CHAPTER 4
Global Positioning System
A Project

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There is no better illustration of the unpredictable payback of fundamental science than the story of Albert Einstein and the Global Positioning System [GPS]...the next time your plane approaches an airport in bad weather, and you just happen to be wondering "what good is basic science," think about Einstein and the GPS tracker in the cockpit, guiding you to a safe landing.

—Clifford Will

1 ■ OPERATION OF THE GLOBAL POSITIONING SYSTEM

Relativistic effects of altitude and speed on clock rates.

General relativity: Crucial to GPS functioning.

Do you think that general relativity concerns only events far from common experience? Think again! Your hand-held Global Positioning System (GPS) receiver "listens" to overhead satellites and tells you where you are at any place on Earth. In this project you show that the operation of the GPS system depends fundamentally on general relativity; as a practical matter the system is useless when general relativity effects are ignored (Box 2).

GPS satellite system

The Global Positioning System includes a network of 24 operating satellites, in circular orbits around Earth with orbital period of 12 hours, distributed in six orbital planes equally spaced in angle (Figure 1). Each satellite carries an operating atomic clock (along with several backup clocks) and emits a timed signal coded to tell the satellite's location. By analyzing signals from at least four of these satellites (Box 1), a receiver on Earth's surface with a built-in computer displays its location (latitude, longitude, and altitude). Consumer receivers provide a horizontal position accurate to approximately 5 meters. The GPS system has revolutionized driving, flying, hiking, exploring, rescuing, mapmaking, and the study of geological motions of Earth's crust, among many other applications.

The timing accuracy required by the GPS is so great that general relativistic effects are central to its performance: First, clocks run at different

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 Latest drafts at dropsite exploringblackholes.com, with an invitation for comments.

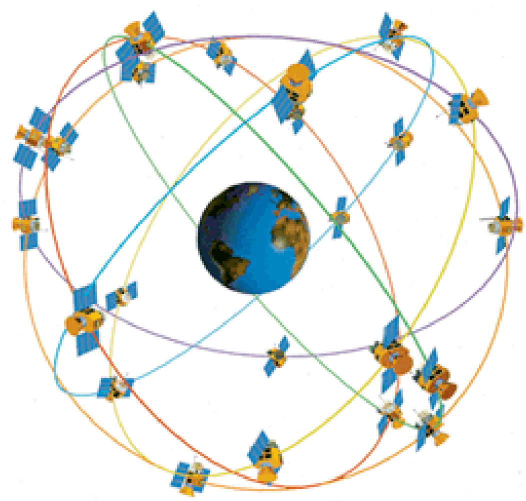

FIGURE 1 Schematic plot of GPS satellites in 12-hour orbits around Earth, not to scale. [From http://www.aero.org/education/primers/gps/whatisgps.html]

General relativity: Position and motion effects

global map rates when they are located at different distances from a center of gravitational attraction. Second, relative motion between the satellite and Earth's surface affect the results; neither the moving satellite nor Earth's surface corresponds to the stationary spherical shell described in Chapter 3, Curving. In this project you investigate these effects.

What is a **project**?

This chapter is a **project**, which differs from a standard chapter by including **QUERIES** throughout the chapter. Projects challenge you to respond to these queries. (QUERY 1 appears on page 6.) Typically, a Query contains several related questions. Answer the queries in order, or as assigned to you, or skip to those that interest you.

2 ■ STATIONARY CLOCKS

Warping of time at different altitudes.

Simplest case: clock on tower; no Earth rotation.

The Global Positioning System depends on the reception by a receiver at Earth's surface of microwave signals from overhead satellites. Begin with the simplest possible case: Earth does not rotate and the higher clock is not in a satellite but rather sits stationary at the top of a (very tall!) tower, at higher map radius r_{H}. The satellite communicates with us on Earth, at lower radius r_{L}, using microwaves, which move at light speed. Calculate the radially downward map velocity of a light signal moving from the top to the bottom of the tower. For these conditions, $d\phi = 0$ and for light, $d\tau = 0$. Then the Schwarzschild metric—Equation 5, page 2 of Chapter 3—yields the following radial map velocity of light:

Box 1. Practical Operation of the Global Positioning System

The goal of the Global Positioning System (GPS) is to determine your position on Earth in three dimensions: east-west, north-south, and vertical (longitude, latitude, and altitude). Signals from three overhead satellites provide this information. Each satellite emits a signal that encodes time of emission and the satellite's position at that time, this position being continually calculated using data uploaded from stations on the ground. The receiver clock times the reception of each signal, then subtracts the emission time to determine the time lapse and hence how far the signal has traveled (at the speed of light). This is the distance the satellite was from your position when it emitted the signal. In effect, the receiver constructs three spheres from these distances, one sphere centered on the emission point of each satellite. By simple triangulation, you stand at the point where the three spheres intersect.

Of course there is a wrinkle: The clock in your hand-held receiver is not nearly so accurate as the atomic clock carried in a satellite. For this reason, the signal from a fourth overhead satellite is employed to correct the clock in your receiver. This fourth signal enables the receiver to process GPS signals as though it contained an atomic clock. These precisely-timed satellite signals are also used directly in many scientific and practical applications.

Signals exchanged by atomic clocks at different altitudes and moving at different speeds are subject to general relativistic effects described using the Schwarzschild metric. Neglecting these effects makes the GPS useless (Box 2). This project analyzes these general relativistic effects.

$$\frac{dr}{dt} = -\left(1 - \frac{2M}{r}\right) \qquad \text{(light moving radially inward: } d\tau = 0,\ d\phi = 0) \qquad (1)$$

Map speed of light $\neq 1$.

Is equation (1) a surprise? For the first time in our study of relativity calculated light speed differs from one meter of distance per meter of time. Chapter 12, Seeing, examines this equation in detail. For now, note that this is a *map* velocity—expressed in our arbitrary Schwarzschild global coordinates—not a physical velocity measured directly by any local observer.

The following derivation makes use of the fundamental difference between map time (which is not measured directly by anyone) and two proper times: the wristwatch time lapse $d\tau_{\rm H}$ at the higher tower map radius $r_{\rm H}$ and the wristwatch time lapse $d\tau_{\rm L}$ at the lower Earth map radius $r_{\rm L}$. As usual, the metric converts from map time (on the right side of the metric) to wristwatch time (on the left side of the metric), which we can choose to be the time lapse between ticks on any clock of our choice.

Tower clock emits two downward flashes.

The clock at the top of the tower emits radially downward two flashes (emission events A and B) differentially close together in map time, $dt_{\rm AB}$. For the stationary tower clock, $dr = 0$ and $d\phi = 0$, the metric tells us the corresponding wristwatch time lapse $d\tau_{\rm H}$ recorded on the tower clock:

$$d\tau_{\rm H} = \left(1 - \frac{2M}{r_{\rm H}}\right)^{1/2} dt_{\rm AB} \qquad (d\phi = 0,\ dr = 0) \qquad (2)$$

Figure 2 traces the radially-downward worldlines of the two flashes emitted by the tower clock at events A and B. These flashes are received by the Earth clock at events C and D with map time separation $dt_{\rm CD}$ (reception events C and D). Equation (1) tells us that these worldlines have identical slopes (the radial map speed of light has the same value) at every intermediate value r of

Map time lapse between flashes same at all radii.

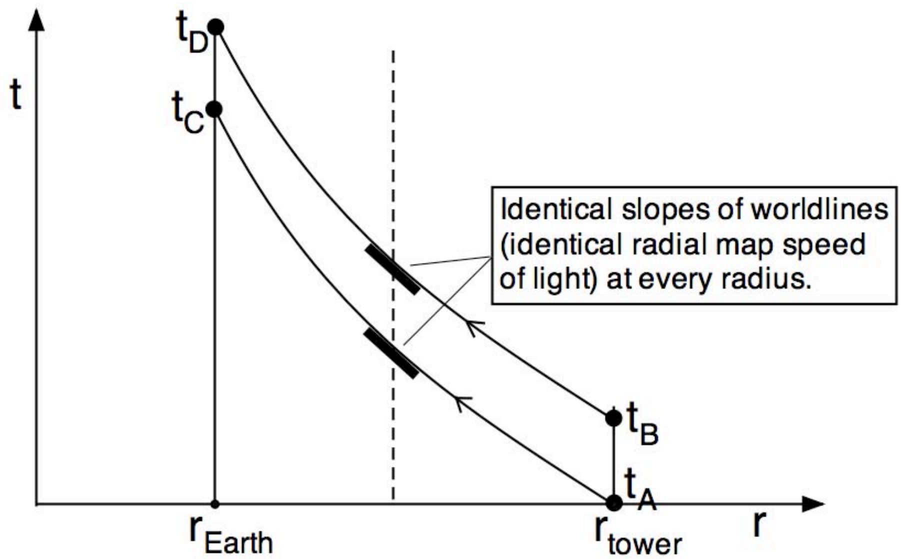

FIGURE 2 Schematic plot in Schwarzschild map coordinates t, r of worldlines of two sequential flashes moving downward from the top of a tower. According to equation (1) the map velocity depends only on map radius r. As a result, the *map* time difference between reception of the flashes is identical to the *map* time difference between emission of the flashes (when emitter and receiver are both at rest in map coordinates). However, the *wristwatch* time between the reception events C and D, measured by the bottom observer, is different from the *wristwatch* time between emission events A and B, measured by the top observer. The figure exaggerates the variation of map velocity of light with radius.

map radius. As a result, the two worldlines are parallel at every radius so the map time between them remains at the initial value $dt_{\rm AB}$. The two flashes arrive at the ground with the same difference in *map* time between them.

$$dt_{\rm CD} = dt_{\rm AB} \tag{3}$$

The lower Earth clock is also at a fixed, single value of the radius $r_{\rm L}$. Therefore its wristwatch time on the ground between the reception of events is similarly given by (2):

$$d\tau_{\rm L} = \left(1 - \frac{2M}{r_{\rm L}}\right)^{1/2} dt_{\rm CD} = \left(1 - \frac{2M}{r_{\rm L}}\right)^{1/2} dt_{\rm AB} \tag{4}$$

The final step in (4) comes from (3). Equations (2) and (4) give us the relation between wristwatch times at the higher tower map radius $r_{\rm H}$ and the lower Earth's map radius $r_{\rm L}$:

$$\frac{d\tau_{\rm H}}{d\tau_{\rm L}} = \left(\frac{1-\dfrac{2M}{r_{\rm H}}}{1-\dfrac{2M}{r_{\rm L}}}\right)^{1/2} \qquad \text{(stationary clocks)} \qquad (5)$$

Shell clock times between flashes different at different radii.

The lapse in Schwarzschild *map* time dt between flashes is the same at the locations of both clocks, but the *wristwatch* time is different as recorded on these different clocks: $d\tau_{\rm H} \neq d\tau_{\rm L}$. Indeed, $r_{\rm L} < r_{\rm H}$, so equation (5) tells us that $d\tau_{\rm H} > d\tau_{\rm L}$; the lapse of wristwatch time on the higher clock is greater than the lapse of wristwatch time on the lower clock. The everyday expression for this inequality is, "A clock at a greater altitude runs faster," in spite of the fact that each Earth and tower observer thinks (correctly!) that his clock runs at its normal rate.

Gravitational red and blue shifts

The wristwatch time lapse on the higher tower clock $d\tau_{\rm H}$ can be envisioned as the measured period $T_{\rm H}$ of a sinusoidal signal emitted from the top of the tower. The measured period $T_{\rm L}$ of the signal as it reaches Earth's surface is therefore observed to be smaller. Frequency is inversely proportional to period, so the observed frequency of the signal increases as it descends. This is called the **gravitational blue shift**. For a signal rising from the Earth's surface to be observed at the top of the tower, the period increases and the frequency decreases; this effect is labeled **gravitational red shift**.

3 ■ APPROXIMATIONS

How small is small?

General relativity effects are small.

The general relativistic effects we study are small. How small? Small compared to what? When *must* we use exact general relativistic expressions? And when we do, are approximations good enough? These questions are so central to the analysis of the GPS that it is useful to begin with a rough estimate of the expected effects, not worrying initially about the crudeness of this approximation. Assume that our stationary tower extends to the height of the GPS satellite and that the satellite rests without moving on the top of that tower ($v = 0$). (And remember that in this initial approximation we assume that the Earth does not rotate.) Write (5) in the form

$$\frac{d\tau_{\rm H}}{d\tau_{\rm L}} = \left(1-\frac{2M}{r_{\rm H}}\right)^{1/2}\left(1-\frac{2M}{r_{\rm L}}\right)^{-1/2} \qquad \text{(stationary clocks)} \qquad (6)$$

You will show in QUERY 7 that the radius of a 12-hour circular orbit is about 26.6×10^6 meters from Earth's center. Inside the front cover are values for the radius and mass of Earth. We now make use of an approximation also written inside the front cover:

$$(1+\epsilon)^n \approx 1 + n\epsilon + O(\epsilon^2) \quad \text{provided} \quad |\epsilon| \ll 1 \quad \text{and} \quad |n\epsilon| \ll 1 \qquad (7)$$

Our approximations are "to first order," that is we neglect the correction term $O(\epsilon^2)$.

QUERY 1: Clock rate difference due to difference in altitude.
Apply approximation (7) to the two parenthetical expressions on the right side of equation (6) with the last term omitted. Multiply out the result to show that to first order:

$$\frac{d\tau_{\mathrm{H}}}{d\tau_{\mathrm{L}}} \approx 1 - \frac{M}{r_{\mathrm{H}}} + \frac{M}{r_{\mathrm{L}}} - \frac{M}{r_{\mathrm{H}}}\frac{M}{r_{\mathrm{L}}} \qquad (v = 0,\ \text{nonrotating Earth}) \tag{8}$$

What are the approximate values of M/r_{Earth} and $M/r_{\mathrm{satellite}}$? Make an argument that the last term on the right side of (8) can be neglected in comparison with the other terms on the right.

QUERY 2: Numerical approximation, stationary clocks.
In the following equation, b stands for all of the terms added to the number one on the right side of equation (8). Substitute numbers into equation (8) and find the numerical value of b:

$$\frac{d\tau_{\mathrm{H}}}{d\tau_{\mathrm{L}}} \approx 1 + b \qquad (v = 0,\ \text{nonrotating Earth}) \tag{9}$$

Small fractional differences in clock rates affect GPS operation.

The number represented by b in equation (9) is an estimate of the fractional difference in rates of signals between stationary clocks at the position of the satellite and at Earth's surface. Is this fractional difference negligible or important to the operation of the GPS? Suppose the timing of a satellite clock is off by 1 nanosecond (10^{-9} second). In 1 nanosecond a light signal (or microwave pulse) propagates approximately 30 centimeters, or about one foot. So a difference of, say, hundreds of nanoseconds will render GPS results inaccurate if we need a location precision of ten meters or so.

QUERY 3: Synchronization discrepancy after one day.
As long as Earth and satellite clocks do not move, the time increments in equation (9) can be as long as we want, leading to the equation

$$\tau_{\mathrm{H}} \approx (1 + b)\,\tau_{\mathrm{L}} \qquad (v = 0,\ \text{nonrotating Earth}) \tag{10}$$

There are approximately 86 400 seconds in one day. (Since this is approximate, which clock records it does not matter.) To an accuracy of one significant digit, the satellite clock and Earth clock go out of synchronism by about 50 000 nanoseconds per day due to their difference in altitude alone. Find the correct value to three-digit accuracy.

The satellite clock will "run fast" by something like 50 000 nanoseconds per day compared with the clock on Earth's surface due to position effects alone. Clearly general relativity is needed for useful operation of the Global

Positioning System, even though the *fractional* difference between clock rates at the two locations (at least the fraction due to difference in radius) is small.

In addition to the effect of position, we must include the effect due to relative motion of satellite and Earth observer. Which way will these effects influence the discrepancy introduced by general relativity: to increase it or decrease it? The satellite clock moves faster than the clock revolving with Earth's surface. Special relativity tells us that (in an imprecise summary), "Speedy clocks run slow." Therefore we expect the effect of motion to *reduce* the amount by which the satellite clock runs fast compared to the Earth clock. In brief, when velocity effects are taken into account, we expect the satellite clock to run faster than the Earth clock by *less* than the estimated 50 000 nanoseconds per day. In Query 8, you check your final result against this prediction.

Velocity effects reduce position effects.

4 ■ MOVING CLOCKS

Relative velocity changes relative clock rates

Now we take account of the effects of motion on the relative rates of Earth clock and satellite clock. Think of the Earth clock as fixed at the equator, so that it moves in a circle as the Earth turns. The satellite clock also circles the Earth, but in its own independent circular orbit, so $dr = 0$ for each clock. In each case $d\tau$ is the wristwatch time between ticks, the time recorded by the given clock. Set $dr = 0$ in the Schwarzschild metric and divide through by the square of the Schwarzschild map time dt^2 to obtain, for either clock in its orbit of radius r,

Add circular motions of Earth and satellite

$$\left(\frac{d\tau}{dt}\right)^2 = \left(1 - \frac{2M}{r}\right) - r^2\left(\frac{d\phi}{dt}\right)^2 = \left(1 - \frac{2M}{r}\right) - v^2 \qquad (\Delta r = 0) \qquad (11)$$

Here $d\tau$ is the wristwatch time between ticks of either clock and $v = rd\phi/dt$ is the instantaneous tangential map velocity of that clock moving at its radius r.

QUERY 4: Clock rate correction formula.
First apply equation (11) to the satellite clock, then apply (11) to the Earth clock. Divide the two sides of the satellite equation by the corresponding sides of the Earth equation. Take the square root of both sides of the result. For both numerator and denominator in the resulting equation, use the approximation (7). In the numerator, set

$$\epsilon_{\mathrm{H}} = -\frac{2M}{r_{\mathrm{H}}} - v_{\mathrm{H}}^2 \qquad \text{(H means satellite)} \qquad (12)$$

Now do the same for the denominator. In the denominator the formula for ϵ_{L} is the same but with L for "lower" as subscripts. Carry out an analysis similar to that in QUERY 1 to retain only the important terms. Show that the result is

$$\frac{d\tau_{\rm H}}{d\tau_{\rm L}} \approx 1 - \frac{M}{r_{\rm H}} - \frac{v_{\rm H}^2}{2} + \frac{M}{r_{\rm L}} + \frac{v_{\rm L}^2}{2} \qquad \text{(satellite directly overhead)} \qquad (13)$$

Newton orbits good enough for GPS analysis.

Now we need numerical values for the quantities on the right hand side of (13). Chapter 9, Orbiting, derives the velocity of a satellite in circular orbit according to general relativity. After completing that chapter you will be able to verify that the following Newtonian description of the radius of any orbit and the satellite speed in that orbit is accurate enough to provide orbital descriptions for the Global Positioning System around Earth. In contrast, the relative differences in clock rates at different orbit radii and satellite speeds are relativistic effects, as we have seen.

QUERY 5: Speed of a clock on the equator.
Earth's center is in free fall as Earth orbits the Sun. The Earth also rotates on its axis, completing one full rotation with respect to the distant stars in what is called a **sidereal day**, which is 86 164.1 seconds long. (Even when we require 6-digit accuracy, which of our clocks measures this time does not matter.) With respect to Earth's center, what is the speed v of a clock at rest on Earth's surface at the equator? Use Newtonian "universal time" t. Express your answer as a fraction of the speed of light.

What is the value of the speed v of the satellite? Newton tells us that the acceleration of a satellite in a circular orbit is directed toward the center and has the magnitude v^2/r, where v is measured in conventional units, such as meters per second. The satellite's mass m multiplied by this acceleration must equal Newton's gravitational force exerted by Earth:

$$\frac{GmM}{r^2} = \frac{mv^2}{r} \qquad \text{(Newton, conventional units)} \qquad (14)$$

Newtonian orbit analysis

Equation (14) provides one relation between the velocity of the satellite and the radius of its circular orbit. A second relation connects satellite velocity and orbit radius to the period of revolution. This period T is 12 hours for GPS satellites:

$$v = \frac{2\pi r}{T} \qquad \text{(Newton. conventional units)} \qquad (15)$$

QUERY 6: Geometric Units
Convert equations (14) and (15) to geometric units: Earth mass M and satellite orbital period T to meters; satellite speed v to the unitless fraction of the speed of light. Then eliminate the radius r between these two equations to find an expression for v in terms of M and T and numerical constants.

5 ■ THE FINAL RECKONING

Effects of altitude AND relative speed on clock rates

GPS SATELLITE COMING OR GOING? Doppler Shifts

Is the GPS satellite approaching the ground receiver or receding from it? If the satellite approaches, the receiver detects a Doppler blueshift of the incoming signal. In contrast, a Doppler redshift occurs when the satellite recedes from the Earth receiver. In this project we carry out calculations for the instant at which the satellite is instantaneously positioned directly above the Earth receiver. In this case the change in the detected signal frequency is due to the relative tangential motion between the two clocks and to their different altitudes. For other relative motions of satellite and receiver, the computer in the hand-held receiver calculates the anticipated Doppler shift and takes it into account in the derived location of the receiver.

QUERY 7: Satellite radius and speed, according to Newton.
Find the numerical value of the speed v (as a fraction of the speed of light) for a satellite in a 12-hour circular orbit. Find the numerical value of the radius r for this orbit—according to Newton and Euclid.

Now we have numerical values for all the terms in equation (13) and can estimate the difference in rates for satellite clocks and Earth clocks.

QUERY 8: Clock rate correction, numerical
Substitute values for the various quantities in (13). To one significant digit the satellite clock appears to run faster than the Earth clock by 40 000 nanoseconds per day. Give your answer to three significant digits.

Section 3 described the difference in clock rates due only to difference in altitude. We predicted at the end of Section 3 that the full general relativistic treatment would lead to a *smaller* difference in clock rates than reckoned for the altitude effect alone. Your result for QUERY 8 verifies this prediction.

Gravitational shifts: no single identifiable cause

An historical aside: Carroll O. Alley, a consultant to the original GPS project, had a hard time convincing the designers not to apply *twice* the correction given in (13): first to account for the difference in clock rates at different altitudes and second to allow for the blue shift in frequency for the signal sent downward from satellite to Earth. There is only one correction; moreover there is no way to identify uniquely what is the “cause” of this correction. Listen to what Clifford Will says about the difference in rates

Box 2. A General Relativity On/Off Switch

Before the first GPS satellite went into orbit, the military who were in charge of the system, were suspicious of the predictions of the academics concerning general relativistic effects on the relative rates of clocks at different altitudes and speeds. To allow for the possibility that general relativity is wrong, the first GPS satellite carried a general relativity on/off switch, set initially to *off*. This setting made no adjustment in the relative rates of satellite clock and Earth clocks.

When the satellite was placed in orbit, the satellite clock drifted with respect to clocks on Earth at the rate predicted by general relativity, within the accuracy of the clocks. This relative clock drift soon made the satellite clock useless for the operation of the GPS. The general relativity switch was then flipped to *on*, and the system operated as designed. Later GPS satellites operate under the assumption that the general relativity corrections of equation (13) always apply.

between one clock emitting flashes from the top of a tower and a second clock receiving the flashes on the ground:

> *A question that is often asked is, Do the intrinsic rates of the emitter and receiver or of the clock change, or is it the light signal that changes frequency during its flight? The answer is that it doesn't matter. Both descriptions are physically equivalent. Put differently, there is no operational way to distinguish between the two descriptions. Suppose that we tried to check whether the emitter and the receiver agreed in their rates by bringing the emitter down from the tower and setting it beside the receiver. We would find that indeed they agree. Similarly, if we were to transport the receiver to the top of the tower and set it beside the emitter, we would find that they also agree. But to get a gravitational red shift, we must separate the clocks in height; therefore, we must connect them by a signal that traverses the distance between them. But this makes it impossible to determine unambiguously whether the shift is due to the clocks or to the signal. The observable phenomenon is unambiguous: the received signal is blue shifted. To ask for more is to ask questions without observational meaning. This is a key aspect of relativity, indeed of much of modern physics: we focus only on observable, operationally defined quantities, and avoid unanswerable questions.*

The conflict described by Clifford Will asserts itself in our different treatments of clock rates at different radii. Box 6 of Chapter 3, Curving, started with equal wristwatch times: $d\tau_{\rm H} = d\tau_{\rm L} = 1$ second, and derived different map times: $dt_{\rm H} \neq dt_{\rm L}$. In contrast, Section 2 of the present chapter notes that the map time lapse between two radially-directed signals does not change as the signals travel between locations: $dt_{\rm H} = dt_{\rm L}$ and from this derives a difference in clock time lapses: $d\tau_{\rm H} \neq d\tau_{\rm L}$. You can show that both methods lead to the same numerical result.

TWO NOTES

NOTE 1. Newtonian orbit radius OK. The approximate analysis in this project assumed that the radius r_{H} of the circular orbit of the satellite and the velocity v of the satellite in that orbit are both correctly computed using Newtonian mechanics. Chapter 9, Orbiting, carries out the Schwarzschild analysis of circular orbits. When you have completed that chapter, you will be able to show that the Newtonian orbit radius and velocity are sufficiently accurate for our purposes in the present Project.

NOTE 2. No latitude effect. Our analysis assumed the speed v of the Earth clock to be that of the speed of the equator. One might expect that this speed-dependent correction would take on different values at different latitudes north or south of the equator, going to zero at the poles where there is no motion of the Earth clock due to rotation of Earth. In practice there is no latitude effect because Earth is not spherical; it bulges a bit at the equator due to its rotation, like a squashed balloon. The smaller radius at the poles increases the M/r_{L} term in (13) by the same amount that the velocity term decreases it. The outcome is that our calculation for the equator applies to all latitudes.

6 ■ SOME APPLICATIONS

Continental drift: millimeters per year.

To be written later, along with a summary.
See government GPS website: `http://www.gps.gov/`
Wikipedia description: `http://en.wikipedia.org/wiki/GPS`

7 ■ REFERENCES

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