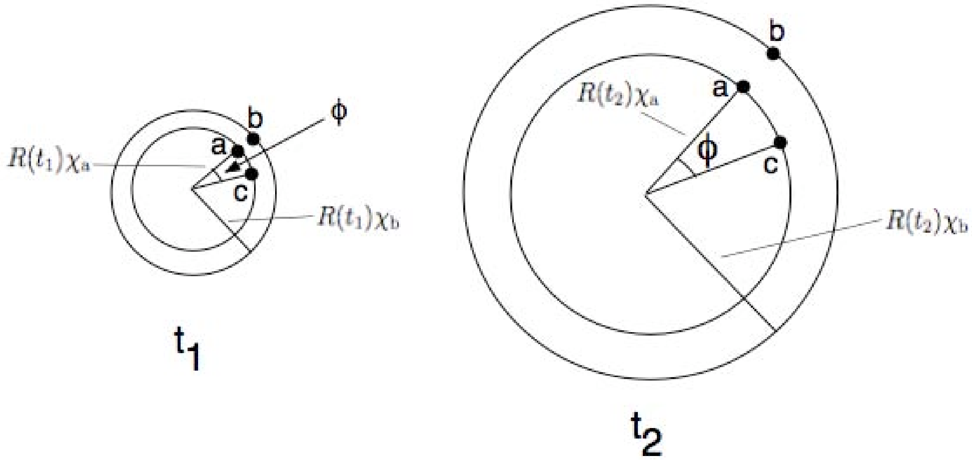


FIGURE 7 One possible radial motion for a galaxy is to remain at rest in the comoving coordinate χ and ϕ and ride outward, following $R(t)$, as the Universe expands. This figure shows the result for a flat Universe. All separations increase by the same ratio, so every observer can analyze galaxy motion with himself at the center and galaxies expanding away from him.

Q_r remains constant and $R(t)$ increases, so p_r decreases. This is called the "cosmological redshift of momentum." The high speed limit on (39) applies to a photon:

$$E = p \propto \frac{1}{R(t)} \qquad \text{(light)} \qquad (40)$$

Constant of tangential motion Q_ϕ

Even though tangential motion is difficult to detect, we can carry out a derivation of tangential motion similar to Q_r for radial motion. One applies the Principle of Maximal Aging to two patches separated in angle instead of radius. The result is

$$Q_\phi \equiv mR^2S^2\frac{d\phi}{d\tau} = RS\left(\frac{mRSd\phi}{d\tau}\right) \equiv RSp_\phi \quad \text{is a constant of motion} \quad (41)$$

Equation (41) provides a definition of local tangential momentum p_ϕ because $RSd\phi$ is a measured disance, from metric (17).

6 ■ MEASURING DISTANCE

Extending a ruler from one lonely outpost.

So much for the theory of how galaxies move in the expanding Universe. What about the facts? On Earth we describe motion by plotting distance vs. time.

Box 5. Edwin P. Hubble

FIGURE 8 Edwin P. Hubble on the cover of Time Magazine, 1948.

Edwin P. Hubble was as important to astronomy as Copernicus. He expanded our view of the Universe from a single home galaxy to many galaxies that are rushing away from one another.

Hubble was born in 1889. In his youth he was an outstanding athlete and one of the first Rhodes Scholars at Oxford University, where he studied the law and Spanish. After returning to the United States he taught Spanish, physics, and mathematics in high school. He served in World War I, after which he earned a Ph.D. at Yerkes Observatory of the University of Chicago.

In 1919 Hubble took up a position at Mount Wilson Observatory where he used the new 100 inch Hooker reflecting telescope, with which he discovered and analyzed redshifts of light from what were then called "nebulae." Hubble showed that nebulae were not objects within our galaxy but galaxies themselves, in motion away from our galaxy. The nearby galaxies he studied recede from us at speeds proportional to their distance (Figure 11).

Before his death in 1953, Hubble made observations with the 200 inch telescope installed on Mount Palomar California in 1948.

Life in the Universe is more complicated. There are two problems: We cannot directly measure distances to objects outside our galaxy, and we cannot directly measure times longer than a few centuries. What hope do we have then for measuring billions of years and billions of light years in the Universe?

First we give up trying to measure time. Instead we measure distance and velocity, both through indirect means. Section 7 discusses velocity measurements through redshift of spectral lines; here we focus on distance.

We cannot use laser ranging or classical surveying methods to measure distances outside our galaxy. The most widely used method employs what is called a **standard candle**. A standard candle is a light source whose intrinsic brightness is known. For that intrinsic brightness (more precisely, luminosity) and the apparent brightness (more precisely, flux density) of the object at Earth, one can determine the distance. However, the expanding Universe complicates the analysis, as detailed in Box 4.

Cepheid variables: standard candles

When Hubble did his observations, the major standard candle was one form of the so-called *Cepheid variable* stars. These are stars whose emitted power varies periodically. Their rate of pulsation depends on their emitted power: the longer the pulsation period, the greater the emitted power of the star.

Hubble found Cepheid variable stars in many nearby galaxies, but he could not detect them in more distant galaxies. To find their approximate distances he classified different galaxies, found the intrinsic brightness of galaxies of a given type that were near enough to allow detection of Cepheid variables they contained, then assumed the same intrinsic brightness for more distant galaxies of the same type.

Hubble: nebulae are distant "island universes."

Hubble's observations in 1923-1924 showed that most spiral nebulae (fuzzy patches of light in the sky) are much farther away than the limits of our galaxy; they were indeed separate "island universes," or what we now call "galaxies." He also classified "elliptical," "lenticular," and "irregular" galaxies, so-called because of their appearance. All lie outside our own Milky Way galaxy. (Interesting fact: "Galaxy" and "lactose" come from the Greek and Latin words for milk.) In summary: The Universe extends far beyond our galaxy.

Cepheid variable stars are too faint to be seen at distances more than a hundred million light years. For more distant sources, the standard candle of choice is a Type Ia supernova. A Type Ia supernova results when a small, dense white dwarf star gradually accretes mass from a binary companion star, finally reaching a mass at which the white dwarf becomes unstable and explodes into a supernova. The "slow fuse" on the gradual accretion process can lead to to an explosion of almost the same size on each such occasion, giving us a "standard candle" of the same intrinsic brightness, provided that the original nuclear burning has left every white dwarf with a similar nuclear composition. If so, the brightness of the explosion as seen from Earth provides a measure of the distance to the supernova. The cosmological red shift of light tells us how fast the supernova is receding (Section 4). Because supernovae (the plural of supernova) are so bright, they can be seen at a very great distance, which brings us information about the Universe much of the way back to the big bang.

Astronomers plot a quantity called distance modulus, $m - M$, where m is the apparent magnitude and M is the absolute magnitude. This difference is related to luminosity distance (Box 6) by the equation

$$m - M = 5\log_{10}\left(\frac{d_{\mathrm{L}}}{10\,\mathrm{pc}}\right) \tag{42}$$

where pc stands for *parsec*, a unit of distance. Ten parsecs is 32.6 light years. Why this peculiar formula? Blame the ancient Greeks, who first quantified the brightness of stars. The key is the realization that M is known (or knowable) for Type Ia supernovae, so that measurements of apparent magnitude m yield the distance modulus and, therefore, the luminosity distance d_{L}.

A graph of apparent magnitude vs. redshift is called a **Hubble Diagram**. Figure 9 shows the Hubble Diagram for Type Ia supernovae. The thin spread in the vertical direction confirms that Type Ia supernovae are good standard candles—they all have the same M (when small corrections are applied to raw measurements) so that magnitude m determines distance.

The implications? First the obvious: Redshift increases with distance. The next section gives an interpretation of this as a result of cosmological expansion. The more subtle and surprising result is that this expansion is speeding up with time. Project 10, Cosmology, elaborates on this second point.

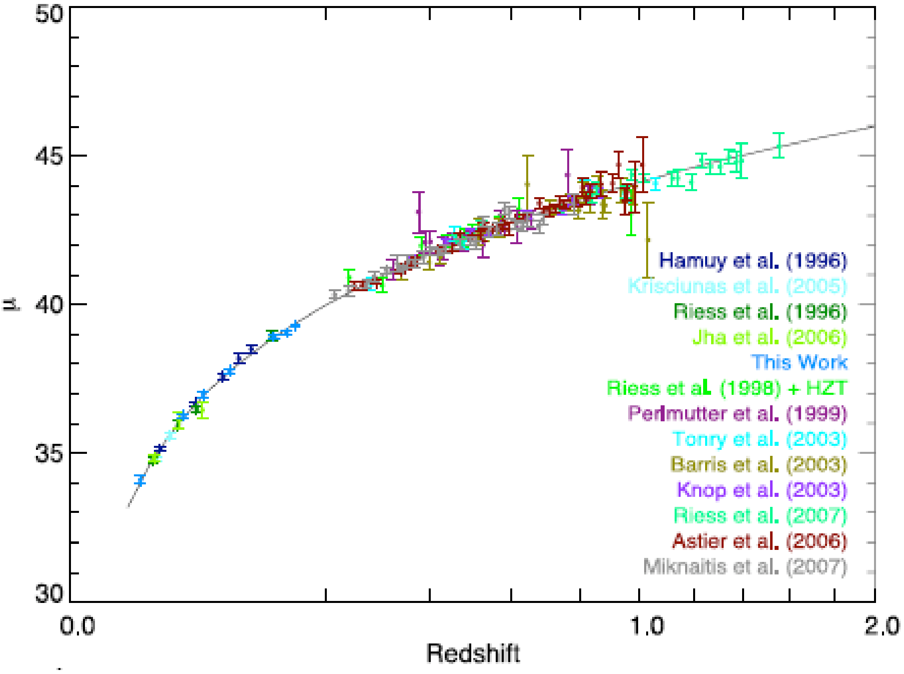


FIGURE 9 Effective magnitude of Type Ia supernovae as a function of their redshift z. The vertical axis is $\mu = m - M$, the difference between apparent magnitude and intrinsic magnitude.

In the future, a second way to measure distances may prove useful in cosmology. From metric (17), objects of known transverse size D at radial coordinate distance χ extend across an angle

$$\theta \approx \frac{D}{S(\chi)R(t_{\mathrm{emit}})} \qquad (\theta^2 \ll 1) \tag{43}$$

In flat spacetime the distance would be $d = D/\theta$ if $\theta \ll 1$. In the expanding Universe, cosmologists define the **angular diameter distance** as:

$$d_{\mathrm{A}} \equiv \frac{D}{\theta} = S(\chi)R(t_{\mathrm{emit}}) = \frac{S(\chi)R(t_0)}{1+z} \tag{44}$$

where we used equation (28). Objects of known transverse size D are called **standard rulers**. Comparing (44) with (52), you can show that $d_{\mathrm{A}} = d_{\mathrm{L}}/(1+z)^2$. Thus, measurements of standard candles and standard rulers for an object of known z yield the same information. The difficulty lies in determining the intrinsic size and luminosities of objects billions of light years away.

7 ■ LAWS OF RECESSION

Recession rate proportional to distance—at least for nearby galaxies.

When Edwin N. Hubble arrived at the Mount Wilson Observatory in California USA in 1919 and began using the new 100-inch telescope, many astronomers believed that the entire Universe consisted of stars in the Milky Way, what we now call "our galaxy." A disturbing feature of this model of the Universe was the behavior of some of the objects they called **nebulae**. We now know that some nebulae are within our galaxy but most are separate galaxies distant from our own. As early as 1912 Vesto Melvin Slipher had shown that light from many nebulae had significant redshifts, implying that they were moving away from us at high speed. But were these nebulae dim objects in our own galaxy or bright objects outside our galaxy? To answer this question, Hubble needed, first, a relation between redshift and recession velocity. Second, he needed a measure of the distance of these nebulae from us. We examine these tasks in turn.

Velocity vs. Redshift

Hubble used special relativity Doppler shift.

Slipher and Hubble used Doppler shift of light to find a relation between redshift z and velocity of recession v. They were astronomers, not general relativists. (General relativity did not exist when Slipher began his work.) For them the nebulae are speeding away from us in static flat space, and the redshift is a Doppler effect that can be analyzed using special relativity. We will show that this simple analysis gives the correct results for nearby nebulae receding from us at speeds much less than that of light.

Figure 10 introduces the Doppler shift for special relativity. A receding source emits two flashes a time Δt apart in the observer's frame. At speed unity, the first flash moves a distance Δt between flash emissions, while the source moves an additional distance $v\Delta t$ away from the observer. The observer will receive the two flashes a time $\Delta t + v\Delta t$ apart. Let the time between flash emissions be the period of oscillation of the source. Then the wavelength for the observer is

$$\lambda_{\text{obs}} = (1+v)\Delta t \qquad \text{(special relativity)} \tag{45}$$

Now, the time between flash emissions in the rest frame of the source is different from the time Δt in the frame of the observer. We say that "the emitting clock runs slow," according to the equation

$$(1-v^2)^{1/2}\Delta t = \Delta t_{\text{source}} = \lambda_{\text{source}} \qquad \text{(special relativity)} \tag{46}$$

The ratio of observed wavelength to the wavelength in the frame of the source is

$$\frac{\lambda_{\text{obs}}}{\lambda_{\text{source}}} = \frac{(1+v)\Delta t}{(1-v^2)^{1/2}\Delta t} = \left(\frac{1+v}{1-v}\right)^{1/2} = 1+z \qquad \text{(special relativity)} \tag{47}$$

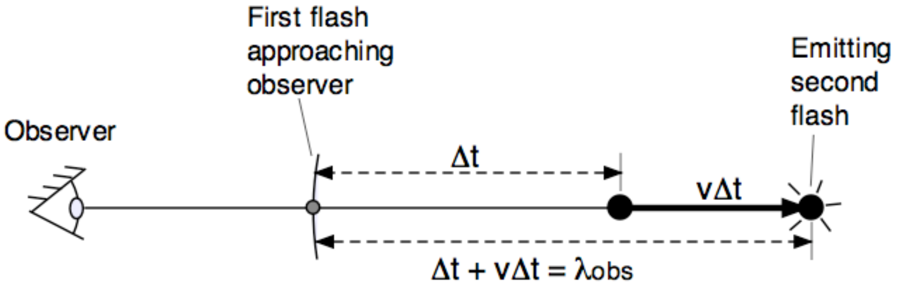


FIGURE 10 Doppler effect in special relativity, used by Hubble to analyze the speed of receding nearby galaxies. Earlier than the time shown in this figure an object emitted one flash, then moved a distance $v\Delta t$ farther away from the observer, and is emitting the second flash at the instant shown. During that time lapse the initial flash moved a distance Δt closer to the observer. Let the time between the two flashes represent one period of a continuous wave. Then the wavelength λ_{obs} detected by the observer has the value shown in the figure. Because the emitter is in motion in this frame, its wristwatch "runs slow" according to special relativity, resulting in a different wavelength $\lambda_{\mathrm{source}}$ in the rest frame of the source.

where we have inserted the definition of redshift z from (28). Nearby galaxies are not moving away from us very fast; for them we may make the approximation:

$$1 + z = (1+v)^{1/2}(1-v)^{-1/2} \approx \left(1 + \frac{v}{2}\right)^2 \approx 1 + v \qquad (v^2 \ll 1) \quad (48)$$

so for slow-moving galaxies the redshift z is equal to the velocity of recession v.

$$v = z \qquad (v^2 \ll 1) \qquad (49)$$

Doppler OK for small z

This Doppler interpretation of the cosmological redshift is valid for $z \ll 1$, because spacetime over such a small distance is well approximated by a single flat patch, for which general relativity reduces to special relativity.

Measuring Distance with a "Standard Candle"

Equation (49) gives the velocity of recession. Hubble also needed to know how far away the emitting star is, σ_{now}. To determine distance we need what is called a **standard candle**, that is a star whose intrinsic brightness is known. From that intrinsic brightness and the apparent brightness of this star at Earth, one can then determine its distance. However, the expanding Universe complicates this analysis, as detailed in Box 6.

Hubble's Law of Recession

Hubble's law of recession

From the redshift of different galaxies, Hubble now knew from (49) their recession velocities. From intrinsic brightness of Cephied variables and galaxies

Box 6. Finding the distance (which distance?) to a standard candle

Consider a star that emits electromagnetic power L (energy per unit time), called **luminosity**, as viewed in its rest frame. We assume that this emission is isotropic, the same in all directions. Place this star at the center of coordinates, $\chi = 0$. Place an observer at a comoving coordinate χ away from the star. In special relativity the power per unit area, also called **flux density** F, reaching an observer at this distant location is:

$$F = \frac{L}{4\pi d^2} \qquad \text{(flat spacetime)} \qquad (50)$$

where d is the distance between star and observer. In an expanding Universe F is modified in several ways. First, the metric contains no distance d, but rather a map coodinate χ and an angular factor $S(\chi)$. Second, the energy reaching the observer is reduced by a factor $(1 + z)$ due to the cosmological redshift. Third, the time lapse this light takes to arrive at the observer is stretched out by another factor $(1 + z)$. The result is

$$F = \frac{L}{4\pi(1+z)^2 R^2(t_0) S^2(\chi)} \qquad (51)$$

We can measure F and z. Suppose we also know the cosmic radial function $R(t_0)$ at the time of observation and the intrinsic power L of the emitter. Astronomers define a **luminosity distance** d_{L} by the equation

$$d_{\mathrm{L}} = \left(\frac{L}{4\pi F}\right)^{1/2} \qquad (52)$$

and report the value of d_{L} for a given star. Cosmologists convert d_{L} to $S(\chi)$ using the formula:

$$S(\chi) = \frac{d_{\mathrm{L}}}{(1+z)R(t_0)} \qquad (53)$$

The quantities d_{L} and $S(\chi)$ are the measures of distance to our standard candle of luminosity L. You should convince yourself that (52) and (53) taken together imply (51).

of a given type he knew their distance. He found a direct proportion between the average recession velocity and distance (Figure 11). He called this result the Redshift-Distance Law. We call it **Hubble's Law**, one of the major results of cosmology in the twentieth century:

$$v = H_0 d_{\mathrm{L}} \qquad \text{(nearby galaxies)} \qquad (54)$$

Hubble constant H_0

Here H_0 is called the **Hubble constant** and refers to its value at the present age of the Universe. The current value of the Hubble constant in units used by astronomers is

$$H_0 = 73 \pm 2\ \frac{\text{kilometer/second}}{\text{Megaparsec}} \qquad (55)$$

where one Megaparsec equals 3.26 million light years. Expressed in metric units, this has the value:

$$H_0 = (8.0 \pm 0.2) \times 10^{-27}\ \text{meter}^{-1} \qquad (56)$$

Robertson-Walker Law of Recession

Recession at great distance and great speed

What happens when we do not make the assumption that emitting galaxies are nearby? We use the Robertson-Walker metric to answer this question. Write the spacelike form of (17) for fixed angle ϕ.

$$d\sigma^2 = R^2(t)d\chi^2 - dt^2 = ds^2 - dt^2 \qquad (d\phi = 0) \qquad (57)$$

At fixed time t_1 this equation can be integrated to give the distance d:

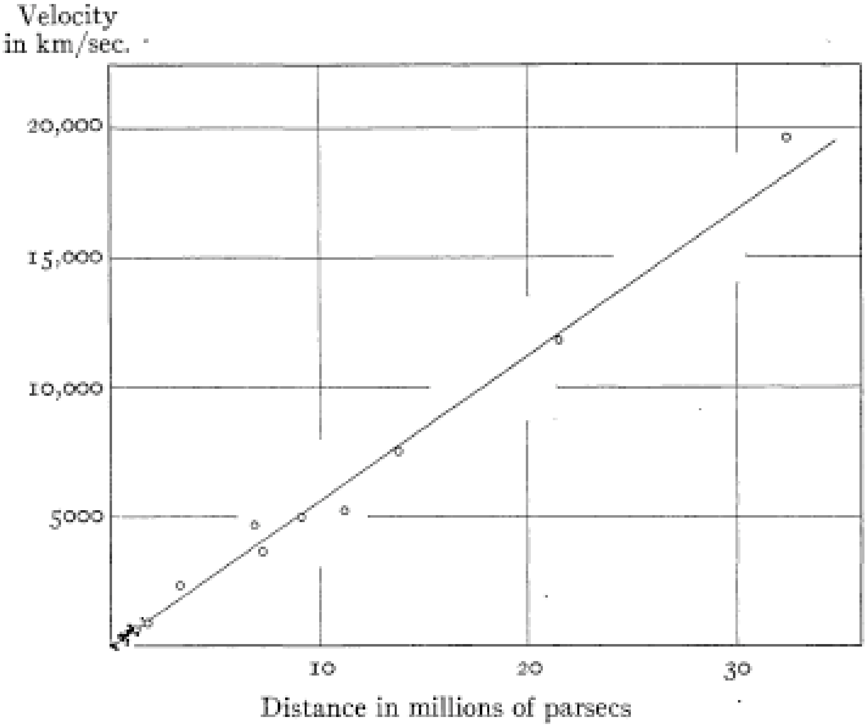


FIGURE 11 A plot of recession velocity as a function of distance by Hubble and Milton Humason (1931). Open circles represent averages of groups of galaxies; solid dots near the origin show individual galaxies from an earlier paper by Hubble. A parsec equals 3.26 light-years, so the most distant group of galaxies is approximately 100 million light-years distant—"nearby" by modern standards. The Hubble constant derived from the slope of the line in this figure is different from the current value, equation (55); see the exercises.

$$d_1 = R(t_1)\chi \qquad (dt = 0) \tag{58}$$

Assume that a distant galaxy is at rest in comoving coordinates χ (and ϕ), so that χ remains constant. Then at a later time t_2, the galaxy is at distance

$$d_2 = R(t_2)\chi \qquad (dt = 0) \tag{59}$$

The recession speed at time t is expressed using elementary calculus:

$$\begin{aligned} v_{\mathrm{r}} &= \lim_{t_2 \to\, t_1} \frac{d_2 - d_1}{t_2 - t_1} = \lim_{t_2 \to\, t_1} \frac{R(t_2) - R(t_1)}{t_2 - t_1}\chi \\ &\equiv \dot{R}\chi = \left(\frac{\dot{R}}{R}\right) R\chi \equiv H(t)d \end{aligned} \tag{60}$$

Hubble parameter

where the **Hubble parameter** $H(t)$ is defined as

$$H(t) \equiv \frac{\dot{R}(t)}{R(t)} \qquad \text{(Hubble parameter)} \qquad (61)$$

We can expect the Hubble parameter to have different values at different times t during the evolution of the Universe. Its value at the present time is given the symbol $H_0 \equiv H(t_0)$.

As noted in Section 6, astronomers cannot measure d directly. Instead they measure d_{L} or d_{A}. When either of these is plotted against redshift z, the resulting relation is linear only for $z \ll 1$. At high redshift the behavior depends on the detailed form of the radial function $R(t)$.

We need radial function $R(t)$.

We have milked about as much information out of the Robertson-Walker metric as we can without knowing the time development of the radial function $R(t)$, which derives from the constituents of the Universe as it expands. The following Project 10, Cosmology, develops this radial function from a combination of observed redshifts (28) using standard candles at different distances and further solutions of Einstein's equations. The result provides our current picture of the history of the Universe and gives us insight into its possible futures.

"Scientists confirmed today that everything we know about the structure of the universe is wrongedy-wrong-wrong."

8 ■ REFERENCES

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Final cartoon by Jack Ziegler in the New Yorker Magazine July 13, 1998. IF we use it, we will get formal permission.

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Figure 4 from From the Sloan Digital Sky Survey:
`http://www.sdss.org/includes/sideimages/quasar_stack.html`

Picture of Edwin Hubble from the cover of Time Magazine, February 9, 1948

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9 ■ EXERCISES

1. Tangential Momentum

Carry out the full derivation of the tangential momentum Q_ϕ in equation (41), including equations similar to (32) through (38) and a figure similar to Figure 7.

2. Energy not a Constant of Motion

Show that a derivation of the energy as a constant of motion is not possible. Begin by varying only the *time* of the central event in Figure 7. What derails this derivation, making it impossible to complete?

3. Transverse Motion

A galaxy is five billion light-years distant. The most sensitive microwave array can detect a displacement angle as small as 50 microarcseconds transverse to the radial direction of sight. (One second of arc is 1/3600 of a degree.) With what transverse speed, as a fraction (or multiple) of the speed of light, must the distant source move in order that its transverse motion be detected in a 100-year human lifetime? Assume the Universe is flat.

5. Hubble's Error

Compare the value of the slope in Figure 11 with the modern value of Hubble's constant given in equations (55) and (56). By what factor was Hubble's result different from the current value of the Hubble constant? [He is very wrong, and this will have to be explained—traced in history.]