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Chapter 15. SEEING
Final Views

- *In which direction(s) does a rain observer look to see a given star?*
- *How does the panorama of the heavens change for the rain observer as she descends?*
- *Is starlight lethal for the rain observer? If so, where on her trip?*
- *How close to the singularity will the rain observer survive?*
- *What is the last thing that the rain observer sees?*

CHAPTER

15

SEEING
Final Views

Edmund Bertschinger & Edwin F. Taylor *

What mystery pervades a well!
The water lives so far -
A neighbor from another world
Residing in a jar

Whose limit none have ever seen,
But just his lid of glass -
Like looking every time you please
In an abyss's face!
. . . . . .

But nature is a stranger yet;
The ones that cite her most
Have never passed her haunted house,
Nor simplified her ghost.

To pity those that know her not
Is helped by the regret
That those who know her, know her less
The nearer her they get.

—Emily Dickinson

15.1 ■ SHELL AND RAIN OBSERVERS VIEW THE SAME BEAM

See the same beam in two different directions.(#secOrbitView)

Goal: Stress-free life in a local inertial frame.

To live on a shell near a black hole is stressful. The local shell frame, dropped from rest to make it inertial, can only be temporary because it soon falls to a lower radius. To stand for longer times on a shell near the black hole torments our mostly-liquid bodies with the crushing gravitational acceleration (Section 6.6). We know how to escape this stress: Just let go! The present chapter describes visual observations from a local frame that is inertial for an extended

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period: the rain frame that drops from rest starting at a great distance from the black hole and falls radially to its center.

Typically, shell and rain observers see a single beam coming from different directions.

Chapter 12 told us which beams—beams with which values of b^*—connect a given distant star to the map location of an observer. In particular, we focused on the primary beam, the most direct beam from distant star to observer location. Chapter 14 told us in what direction a *shell* observer looks to see that beam. The present chapter finds the direction in which the *rain* observer looks to see that *same* beam. We will find that the rain viewing direction is typically quite different from the shell viewing direction, even though both observers look at the same beam. With a little intuition gleaned from special relativity—namely that observers in relative motion view the same events differently—we will not be too surprised.

Objection 15.1. *Didn't we already go "Inside the Black Hole" in Chapter 7? What's new that justifies a second dive to oblivion?*

We did not plot global trajectories of light until Chapter 12, so in Chapter 7 we could not predict the rain observer's panorama of the heavens in all directions, just her view inward and outward along the radial line, whose inward and outward directions are the same in Schwarzschild coordinates, global rain coordinates, the local shell frame, and the local rain frame.

15.2 ■ RAIN FRAME VIEW OF DISTANT STARS AND OF THE BLACK HOLE

Enjoy the view; put off the catastrophe.(#sec:RainView)

Ride the rain frame through the event horizon.

Chapter 14 described the panorama seen by a shell spectator. But shells cannot exist inside the event horizon, and we want to dive on inward to the singularity. From the infinite number of possible free-fall frames, we choose the simplest, the *rain frame* (Section 7.2), which drops from rest at a great distance and falls radially inward along the $\phi = 0$ axis where we have placed our shell observers. Equation (7.40) assures us that when we ride in a rain frame, tidal forces do not trouble us until the final 2/9 second (on our wristwatch) before arrival at the singularity.

Like the shell observer, the rain observer records observations on her personal planetarium, identical in design to the one for the shell observer in Figure 14.1. Her personal planetarium remains stationary in the rain frame as she descends. Her panorama of distant stars and view of the black hole both change dramatically as she plummets inward. We now assemble the tools to describe her startling vision. In Query 15.1 you derive local rain frame coordinates from the analysis in Chapter 7.

QUERY 15.1. Local rain frame coordinates. (#query29)

Find local rain frame coordinates. The second of equations (7.5) uses the Lorentz transformation to derive the expression for $\Delta y^*_{\text{rain}} \equiv y_{\text{rain}}/M$ in terms of Schwarzschild map coordinates and increments: (#Deltayrain)

$$\Delta y^*_{\text{rain}} = \left(1 - \frac{2}{\bar{r}^*}\right)^{-1} \Delta r^* + \left(\frac{2}{\bar{r}^*}\right)^{1/2} \Delta t^* \tag{15.1}$$

A. From equation (7.18) derive the following expression for an increment Δt^* of Schwarzschild map time in terms of increments of global rain coordinates: (#RainDiff)

$$\Delta t^* = \Delta T^* - \left(\frac{2}{\bar{r}^*}\right)^{1/2} \left(1 - \frac{2}{\bar{r}^*}\right)^{-1} \Delta r^* \tag{15.2}$$

Combine equations (15.1) and (15.2) to show that the local rain frame increment Δy^*_{rain} is: (#DeltayrainB)

$$\Delta y^*_{\text{rain}} = \Delta r^* + \left(\frac{2}{\bar{r}^*}\right)^{1/2} \Delta T^* \tag{15.3}$$

B. From the *definition* of rain time T^* in Section 7.4, show that (#raintime)

$$\Delta t^*_{\text{rain}} \equiv \Delta T^* \tag{15.4}$$

C. Finally, from the equality of distances measured perpendicular to the direction of relative motion in the Lorentz transformations of special relativity, show that, for a rain inertial frame, (#rainx)

$$\Delta x^*_{\text{rain}} = \Delta x^*_{\text{shell}} = \bar{r}^* \Delta \phi \tag{15.5}$$

Equations (15.3) through (15.5) express local rain frame coordinate increments in terms of global rain coordinates and their increments. Coefficients in these equations are real everywhere except at the singularity; so local rain coordinates are valid everywhere else, both inside and outside the event horizon.

Next find a relation between the direction of a star θ_{rain} seen by the rain observer at radius r^* and the b^*-value of the light beam from a particular star that arrives at that radius. Sample Problem 15.1 carries out that task, and Figure 15.3 plots the results.

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Objection 15.2. *Why can't we just use equations (1.55) and (1.57) to carry out a special relativity Lorentz transformation from the known viewing angles in the local inertial shell and the desired viewing angles in the overlapping local inertial rain frame?*

We can if we limit ourselves to observations outside the event horizon. (See exercise at the end of this chapter.) But no stationary shell can exist inside the event horizon, so there we cannot transform from a shell viewing angle to a rain viewing angle.

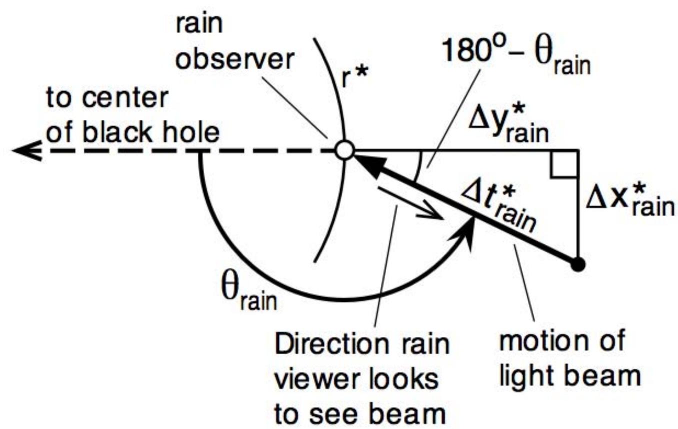

FIGURE 15.1 Sample Problem 15.1: A beam reaches the rain observer as she dives past the shell of radius r^*. The rain observer sees the beam by looking at the angle $\theta_{\rm rain}$, in the direction *opposite* to the direction of motion of the beam.

(#LaserRain)

SAMPLE PROBLEM 15.1. Relate $\theta_{\rm rain}$ to Impact Parameter b^*.

The rain observer falling past radius r^* sees a beam from a distant star at the angle $\theta_{\rm rain}$ with respect to the radially inward direction. Find an expression for $\theta_{\rm rain}$ in terms of r^* and the b^*-value of that light beam.

SOLUTION

This solution follows the same outline as Sample Problem 14.1, but is more complicated. Begin with Figure 15.1. The incoming beam in Figure 15.1 represents any one of the multiple beams arriving at the observer from a single star. Light moves with speed one in the local inertial rain frame, $\Delta s^*_{\rm rain}/\Delta t^*_{\rm rain} = \Delta s_{\rm rain}/\Delta t_{\rm rain} = 1$, so (#lightspeedrain)

$$(\Delta x^*_{\rm rain})^2 + (\Delta y^*_{\rm rain})^2 = (\Delta s^*_{\rm rain})^2 = (\Delta t^*_{\rm rain})^2 \qquad \text{(light)} \qquad (15.6)$$

Use Figure 15.1 and equations (15.4) through (15.6) to calculate $\sin\theta_{\rm rain}$ and $\cos\theta_{\rm rain}$. In Figure 15.1 the light moves inward and upward, so $\Delta x_{\rm rain}$ is positive: (#sinrain)

$$\sin\theta_{\rm rain} = \sin(180^\circ - \theta_{\rm rain}) = \lim_{\Delta t^*_{\rm rain}\to 0} \frac{\Delta x^*_{\rm rain}}{\Delta t^*_{\rm rain}} = \frac{r^* d\phi}{dT^*} \qquad (15.7)$$

To calculate $\cos(180^\circ - \theta_{\rm rain})$ in Figure 15.1 we need a positive value of $\Delta y^*_{\rm rain}$. But the light moves inward in Figure 15.1, making $\Delta y_{\rm rain}$ negative. So put a minus sign in front of this symbol when we substitute from equation (15.3):(#waytodrdTB)

$$\cos(180^\circ - \theta_{\text{rain}}) = \lim_{\Delta t^*_{\text{rain}} \to 0} \frac{-\Delta y^*_{\text{rain}}}{\Delta t^*_{\text{rain}}} = -\frac{dr^*}{dT^*} - \left(\frac{2}{r^*}\right)^{1/2} \tag{15.8}$$

From (15.8) plus a trigonometric identity:(#waytodrdTC)

$$\cos\theta_{\text{rain}} \equiv -\cos(180^\circ - \theta_{\text{rain}}) = \frac{dr^*}{dT^*} + \left(\frac{2}{r^*}\right)^{1/2} \tag{15.9}$$

Rederive $b = L/E$, equation (12.3), from expressions for E/m and L/m in global rain coordinates. Equation (7.24) gives the energy of a free stone in unitless coordinates: (#Erain).

$$\frac{E}{m} = \left(1 - \frac{2}{r^*}\right)\frac{dT^*}{d\tau^*} - \left(\frac{2}{r^*}\right)^{1/2}\frac{dr^*}{d\tau^*} \qquad \text{(stone: global rain coordinates)} \tag{15.10}$$

Box 9.1 demonstrates that the expression for angular momentum of a stone is the same in global rain coordinates as it is in Schwarzschild coordinates: (#Lrain)

$$\frac{L^*}{m} \equiv r^{*2}\frac{d\phi}{d\tau^*} \qquad \text{(stone: global rain coordinates)} \tag{15.11}$$

From (15.11) and (15.10), derive $b = L/E$ for light: (#LoverE)

$$b^* = \frac{L^*}{E} = \frac{r^{*2}d\phi}{\left(1 - \frac{2}{r^*}\right)dT^* - \left(\frac{2}{r^*}\right)^{1/2}dr^*} \qquad \text{(light: rain map coordinates)} \tag{15.12}$$

Solve (15.7) for $r^*d\phi$, substitute into (15.12), and rearrange:(#rbsin)

$$\frac{r^*\sin\theta_{\text{rain}}}{b^*} = \left(1 - \frac{2}{r^*}\right) - \left(\frac{2}{r^*}\right)^{1/2}\frac{dr^*}{dT^*} \tag{15.13}$$

From equation (15.9) we have: (#drdtasterisk)

$$\frac{dr^*}{dT^*} = -\left(\frac{2}{r^*}\right)^{1/2} + \cos\theta_{\text{rain}} \tag{15.14}$$

Substitute this expression into (15.13), put $\cos\theta_{\text{rain}}$ on one side, square both sides, convert to $\sin\theta_{\text{rain}}$ using the identity $1 - \sin^2\theta_{\text{rain}} = \cos^2\theta_{\text{rain}}$, and rearrange the results to obtain a quadratic equation in $\sin\theta_{\text{rain}}$ in standard form: (#quadinsine)

$$\left(\frac{r^{*2}}{b^{*2}} + \frac{2}{r^*}\right)\sin^2\theta_{\text{rain}} - \frac{2r^*}{b^*}\sin\theta_{\text{rain}} + \left(1 - \frac{2}{r^*}\right) = 0 \tag{15.15}$$

Solve for $\sin\theta_{\text{rain}}$: (#sinetheta)

$$\boxed{\sin\theta_{\text{rain}} = \frac{b^*r^{*2} \pm b^*\left(2r^{*3} - 2r^*b^{*2} + 4b^{*2}\right)^{1/2}}{r^{*3} + 2b^{*2}}} \tag{15.16}$$

On the right-hand side the upper (plus) sign corresponds to incoming beams and the lower (minus) sign to outgoing beams.

Equation (15.16) alone is ambiguous, because $\sin\psi = \sin(180^\circ - \psi)$ for an arbitrary value of ψ. So derive a supplementary equation for $\cos\theta_{\text{rain}}$ as follows: Substitute for

$\sin\theta_{\rm rain}$ from (15.16) into (15.13) and solve for dr^*/dT^*. Substitute the resulting expression for dr^*/dT^* into (15.14), with the result: (#costheta)

$$\cos\theta_{\rm rain} = \left(\frac{r^*}{2}\right)^{1/2}\left[\frac{2b^{*2} \mp r^*\left(2r^{*3} - 2r^*b^{*2} + 4b^{*2}\right)^{1/2}}{r^{*3} + 2b^{*2}}\right] \qquad (15.17)$$

The upper (minus) sign corresponds to incoming beams and the lower (plus) sign to outgoing beams.

A black hole is symmetric with respect to a radial line. Therefore equations (15.16) and (15.17) must give the same results whether the rain observer looks right or left with respect to the radially inward direction. When $\theta_{\rm rain}$ reverses sign, b^* also reverses sign, so $\sin\theta_{\rm rain}$ in (15.16) is identical whether the observer looks to the right or to the left. The impact parameter b^* appears only squared in (15.17), and the cosine function has the same sign for positive and negative $\theta_{\rm rain}$. *More:* The cosine function gives a unique value of the angle in the range between $0^\circ \leq \theta_{\rm rain} \leq 180^\circ$, so we need only the cosine function (15.17) to determine rain observation angles uniquely.

Figure 15.2 plots the results of (15.17) for starlight. The display along the top is the shell speed of the rain frame, from (6.22): (#vshellspeed)

$$v_{\rm shell} = \left(\frac{2}{r^*}\right)^{1/2} \qquad \text{(shell speed of rain diver, } r^* \geq 2) \qquad (15.18)$$

Along the bottom is the unitless wristwatch time τ^* to the center at various radii, from (7.13): (#timecenter)

$$\tau^*[r^* \rightarrow 0] = \frac{\left(2r^{*3}\right)^{1/2}}{3} \qquad \text{(rain diver remaining time to center)} \qquad (15.19)$$

The rain observer sees the same visual distortions on all sides of its radial direction of motion. Therefore we can rotate the plane of Figure 15.3 around its horizontal axis to construct our three-dimensional view of the heavens.

Rain observer's 3-dimensional view of the heavens

Figure 15.3 tells us that just before she reaches the singularity at $r^* = 0$, every curve with $|b^*| < \pm b^*_{\rm critical}$ converges on one of the two points $\theta_{\rm rain} = \pm 90^\circ$. *Meaning:* All images of every star in the heavens appear in a bright ring at 90°, perpendicular to her direction of motion, a ring that bisects her sky. For her the black hole covers the forward half of sky and behind her the sky is also black, empty of stars.

QUERY 15.2. Compare rain and shell observer images. (#query2211)
Use Figures 15.2, 14.3, and 12.10 to answer the following questions.

A. Compare the angular size of the black hole seen by rain observers at radii $r^* = 6,\ 4,\ 3,\ 2$ with its angular size seen by shell observers at the same radii.
B. Compare the angular direction of the primary image of a star at $\phi_\infty = 90^\circ$ seen by the rain observer as she sequentially passes $r^* = 6,\ 4,\ 3,\ 2$ with its angular direction seen by shell observers at the same radii.

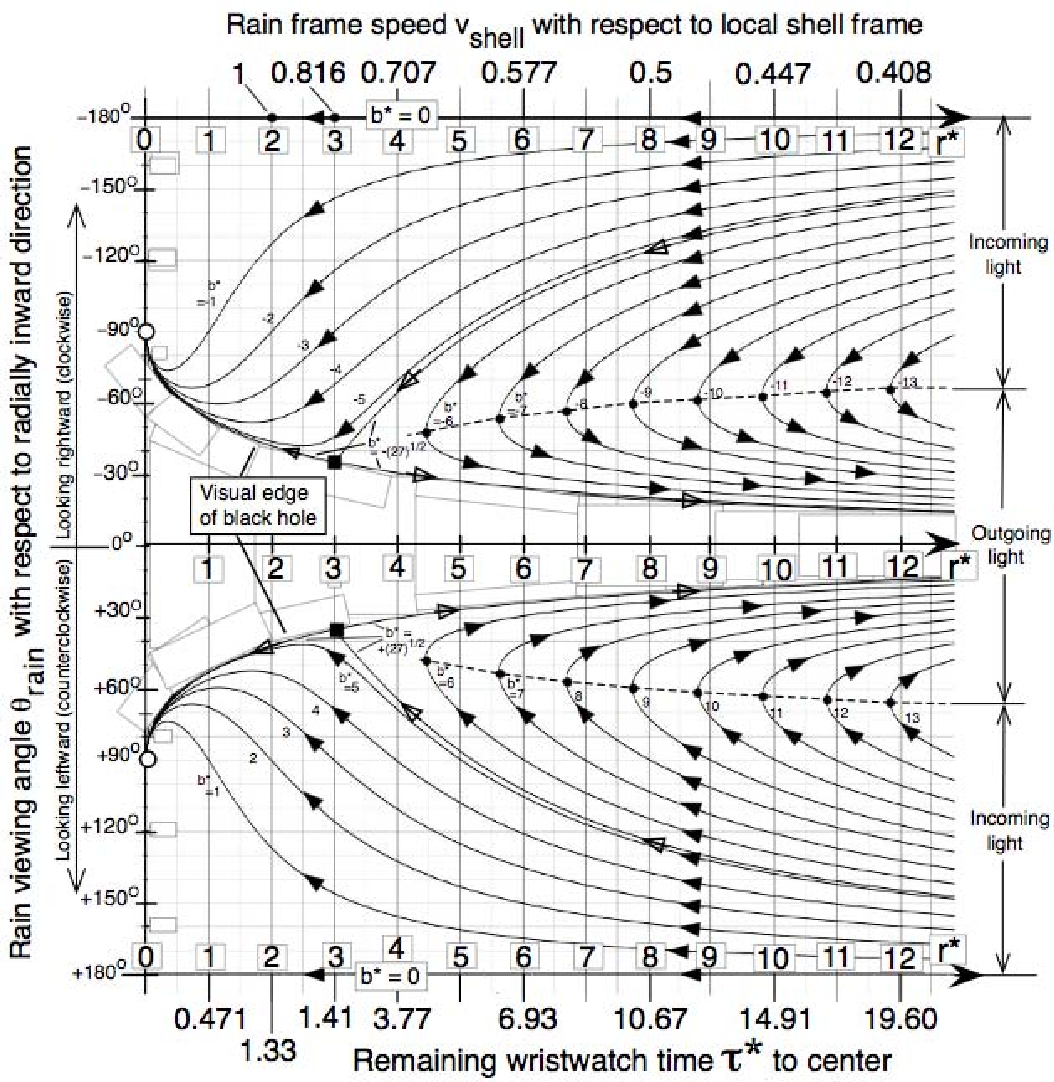

FIGURE 15.2 The angle $\theta_{\rm rain}$ with respect to the radially inward direction in which a rain viewer falling past radius r^* sees starlight beams with different values of impact parameter b^*, from equation (15.17). Boxed integers along top, middle, and bottom axes show the radius r^*. The two little black squares show $\pm b^*_{\rm critical}$ beams on the tangential light sphere. Note the forward shift of viewing angles compared with shell viewing angles at the same radius in Figure 14.6.

(#RainViewingStars)

QUERY 15.3. Aberration. (#query4411)

A. In Figure 14.3, find the turning point for the beam with $b^* = -9$. At what angle does the shell observer look to see this beam? At what radius is this shell observer? Now look at the turning point of the same beam in Figure 15.2. At what angle does the rain observer look to see that beam as she passes the same radius? Is this viewing angle farther forward in her direction of motion or farther backward than the shell viewing angle of the same beam?

B. You stand still on a road and notice that rain is coming down vertically in your frame. A car speeds past you. For the driver of the car, does the rain come toward him from a direction that is farther forward in his direction of motion or farther backward than it does for you on the ground?

Comment: Light does not move in the same way as raindrops do, but the effects described in Item A and Item B are qualitative similar to one another. These effects are called **aberration**.

C. The rain observer described in Item A sees a beam directly overhead, at $\theta_{\text{rain}} = 90°$. At what angle does the shell observer at the same radius look to see this beam?

QUERY 15.4. Turning Point Radius (#query2110)

A. Show that when the square root expression in equations (15.16) and (15.17) equals zero, the radius is the turning point r^*_{tp}, located by equation (12.20).

B. Use (12.20) a second time to show that at the turning point the common denominator in (15.16) and (15.17) has the value $r^{*3}_{\text{tp}} + 2b^{*2} = b^{*2} r^*_{\text{tp}}$.

C. Show that Items A and B imply that $\sin\theta_{\text{rain, tp}} = r^*_{\text{tp}}/b^*$ and $\cos\theta_{\text{rain, tp}} = (2/r^*_{\text{tp}})^{1/2}$. Use (12.20) a third time to check the identity $\sin^2\theta_{\text{rain, tp}} + \cos^2\theta_{\text{rain, tp}} = 1$ for these expressions.

D. Use the results of Item C to spot-check the angle $\theta_{\text{rain,tp}}$ in Figure 15.2 for representative values of b^* and r^*_{tp}.

E. In Item C, it looks like $\cos\theta_{\text{rain, tp}} \to \infty$ as $r^*_{\text{tp}} \to 0$ and $\sin\theta_{\text{rain, tp}} \to \infty$ as $r^*_{\text{tp}} \to \infty$. How can values of a sine and cosine be greater than one? Explain why this is not a problem.

F. From Query 13.4, show that turning point radius r^*_{tp} for each value of b^* is the same for local shell viewers as for local rain frame viewers.

QUERY 15.5. Star images near the singularity (#query2116)
Look at the left column of sequential rain observer views in Figure 15.7. Notice that at $r^* = 5$ and $r^* = 2.13$ star images D, E, and F are farther forward toward the black hole than they are in the final view at $r^* = 1/4$. Show how Figure 15.2 predicts this effect, and also predicts that the images retreat to $\theta_{\text{rain}} \to 90°$ as $r^* \to 0$.

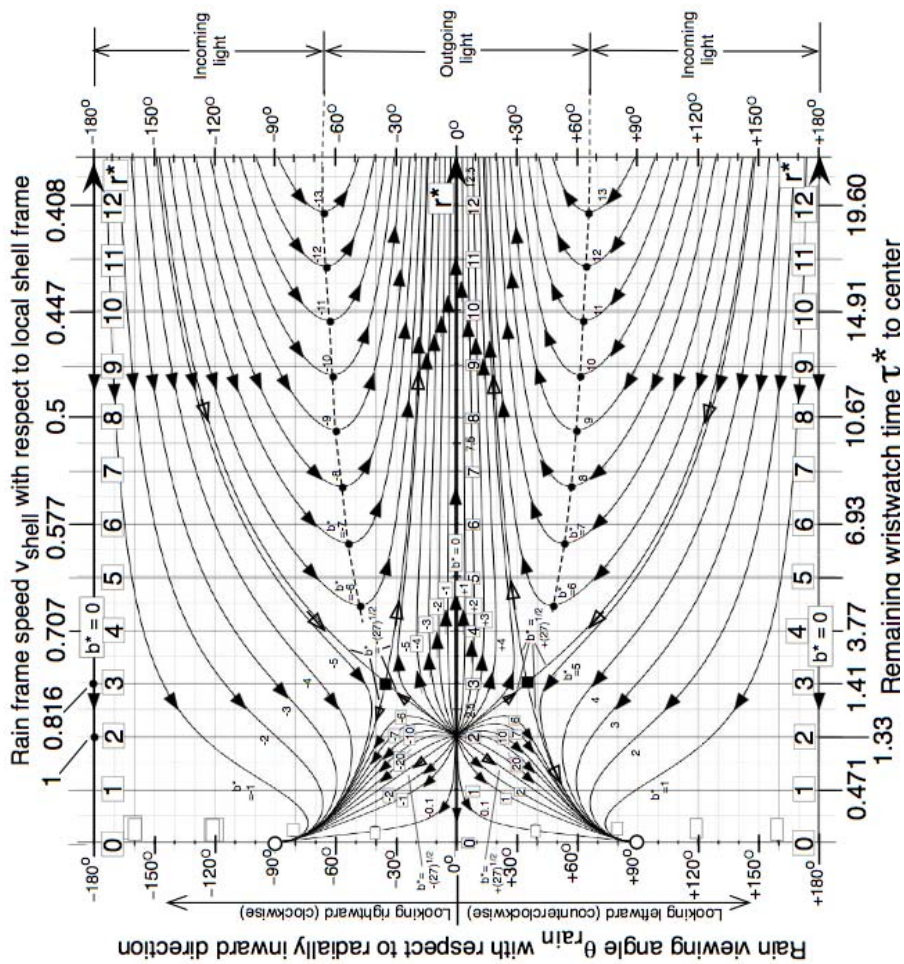

FIGURE 15.3 Figure 15.2 augmented to include beams from *all* light sources, not just those from distant stars.

(#RainViewingAngle)

15.3 ■ RAIN FRAME VIEW OF ALL LIGHT SOURCES

Fire lasers at in all directions.(#sec:RainView)

Beams from all sources

Figure 15.2 limits itself to starlight beams and the visual edge of the black hole. But shell and rain observers can fire lasers in various local directions; equations (15.16) and (15.17) describe such beams also. Figure 15.3 includes light beams from all sources, not just beams from distant stars.

Seeing starlight down to the singularity

What is that spray of beams diverging from the point $(\theta_{\rm rain} = 0, r^* = 2)$ on this diagram? Compare this spray with the beams emitted from near the point labeled B, at $(\theta_{\rm shell} = 0,\ r^* = 2)$ in shell Figure 14.6; Figure 14.7 plots some of these trajectories. Shells exist only outside the event horizon, so shell observers see beams only down to that limit. In contrast, the rain observer can also view beams that pass inward through the horizon and drop all the way to the singularity—as she does the same.

Ring around the sky

As she approaches the singularity, the rain observer sees a ring of star images that encircles her at $\theta_{\text{rain}} \to 90°$ with respect to her inward direction. All beams with $|b^*| < b^*_{\text{critical}}$ from every source contribute to that sky-bisecting ring. Everywhere else her sky is black (except for the single beam with $b^* = 0$).

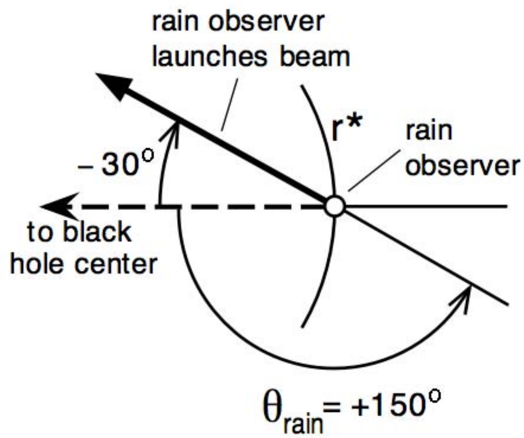

FIGURE 15.4 Sample Problem 15.2. Instead of receiving the beam shown in Figure 15.1 as she falls, the rain observer launches a beam in the same direction.

(#RainLaunch)

SAMPLE PROBLEM 15.2. Observer launches beams.

PROBLEM A. Suppose that the rain observer does not *receive* the beam shown in Figure 15.1, but instead *launches* a beam in the direction in which the incoming beam was traveling. Let the (clockwise) launch angle be $-30°$as shown in Figure 15.4. *Question:* Will the launched beam escape the black hole or plunge into it? Answer this question for the five beams that rain observer launches as she passes shells at $r^* = 10$, 6, 3, 2, and 1.

SOLUTION A FOR $r^* = 10$. The beam that the rain observer launches is an extension of the *starlight* beam that she would have received received from the angle $\theta_{\text{rain}} = +150°$, as shown in Figure 15.4. So we want to determine the fate of the beam that a rain observer passing $r^* = 10$ sees at this rain angle. In Figure 15.3, find the intersection of the vertical line $r^* = 10$ and the horizontal line $\theta_{\text{rain}} = +150°$. The beam at that intersection is $b^* = 4$. That beam ultimately plunges into the black hole.

SOLUTIONS A FOR $r^* = 6$, 3, 2, 1. The intersections of the horizontal line at $\theta_{\text{rain}} = +150°$ with vertical lines at these smaller radii all lie in the region in which beams plunge into the black hole.

PROBLEM B. The rain observer in Figure 15.1 launches a beam in the direction *opposite* to that in Figure 15.4. *Question:* Will the launched beam escape the black hole or plunge into it? Answer this question for the five beams that rain observer launches as she passes shells at $r^* = 10$, 6, 3, 2, and 1.

SOLUTION B FOR $r^* = 10$. The rain observer launches the beam at an angle $+150°$. That is an extension of the *starlight* beam that would have approached her at the angle $\theta_{\rm rain} = -30°$. In Figure 15.3 the vertical line $r^* = 10$ crosses the horizontal line $\theta_{\rm rain} = -30°$ near the curve for $b^* = -8$. The $b^* = -8$ beam moves outward to escape the black hole.

SOLUTIONS B FOR $r^* = 6, 3, 2, 1$. The intersections of the horizontal line $\theta_{\rm rain} = +150°$ with vertical line at $r^* = 6$ is also in the set of b^*-curves for outgoing beams. The corresponding intersection with $r^* = 3$ is just below the square box representing the tangent light sphere, but still picks out an outgoing beam. In contrast, at the smaller radii $r^* = 2$ and $r^* = 1$ the beams at the intersections are all moving inward and plunge into the black hole.

QUERY 15.6. Laser blazing (#query4116)
A rain observer passing $r^* = 4$ fires a laser pulse at an angle of $+70°$. A second rain observer passes $r^* = 2$ as the flash arrives there.

A. Show that the emitted beam is an extension of the *starlight* beam that the first rain observer would have seen coming from behind her at the angle $\theta_{\rm rain} = -110°$.
B. Show that the emitted beam has impact parameter $b^* = -3$.
C. Show that the second rain observer sees this beam at the angle $\theta_{\rm rain} \approx 66°$.

QUERY 15.7. Capture ranges (#query3116)
A rain observer passing radius r^* emits beams in a range of angles. This range includes *all* of the beams that fall into the black hole.

A. Show that when $r^* = 12$, this range of angles is from $-34°$ to $+34°$ with respect to the radially inward direction.
B. What is this range of angles for $r^* = 6$? [My answer: $-77°$ to $+77°$ with respect to the radially inward direction.]
B. What is this range of angles for $r^* = 3$? [My answer: $-145°$ to $+145°$ with respect to the radially inward direction.]
C. What is this range of angles for $r^* = 1$? [My answer: Trick question! ALL beams emitted at $r^* = 1$ fall into the black hole]

15.4 ■ RAIN FRAME ENERGY OF STARLIGHT

Falling through gravitationally blueshifted light.(#sec:ChangingEnergies)

Section 14.6 analyzed the energy of starlight measured by the shell observer and found this energy to be lethal near the event horizon. The rain observer, however, "runs away" from incoming starlight, so we expect that this starlight will be less damaging to the rain observer than to the shell observer at the same radius. The shell observer is outside the event horizon, while the rain observer descends to the singularity. On the way there she encounters light that has fallen farther inward, so may be even more dangerous than it was outside the horizon.

Will the rain observer receive a lethal dose of high-energy starlight during her descent? To pursue this question we need to analyze light energy in global rain coordinates. Equation (15.10) gives the map energy of a stone in global rain coordinates. The special relativity expression for rain frame energy with the substitution $\Delta t_{\text{rain}} \equiv \Delta T$ from (15.4) yields (#rainen)

Rain observer in danger?

$$\frac{E_{\text{rain}}}{m} = \lim_{\Delta\tau^* \to 0}\left(\frac{\Delta t^*_{\text{rain}}}{\Delta\tau^*}\right) = \lim_{\Delta\tau^*\to 0}\left(\frac{\Delta T^*}{\Delta\tau^*}\right) = \frac{dT^*}{d\tau^*} \qquad \text{(stone)} \quad (15.20)$$

We want to modify this expression to describe light. Substitute $dT^*/d\tau^*$ from (15.20) into (15.10) and solve for E_{rain}/m: (#rainenB)

$$\frac{E_{\text{rain}}}{m} = \left(1 - \frac{2}{r^*}\right)^{-1}\frac{E}{m}\left[1 + \left(\frac{2}{r^*}\right)^{1/2}\frac{m}{E}\frac{dr^*}{d\tau^*}\right] \qquad \text{(stone)} \quad (15.21)$$

Rewrite equation (12.5) in unitless coordinates: (#eq:9A)

$$\left(\frac{dr^*}{d\tau^*}\right)^2 = \left(\frac{E}{m}\right)^2 - \left(1 - \frac{2}{r^*}\right)\left[1 + \frac{(L^*/m)^2}{r^{*2}}\right] \qquad \text{(stone)} \quad (15.22)$$

Substitute this expression for $dr^*/d\tau^*$ into (15.21) and set up the expression for the *second term* in the square bracket in this equation: (#rainenC)

$$\left(\frac{2}{r^*}\right)^{1/2}\frac{m}{E}\frac{dr^*}{d\tau^*} \qquad \text{(piece of (15.21) for a stone)} \qquad (15.23)$$

$$= \pm\left(\frac{2}{r^*}\right)^{1/2}\left\{1 - \left(1 - \frac{2}{r^*}\right)\left(\frac{m^2}{E^2} + \frac{L^{*2}}{E^2 r^{*2}}\right)\right\}^{1/2}$$

This equation is for a stone. For light, go to the limit of small mass and high speed: $m \to 0$ and $L^*/E \to b^*$. (#rainenD)

$$\left(\frac{2}{r^*}\right)^{1/2}\frac{m}{E}\frac{dr^*}{d\tau^*} \qquad \text{(piece of (15.21) for a stone . . .)} \quad (15.24)$$

$$\to \pm\left(\frac{2}{r^*}\right)^{1/2}\left\{1 - \left(1 - \frac{2}{r^*}\right)\frac{b^{*2}}{r^{*2}}\right\}^{1/2} \qquad \text{(. . . in the light limit)}$$

The overall sign is the same as the sign of dr^*. Therefore the upper (plus) sign is for outgoing light, while the lower (minus) sign is for incoming light. Multiply both sides of (15.21) by m/E and into the resulting expression

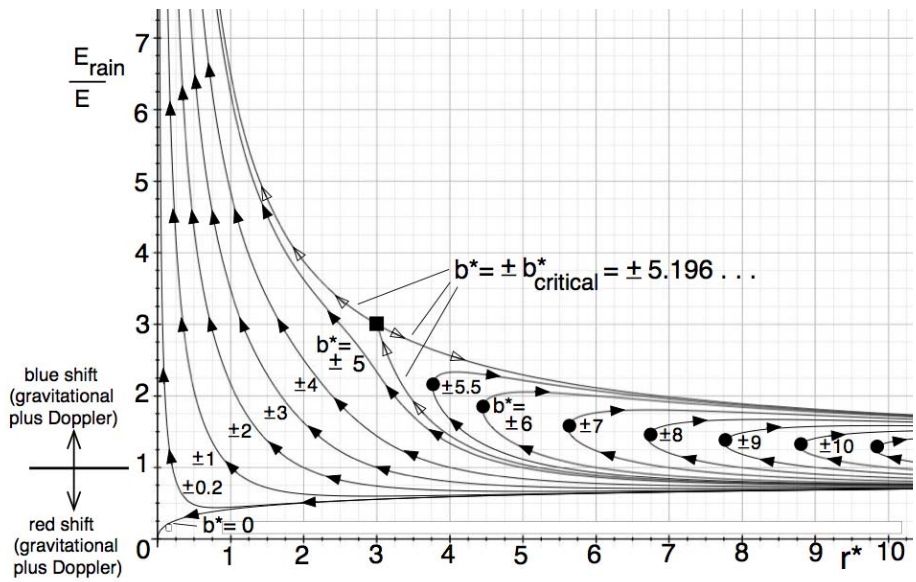

FIGURE 15.5 Ratio $E_{\rm rain}/E$ of starlight measured by the rain observer, from equation (15.25). (#ErainFig)

Energy of starlight for rain observer.

substitute the second line of (15.24) to obtain an expression for the energy of light—as a function of b^* and r^*—measured in the descending rain frame. After some rearrangement, this result is:(#rainframeen)

$$\frac{E_{\rm rain}}{E} = \frac{r^* \pm 2^{1/2}\left\{r^* - (r^*-2)\left(\dfrac{b^*}{r^*}\right)^2\right\}^{1/2}}{r^*-2} \qquad \text{(light)} \qquad (15.25)$$

For incoming starlight, choose the minus sign, but if the starlight deflects and returns outward, change to the plus sign as it passes the turning point. Starlight beams with $|b^*| < b^*_{\rm critical}$ have no turning point, their sign in (15.25) remains minus. The impact parameter b^* appears as a square in (15.25); the beam has the same energy whether it moves clockwise or counterclockwise around the black hole.

Rain energies of inward- and outward-moving starlight beams

Figure 15.5 plots results of equation (15.25) for starlight. We expect the rain-observed energy of an *incoming* beam to depend on two competing effects: the gravitational blue shift (increase in energy) of the falling light vs. the Doppler downshift in energy as the rain observer "runs away" from the in-falling light. In contrast, starlight that has passed its turning point and heads outward again moves opposite to the incoming rain observer, so she will measure its energy to be Doppler up-shifted. Figure 15.5 shows that there is a net blue shift for some starlight beams and parts of other beams, and a net red shift for others.

QUERY 15.8. Rain frame energy at large radius (#query2072)

Check the limiting case of E_{rain} for large radius. Show that E_{rain} takes the expected value, namely (#approxinfty)

$$\lim_{r^* \to \infty} E_{\text{rain}} = E \qquad \text{(light)} \tag{15.26}$$

QUERY 15.9. Rain frame energy less than shell energy? (#query2972)

Think about the ratio $E_{\text{rain}}/E_{\text{shell}}$ at a given radius r^* outside the horizon. Without using equations, predict the conditions for which $E_{\text{rain}}/E_{\text{shell}} < 1$ and the conditions for which $E_{\text{rain}}/E_{\text{shell}} > 1$. Under what conditions will $E_{\text{rain}}/E_{\text{shell}} = 1$.

QUERY 15.10. Chasing the light (#query2272)

A. Consider a starlight beam that moves with $b^* = 0$ radially inward. Explain why a rain observer measures the energy of the beam to be $E_{\text{shell}} > E$, while a shell observer measures its energy to be $E_{\text{shell}} > E$

B. For a starlight beam at its turning point, do you expect E_{rain} to be less than or greater than E? What does Figure 15.5 say about this question? Plug the expression for r^*_{tp} into (15.25) and explain the result. Compare it with a similar formula result for E_{shell}/E.

QUERY 15.11. Shell energy and rain energy (#query2278)

A. Consider a beam that moves radially inward with $b^* = 0$. Explain why a rain observer measures the energy of the beam to be $E_{\text{shell}} > E$, while a shell observer measures its energy to be $E_{\text{shell}} > E$.

B. For a beam at its turning point, do you expect E_{rain} to be less than or greater than E? What does Figure 15.5 say about this question? Plug the expression for r^*_{tp} into (15.25) and explain the result. Compare it with the result of similar formula for E_{shell}/E.

QUERY 15.12. *Optional:* **Trouble at the event horizon?** (#query3111)
The denominator $r^* - 2$ in (15.25) goes to zero at the event horizon. Does this mean that starlight has infinite energy there? Use our standard approximation (inside the front cover); set $r^* = 2(1+\epsilon)$, where $0 < \epsilon \ll 1$ and verify that E_{rain}/E is not infinite at the event horizon. Show that your approximation at $r^* = 2$ correctly predicts values of E_{rain}/E for two or three of the b^*-curves in Figure 15.5.

QUERY 15.13. Killer starlight? (#query2279)

A. Find an expression $E_{\text{rain,MAX}}/E$ for the maximum rain energy of starlight at every radius.
B. Suppose the rain observer approaches the singularity: Let $r^* = \epsilon$, where $0 < \epsilon \ll 1$ and show that for starlight $\lim_{r^*\to 0}(E_{\text{rain,MAX}})/E = b^*_{\text{critical}}/r^*$.

Lethal starlight is bad, but tides are worse.

In Query 15.8 you show that close to the singularity the maximum energy of starlight measured by the rain observer increases as the *inverse first power* of the decreasing radius r^*. The other mortal danger to the rain observer comes from relative tidal accelerations: Equations (10.3) through (10.5) tell us that tidal accelerations increase as the *inverse third power* of the decreasing radius, which is proportionally faster than the increase in the energy of incoming starlight.

Which will finally be lethal for the rain observer: killer starlight or killer tides? Inverse third power tidal accelerations appear to be the primary candidates. Analyzing tidal acceleration is straightforward as well: its effects are simply mechanical. In contrast, we have trouble predicting results for light: they depend not only on energy of the light but also on its intensity and the rain observer's wristwatch time of exposure. This book says nothing about the focusing properties of curved spacetime near the black hole—an advanced topic—so we lack the tools to predict results of the rain observer's accumulated exposure to starlight as she descends.

We know of nothing that can shield us from tidal effects. In contrast, we have a lot of experience protecting humans against radiation of different wavelengths. Perhaps a specially-designed personal planetarium will allow the rain observer to survive all the way down to her tidal limit. This is our assumption in the description of the final fall in Section 15.5.

Figure 15.5 plots the energy of incoming and outgoing *starlight*. But shell and other observers can also launch laser beams in various directions. Equation (15.25) also predicts the rain energy of these laser beams. Figure 15.6 plots E_{rain}/E for light from all sources. The beams that loop in and out of Figure 15.6 from above, with $b^* = \pm 5.5$, ± 6, and ± 7, are some of those in the spray of beams shown in Figure 15.3. Query 15.8 analyzes two laser beams aimed radially outward.

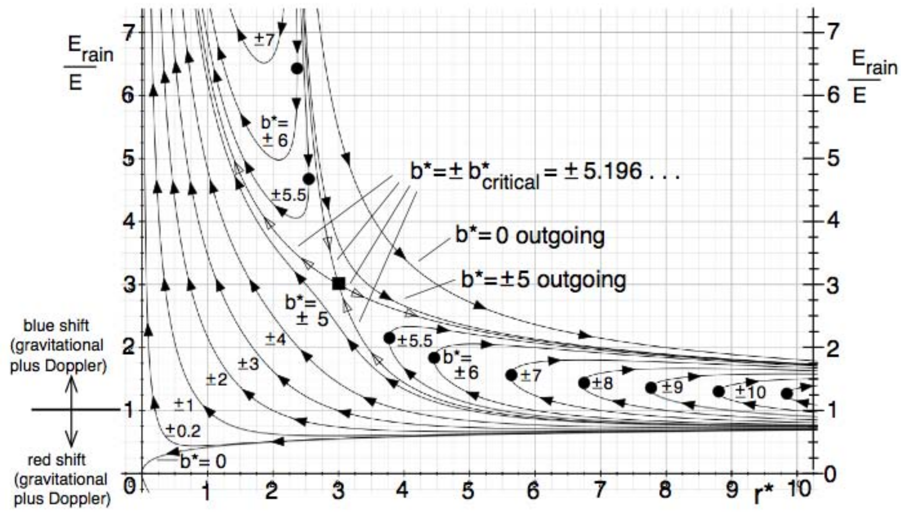

FIGURE 15.6 Ratio E_{rain}/E of light beams from all sources measured by the rain observer, from equation (15.25).

(#ErainFigAll)

QUERY 15.14. Lasers shine outward. (#query2199)

Figure (15.6) displays the rain energy of light from all sources, in particular, the outgoing laser beam with $b^* = 0$. Answer the following two questions about such a beam:

A. A shell observer at $r^* = 2.5$ shines a laser pulse with shell energy E_{shell} radially outward. A rain observer receives this pulse as she hurtles inward past the shell at $r^* = 10$. What rain energy does she measure for this pulse? *Hint:* Use equation (14.11) to find the map energy.

B. A shell observer at r_1^* shines radially outward red light of wavelength $\lambda_{\text{shell}} = 700$ nanometers in his frame. A descending rain viewer, as she falls inward past a shell at larger radius r_2^*, sees the outgoing light as blue, with wavelength $\lambda_{\text{rain}} = 400$ nanometers in her frame. Find the value of r_2^* as a function of r_1^*. *Briefing:* For a photon in flat spacetime and in conventional units, $E = hf = hc/\lambda$, where h is Planck's constant, f is the frequency, and λ is the wavelength. Remember that $E_{\text{shell}} \neq E_{\text{rain}} \neq E$ except at a great distance from the black hole.

QUERY 15.15. Targeting infinity (#query7199)

A rain observer near a black hole launches a laser beam that escapes to a great distance. Far from the black hole $E_{\text{rain}} = E_{\text{shell}} = E$ there, where E is the map energy of the beam.

Some of the beams analyzed in Sample Problem 15.2 escape to a great distance. Let E_{rain,r^*} be the rain energy of the emitted beam measured by the rain observer who emits it. In terms of this energy, what is the energy at a great distance of the beams in Sample Problem 15.2 that escape?

Example: Examine the Solution B for $r^* = 10$. The beam has $b \approx -8$ and is headed outward. In Figure 15.6 the vertical line at $r^* = 10$ intersects the outgoing $b^* = \pm 8$ curve at about $E_{\text{rain},10}/E = 1.6$. This means that at a great distance the beam has energy $E = 1.6 E_{\text{rain},10}$.

Carry out a similar analysis for the rest of the beams in Sample Problem 15.2 that escape to a great distance.

QUERY 15.16. Negative energy of light? (#query3199)

In equation (15.25) let $r^* = \epsilon$, where $0 < \epsilon \ll 1$. Show that one choice of the plus-or-minus sign yields a negative value—which can be a very large negative value—for E_{rain}/E. Is $E_{\text{rain}} < 0$ possible? Is $E < 0$ possible? In the exercises you will examine consequences of negative E_{rain}/E.

15.5 ■ THE FINAL FALL

Free-fall to the center(#sec:FinalPlunge)

Inside the horizon: time of a long movie

We celebrate with the final parade of an all-star cast. Let's follow general relativists Richard Matzner, Tony Rothman, and Bill Unruh looking at the starry heavens as we free-fall straight down into a black hole so massive, so large that even after crossing the event horizon we have nearly two hours of existence ahead of us—the time of a long movie—to behold the whole marvelous ever-changing spectacle. Almost everything we have learned about relativity—both special and general—contributes to our appreciation of this mighty sequence of panoramas.

Panoramas Seen by the Rain Frame Observer

—*Adapted from Matzner, Rothman, and Unruh. Some numerical values calculated by Luc Longtin.*

Fall into a one-billion-solar-mass black hole.

With this background, we now imagine a free-fall journey into a billion-solar-mass black hole ($M = 10^9 M_{\text{Sun}} = 1.477 \times 10^9$ kilometers $= 1.6 \times 10^{-4}$ light-years—about one-third of the estimated mass of the black hole at the center of galaxy M87). The map radius of the event horizon—double the above figure—is about the size of our solar system. We adjust our launch velocity to match the velocity which a rain frame, falling from rest at infinity, would have at our inward shell launch point at the initial map radius of $r = 5\,000M$. Our resulting inward shell launch velocity, $v = -(2M/r)^{1/2}$ with respect to a local shell observer, is equal to two percent of the speed of light. We record each stage in the journey by giving both the time-to-crunch on our wristwatch and our current map radius r.

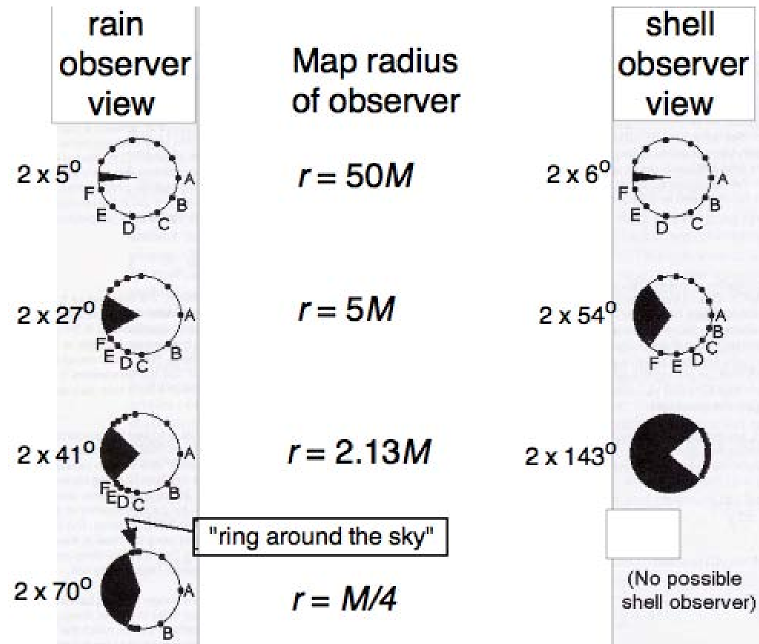

FIGURE 15.7 Pie charts showing rain directions to stars and the visual edge of the black seen in sequence by a single rain frame viewer (left-hand column) and a set of stationary shell spectators (right-hand column). Dots labeled A through F represent directions of stars evenly spaced around the sky for a viewer remote from the black hole. In the final instants of her journey, the sky behind the rain traveler is black, nearly empty of stars, and the black hole covers the sky ahead of her. Cleaving the forward half of the firmament from the backward half is a bright ring around the sky. This figure does not show multiple images of stars. (Figure based on the work of M. Sikora, courtesy of M. Abramowicz.)
(#PlungingView)

26 years to the end

The beginning of the journey, 26 years before the end. At this point the black hole is rather unimpressive. There is a small region (about 1 degree across—i.e., twice the size of the Moon seen from Earth) in which the star pattern looks slightly distorted and within it (covering about one-tenth of a degree) a disk of total blackout. Careful examination shows that a few stars nearest the rim of the blacked-out region have second images on the opposite side of the rim. Had these images not been pointed out to us, we probably would have missed the black hole entirely.

300 days

Three hundred days before the end, at $r = 500M$. Some noticeable change has occurred. The dark circular portion of the sky has now grown to one full degree in width.

One week

One week before the end, at $r = 41M$. The image has grown immensely. There is now a pure dark patch ahead with a diameter of about 22 degrees (approximately the size of a dinner plate held at arm's length). The original star images that lay near the direction of the black hole have been pushed away from their original positions by about 15 degrees. Further, between the dark patch itself and these images lies a band of second images of each of these stars. Looking near the darkness with the aid of a telescope, we can even see faint second images of stars that actually lie behind us! This light has looped once around the black hole on its way to our eye (Figure 12.9). From this point on, Doppler shift and gravitational blue shift radically change the observed frequencies of light that originate from different stars.

12 hours

Twelve hours before the end, at $r = 7M$. A sizeable portion of the sky is now black in front of us; the diameter of the black hole covers a 44-degree angle, over 10 percent of the entire visual sphere.

2 hours

Two hours before the end. We are now at $r = 2.13M$, just outside the event horizon and our speed is 97 percent that of light as measured in the local shell frame. Visual changes of angle are now extremely important. Anything we see after an instant from now will be a secret taken to our grave, because we will no longer be able to send any information out to our surviving colleagues. Although we will be "inside" the black hole, not all of the sky in front of us appears entirely dark. Our high speed causes light rays to arrive at our eyes at extreme forward angles. Even so, a disk subtending a total angle of 82 degrees in front of us is fully black—a substantial fraction of the forward sky.

Secondary images

Behind us we see the stars grow dim and spread out; for us their images are not at rest, but continue to move forward in angle to meet the advancing edge of the black hole. This apparent star motion is again a forward-shift due to our speed. But there is a more noticeable feature of the sky: We can now see second images of all the stars in the sky surrounding the black hole. These images are squeezed into a band about 5 degrees wide around the image of the black hole. These second images are now brighter than were the original stars. Surrounding the ring of second images are the still brighter primary images of stars that lie ahead of us, behind the black hole. The band of light caused by both the primary and secondary images now shines with a brightness ten times that of Earth's normal night sky.

2 minutes

Approximately two minutes before oblivion: $r = M/7$. The black hole now subtends a total angle of 150 degrees from the forward direction—almost the entire forward sky. Behind us star images are getting farther apart and rushing forward in angle. Only 20 percent of star images are left in the sky behind us. In a 10-degree-wide band surrounding the outer edges of the black hole, not only second but also third and some fourth images of the stars are now visible. This band running around the sky now glows 1000 times brighter than the night sky viewed from Earth.

Final seconds

The final seconds. The sky is dark everywhere except in that rapidly thinning band. This luminous band—glowing ever brighter—runs completely around the sky perpendicular to our direction of motion. At 3 seconds before oblivion it shines brighter than Earth's Moon. New star images rapidly appear along the inner edge of the shrinking band as higher and higher-order images become visible from light wrapped many times around the black hole. The stars of the visible Universe seem to brighten and multiply as they compress into a thinner and thinner ring transverse to our direction of motion.

Awesome ring bisecting the sky

Only in the last 2/9 of a second on our wristwatch do tidal forces become strong enough to end our journey and our view of that awesome ring bisecting the sky.

15.6 ■ REFERENCES

Initial quote: Emily Dickinson, Franklin Number 1433, Version A, about 1877, in *The Poems of Emily Dickinson, Variorum Edition*, Edited by R. W. Franklin, Cambridge Massachusetts, The Belknap Press of Harvard University, 1998, Volume III, page 1253.

Description of final dive (Section 15.5) and Figure 15.7 are adapted from Richard Matzner, Tony Rothman, and Bill Unruh, "Grand Illusions: Further Conversations on the Edge of Spacetime," in *Frontiers of Modern Physics: New Perspectives on Cosmology, Relativity, Black Holes and Extraterrestrial Intelligence*, edited by Tony Rothman, Dover Publications, Inc., New York, 1985, pages 69–73. Luc Longtin provided corrections for "times before oblivion" in Section 15.5 and calculated numbers for Figure 15.7.

15.7 ■ EXERCISES

FOLLOWING NEW IDEAS FOR FINAL EXERCISES, PLUS SOME EXERCISES FROM FIRST EDITION:

EB: IN EXERCISES FOR THIS CHAPTER, HAVE STUDENTS VERIFY (AND COMPLETE TO MORE SIGNIFICANT DIGITS) THE NUMERICAL RESULTS OF THIS FINAL SECTION. (EFT: INCLUDING QUALITATIVE FACTS, SUCH AS THAT SECONDARY RINGS ARE INSIDE PRIMARY RINGS)

EXERCISE ON CONVERTING SHELL VIEWING ANGLES TO RAIN VIEWING ANGLES OUTSIDE THE HORIZON. FOR ANGLE CONVERSIONS (DIRECTION OF MOTION OF BEAM VS DIRECTION IN WHICH OBSERVER LOOKS TO SEE BEAM), SEE SeeingOrbiterRain110511v1 AND EARLIER DRAFTS.

5. Negative energy of light?

Sometimes scientific results follow what Murray Gell-Mann calls the Totalitarian Principle: "Everything not forbidden is compulsory," a slogan from T. H. White's book *The Once and Future King*. In this exercise you show that light can have negative energy inside the event horizon, but since this light cannot influence the outside world, its negative energy is not forbidden. Therefore is it compulsory?

A. Solve equation (15.20) for $d\tau^*$ and substitute it into (15.21). Multiply through by m to yield a result valid for both light and a stone: (#rainenY)

$$\left[1 - \frac{2}{r^*} - \left(\frac{2}{r^*}\right)^{1/2} \frac{dr^*}{dT^*}\right] E_{\mathrm{rain}} = E \qquad \text{(stone or light)} \quad (15.27)$$

B. Equation (7.26) gives the radial velocity of light in global rain coordinates. Show that in unitless coordinates that equation becomes: (#rainenX)

$$\frac{dr^*}{dT^*} = -\left(\frac{2}{r^*}\right)^{1/2} \pm 1 \qquad \text{(light: radial motion)} \quad (15.28)$$

where the sign is plus for light directed radially outward, minus for light directed radially inward.

C. The rain observer crosses inward through the event horizon, then shines her flashlight radially outward. Choose the plus sign in (15.28) and substitute that equation into (15.27). Simplify. You may find it helpful to make the substitution $[1 - 2/r^*] = [1 - (2/r^*)^{1/2}] \times [1 + (2/r^*)^{1/2}]$. Show that the result is: (#rainW)

$$\left[1 - \left(\frac{2}{r^*}\right)^{1/2}\right] E_{\mathrm{rain}} = E \qquad \text{(light directed radially outward)} \quad (15.29)$$

D. Does the outward-directed light flash move outward or inward in local rain frame coordinates? outward or inward in global rain map coordinates?

E. Equation (15.29) allows both E and E_{rain} to be positive outside the event horizon. But inside the event horizon, one of them must be negative. Which one? Which expression for energy do you feel *must* be positive: Special relativity energy E_{rain} of flat spacetime or map energy E of curved spacetime?

F. If you choose local frame energy E_{rain} to be negative, does that mean that the laws of physics suddenly change for the rain observer as she crosses inward through the event horizon? Are you comfortable with that prediction?

G. If you choose map energy E to be negative inside the event horizon, is there any danger that this negative energy can be "set loose" outside the horizon?

H. Suppose that a virtual positron-electron pair appears just outside the event horizon. The positron escapes to infinity, an element of *Hawking radiation*; the electron moves inward across the event horizon with positive map energy. Could the negative-energy outward-directed light flash cancel the electron's positive energy, resulting in zero net change in the mass of the black hole as a result of this process?

I. What other apparent paradoxes follow from negative energy inside the horizon? Can you clear up these apparent paradoxes?

EB SUGGESTION FOR EXERCISE: EXPANDING INTERPRETATION OF FIGURES **??** AND **??**: ASK THEM TO TRACE DIRECTION OF BEAM IN THESE TWO FIGURES FOR THE CASES IN FIGURE 15.8.

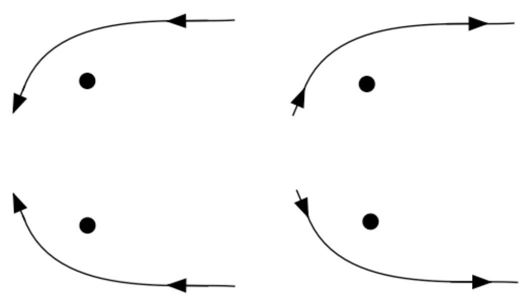

FIGURE 15.8 EB suggestion for exercise (#BeamDirection)

SIMILAR EXERCISE FOR FIGURES **??** AND **??**?

Measuring Your Distance from a Black Hole

You are the pilot of a spaceship, rockets blasting to keep you at rest near a black hole. You would like to know how close you are to it. What experiment can you do to find out? Assume that the mass M of the black hole is known. Can you use laser or radar signals or the view of the stars to tell you what your map radius r^* is with respect to the black hole? Do you need the presence of other spaceships, at rest or in orbit, in order determine your distance?

Crossing the Event Horizon

Pete Brown disagrees with the statement in an earlier chapter, "No special event occurs as you fall through the Schwarzschild event horizon." He says, "Suppose you go feet-first through the event horizon. Since your feet hit the event horizon before your eyes, then your feet should disappear for a short time. When your eyes pass across the event horizon, you can then see what is inside, including your feet again. So tie your sneakers tightly or you will lose them in the dark!" Is Pete right?

4. Visual size of a black hole

We can "see" the black hole only by what it does to light from background stars. Light from every star contributes to the tangential light sphere at map radius $r = 3M$. Light that escapes from the tangential light sphere moves, at a remote location, along a line a distance b_{critical} from a parallel line that goes through the center of the black hole. Compare Figure 15.9 with beam number 2 in Figure **??**.

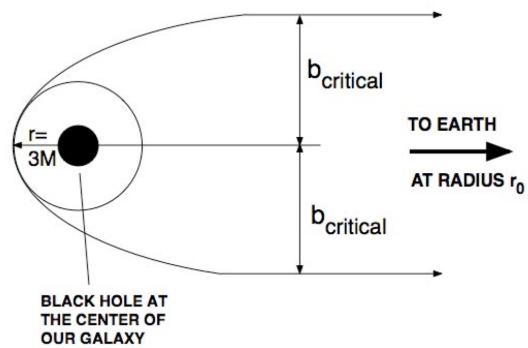

FIGURE 15.9 Schematic diagram showing the visual size of the black hole Sagittarius A* at the center of our galaxy, assumed (probably incorrectly) to be nonspinning. Effectively parallel straight beams that arrive at a distant viewer form a three-dimensional cylinder of diameter $2b_{\text{critical}}$ (shown here in cross section) that in principle the viewer perceives as a ring. CHANGE r0 to rfar.

(#VisualSizeBH)

The Earth can be thought of as sitting on a distant shell surrounding the black hole at the center of our galaxy, called Sagittarius A* (abbreviation: SgrA*), of mass $\approx 3.7 \times 10^6 M_{\text{Sun}}$ at a distance $\approx 26\,000$ light years from Earth. Assume (probably incorrectly) that SgrA* is a nonspinning black hole. Derive and justify an expression for the angular size θ_{Earth} of this black hole as observed from Earth.

A. Let the Earth rest on a shell concentric to SgrA* at a Schwarzschild map raius r. From (**??**) derive an expression for $\sin\theta_{\text{shell critical}}$. Approximate both sides of the resulting expression for $r \gg M$ to obtain (#SagA)

$$\theta_{\text{Earth}} \approx \frac{(27)^{1/2} M}{r} \qquad (r \gg M) \qquad (15.30)$$

B. Insert the values $M_{\text{SgrA}} \approx 3.7 \times 10^6 M_{\text{Sun}}$ and Earth's distance from the black hole of $r \approx 26\,000$ light years to find (#Sagradians)

$$\begin{aligned} \theta_{\text{Earth}} &\approx 1.2 \times 10^{-10} \quad \text{radian} \\ &\approx 6.8 \times 10^{-9} \quad \text{degree} \\ &\approx 2.4 \times 10^{-5} \quad \text{arcsecond} \end{aligned} \tag{15.31}$$

Astrophysicist Sheperd Doeleman and his MIT colleagues have set up a "virtual" microwave telescope the size of the United States by connecting a large number of radio observatory dishes assembled in what is called a Very Large Baseline Array (VLBA). These receivers are sensitive to waves of wavelength 1.3 millimeter. Their preliminary results (September 2008) measure the angular diameter of Sagittarius A* to be smaller than 3.7×10^{-5} arcsecond, compared with the prediction of twice the value from (15.31), namely 4.8×10^{-5} arcsecond for a Schwarzschild black hole. Chapters 15 and 16 on the spinning black hole show that rapid spin *decreases* the visual angular size of a black hole.