

CHAPTER 15

1 Spinning: Local Inertial Frames

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2 *The black hole makes you move. Why? General relativity has no*
 3 *answer. Curvature of spacetime is a description, not an*
 4 *explanation. Every free-fall acceleration described in this book, no*
 5 *matter how violent, is like gravity—it is gravity. Except for tides,*
 6 *you feel no force whatsoever as you float through churning*
 7 *spacetime. You are entitled to interpret your state as that of “rest.”*
 8 *Motion or rest? General relativity describes these distinctions in*
 9 *exquisite detail; then its task is done. In our opinion, science never*
 10 *answers the question, “Why?”*

—the authors

11 ■ EVERY MEASUREMENT IN A LOCAL FRAME

12 *Think globally; measure locally.*

Observations from
local inertial frames

14 General relativity predicts that when an isolated spinning star collapses to a black
 15 hole, gravitational radiation quickly (in a few milliseconds as seen by a distant
 16 observer!) smooths any irregularities in rotation. Thereafter the Kerr metric
 17 describes curved empty spacetime around the resulting spinning black hole. The
 18 present chapter constructs a variety of local inertial frames and predicts what we
 19 observe as we ride in these frames. It also answers further questions, such as: How
 20 can we extract useful energy of rotation from a Kerr black hole? How vulnerable to
 21 killer tides are riders in different frames? Can we move inward across the event
 22 horizon?

23 The present chapter makes heavy use of the results of Chapter 14, Spinning:
 24 The Kerr Map. We refer to figures and equations in that chapter with the notation
 25 14.xx. For example “equation (14.6)” means “equation (6) of Chapter 14,” while
 26 “Figure 14.2” means Figure 2 of that chapter.

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2 ■ LOCAL RING FRAME

28 *Suppress the Kerr metric cross term.*

29 *Problem:* The $dt d\phi$ cross term in the Kerr metric prevents us from constructing a
 30 local inertial frame by our usual method. *Solution:* Climb onto the
 31 zero-angular-momentum ring of Figure 14.5. Equation (14.40) tells us the angular
 32 rotation rate of this ring in Kerr map coordinates (#omegaringB)

$$\frac{d\phi}{dt} = \frac{2Ma}{rR^2} \equiv \omega_{\text{ring}}(r) \quad (\text{stone at rest on } L/m = 0 \text{ ring}) \quad (1)$$

Recast Kerr
metric.

33 where the notation $\omega_{\text{ring}}(r)$ reminds us that the angular rotation rate is different for
 34 different r . Use equation (1) to write the Kerr metric (14.8) in the following form:
 35 (#KerrToRingB)

$$d\tau^2 = \left(\frac{rH}{R}\right)^2 dt^2 - \frac{dr^2}{H^2} - R^2 [d\phi - \omega_{\text{ring}}(r)dt]^2 \quad (\text{Kerr metric}) \quad (2)$$

36
37

QUERY 1. Verifying the Kerr metric

Substitute expressions (1) into metric (2), expand the resulting expression, collect terms, and show that the
 result is the earlier form of the Kerr metric, equation (14.8). As a first step in this process, show that
 (#coeff) 41

$$\left(\frac{rH}{R}\right)^2 = 1 - \frac{2M}{r} + [R\omega_{\text{ring}}(r)]^2 \quad (3)$$

where $R\omega_{\text{ring}}(r)$ is the map speed of the ring, and we need to use expressions (14.5) and (14.7) for R^2 and
 H^2 respectively 43

44

Climb onto zero-
angular-momentum
ring.

45 Metric (2) is still the original Kerr metric, just written in a different form. Now
 46 envision a tiny flat patch riding on this ring; on this patch we erect a local
 47 frame—a latticework of rods and clocks in which we can apply special relativity.
 48 This frame has an average value of map radius \bar{r} , with which we make the following
 49 definitions. From equation (14.7): (#defHbar)

$$\bar{H}^2 \equiv H^2(\bar{r}) = 1 - \frac{2M}{\bar{r}} + \frac{a^2}{\bar{r}^2} \quad (4)$$

50 From equation (14.6): (#defRbar)

$$\bar{R}^2 \equiv R^2(\bar{r}) = \bar{r}^2 + a^2 + \frac{2Ma^2}{\bar{r}} \quad (5)$$

51 and, using definition (5), (#defomegabbar)

$$\bar{\omega}_{\text{ring}} \equiv \omega_{\text{ring}}(\bar{r}) = \frac{2Ma}{\bar{r}\bar{R}^2} \quad (6)$$

52 Now write (2) in the approximate form: (#approxKerr)

$$\Delta\tau^2 \approx \left(\frac{\bar{r}\bar{H}}{\bar{R}}\right)^2 \Delta t^2 - \frac{\Delta r^2}{\bar{H}^2} - \bar{R}^2 [\Delta\phi - \Delta\bar{\omega}_{\text{ring}}\Delta t]^2 \quad (7)$$

Local ring frame
coordinates

53 The coefficients in (7) are now constants, which permits us to define coordinates of
54 a local riding-on-the-ring inertial frame in our usual manner: (#ringt)

$$\Delta t_{\text{ring}} \equiv \frac{\bar{r}\bar{H}}{\bar{R}} \Delta t \quad (8)$$

$$\Delta y_{\text{ring}} \equiv \frac{\Delta r}{\bar{H}}$$

$$\Delta x_{\text{ring}} \equiv \bar{R} (\Delta\phi - \bar{\omega}_{\text{ring}}\Delta t)$$

55 Interpretation of the last equation of (8): Δx_{ring} is the incremental distance along
56 the ring *measured by someone riding on the ring*. (Proof: A stone at rest with
57 respect to the ring rider has $\Delta\phi/\Delta t = \bar{\omega}_{\text{ring}}$ and $\Delta x_{\text{ring}} = 0$.) Substitute equations
58 (8) into the approximate Kerr metric (7) to obtain the standard metric in the local
59 inertial ring frame: (#frame8)

$$\Delta\tau^2 \approx \Delta t_{\text{ring}}^2 - \Delta y_{\text{ring}}^2 - \Delta x_{\text{ring}}^2 \quad (9)$$

60 As usual, we choose the positive Δy_{ring} direction to be radially outward and the
61 positive Δx_{ring} to be tangential to the ring in the direction of increasing ϕ .

Ring frames outside
event horizon

62 **NO LOCAL RING FRAMES BETWEEN HORIZONS.** Ring frames exist only in
63 regions where coefficients in equations (8) are all real, namely for $H^2 \geq 0$.
64 Figure 14.1 shows that no ring frame can exist between the event horizon
65 and the Cauchy horizon. (Figure 14.1 also shows that H^2 becomes positive
66 again inside the Cauchy horizon; can we circle in a stable orbit there?
67 Answer in Chapter 16: Spinning: Explorations. In the present chapter we
68 construct zero-angular-momentum rings only outside the event horizon.)

69

QUERY 2. Install a Kerr map clock on your ring.

Use (8) to write *detailed* instructions for the ring rider on how to install a global “Kerr map clock” at rest in his local inertial ring frame. Is it important that his local Kerr map clock be synchronized with the global Kerr map clock?

74

75 The local ring frame is inertial, so the expression for ring energy of a stone has
76 the special relativity form: (#Ering)

$$\frac{E_{\text{ring}}}{m} = \lim_{\Delta\tau \rightarrow 0} \frac{\Delta t_{\text{ring}}}{\Delta\tau} = \frac{1}{(1 - v_{\text{ring}}^2)^{1/2}} \quad (\text{stone}) \quad (10)$$

77 where we take the calculus limit to describe the stone’s instantaneous velocity v_{ring} ,
78 which can be in any direction with respect to the local ring frame.

79

QUERY 3. Ring energy from map constants of motion

Combine the middle expression in (10) with (8) and (14.35) for $dt/d\tau$ to show that: (#frame12)

$$\begin{aligned} \frac{E_{\text{ring}}}{m} &= \lim_{\Delta\tau \rightarrow 0} \left[\frac{\Delta t_{\text{ring}}}{\Delta t} \frac{\Delta t}{\Delta\tau} \right] = \frac{R}{rH} \left(\frac{E}{m} - \frac{2Ma}{rR^2} \frac{L}{m} \right) \quad (\text{stone}) \quad (11) \\ &= \frac{R}{rH} \left(\frac{E}{m} - \omega_{\text{ring}} \frac{L}{m} \right) \end{aligned}$$

82



83 *Remind me why we keep going to the calculus limit. I thought we were applying*
84 *physics in a local inertial frame, where we always use approximate expressions*
85 *signaled by notations Δ and \approx and subscript labels such as “ring.”*



86 *You’re right: The local, limited nature of these frames determines how large spans*
87 *of space and time can be in which special relativity is “good enough” to describe*
88 *a given experiment or observation. However, as always in both Newtonian and*
89 *relativistic mechanics (both general and special) nothing limits how small an*
90 *increment we can use to express the instantaneous value of a stone’s energy*
91 *E_{ring} , along with other quantities such the components of its momentum and*
92 *velocity. By its very nature, calculus relentlessly drives this limit toward zero.*

Ring frame
description
of motion

93 Can we set up a calculus ring-frame expression corresponding to (11) for the
94 angular momentum of a stone? No! Map angular momentum of a stone *with respect*
95 *to a ring* has no meaning whatsoever for the ring observer, who measures only local
96 quantities. However, the tangential and radial components of ring *linear* momentum
97 of a stone does interest him. Use the differential form of the third expression in (8),
98 along with equations (14.35) and (14.36). After a lot of algebra, the outcome is
99 quite simple: (#pringx)

$$\begin{aligned} \frac{p_{\text{ring}, x}}{m} &\equiv \lim_{\Delta\tau \rightarrow 0} \frac{\Delta x_{\text{ring}}}{\Delta\tau} = R \frac{d\phi}{d\tau} - \frac{2Ma}{rR} \frac{dt}{d\tau} \quad (12) \\ &= \frac{1}{R} \frac{L}{m} \quad (\text{stone}) \end{aligned}$$

100 This result makes sense: When the stone is at rest on the zero-angular-momentum
 101 ring, its map angular momentum is zero and so is its x -momentum as measured by
 102 the ring observer. We also have, for a moving stone, (#pyring)

$$\frac{p_{\text{ring},y}}{m} = \lim_{\Delta\tau \rightarrow 0} \frac{\Delta y_{\text{ring}}}{\Delta\tau} = \frac{1}{H} \frac{dr}{d\tau} \quad (\text{stone}) \quad (13)$$

103 where $dr/d\tau$ is given by (14.41). The tangential component of the ring velocity is:
 104 (#frame15A)

$$\begin{aligned} v_{\text{ring},x} &\equiv \lim_{\Delta t_{\text{ring}} \rightarrow 0} \frac{\Delta x_{\text{ring}}}{\Delta t_{\text{ring}}} \equiv \lim_{\Delta t_{\text{ring}} \rightarrow 0} \frac{(\Delta x_{\text{ring}}/\Delta\tau)}{(\Delta t_{\text{ring}}/\Delta\tau)} \\ &= \frac{p_{\text{ring},x}}{E_{\text{ring}}} = \frac{L}{RE_{\text{ring}}} \quad (\text{stone}) \end{aligned} \quad (14)$$

105 The radial component of the ring velocity is: (#frame15B)

$$\begin{aligned} v_{\text{ring},y} &\equiv \lim_{\Delta t_{\text{ring}} \rightarrow 0} \frac{\Delta y_{\text{ring}}}{\Delta t_{\text{ring}}} \equiv \lim_{\Delta t_{\text{ring}} \rightarrow 0} \frac{(\Delta y_{\text{ring}}/\Delta\tau)}{(\Delta t_{\text{ring}}/\Delta\tau)} \\ &= \frac{p_{\text{ring},y}}{E_{\text{ring}}} = \frac{r}{R} \left(\frac{E}{m} - \frac{2Ma}{rR^2} \frac{L}{m} \right)^{-1} \frac{dr}{d\tau} \quad (\text{stone}) \end{aligned} \quad (15)$$

Map constants
of motion

106 where $dr/d\tau$ is given by (14.41). Now we can use the ring energy and the x -velocity
 107 of a stone to determine its map constants of motion E/m and L/m . Solve (11) for
 108 E/m : (#Efromring)

$$\begin{aligned} \frac{E}{m} &= \left\{ \frac{rH}{R} + \frac{2Ma}{rR} v_{\text{ring},x} \right\} \frac{E_{\text{ring}}}{m} \\ &= \left\{ \left[1 - \frac{2M}{r} + (R\omega_{\text{ring}})^2 \right]^{1/2} + R\omega_{\text{ring}} v_{\text{ring},x} \right\} \frac{E_{\text{ring}}}{m} \end{aligned} \quad (16)$$

109 In the first line of (16) we used L/m from (14): (#Lfromring)

$$\frac{L}{m} = R v_{\text{ring},x} \frac{E_{\text{ring}}}{m} = R \frac{p_{\text{ring},x}}{m} \quad (\text{stone}) \quad (17)$$

110 In the second line of (16) we used (1) and (3)

111

QUERY 4. Ring energies and speeds

Look at the orbits of stones pictured in Figure 14.9 (prograde orbit) and Figure 14.10 (retrograde orbit). Captions to those figures carry detailed orbit information. Set up a table for the desired entries.

For each of the two figures, tabulate values for E_{ring} , $v_{\text{ring},x}$, and $v_{\text{ring},y}$ in a ring frame at each of the following radii: the upper turning point, the lower turning point, and at a map radius halfway between the two.

117

EB: THIS QUERY SEEMS BORING TO ME: WHERE DOES IT GO? WHO CARES? DO YOU WANT ME TO EXTEND THIS KIND OF QUERY TO REVERSING ORBIT (FIGURE 14.12) AND TO LIGHT (FIGURES 14.16 THROUGH 14.18) AS SUGGESTED ON AN EARLIER DRAFT—COPY ON FOLLOWING PAGE?

122

3.2 THE PENROSE PROCESS

124 *Milking energy of rotation*

Negative map energy

125 Equation (16) carries the possibility that map energy E/m can be negative for
 126 negative $v_{\text{ring}, x}$, that is for a stone moving along the ring in a direction opposite to
 127 the direction of rotation of the black hole. In 19XX Roger Penrose harnessed this
 128 idea to display a way to tap the energy of rotation of a spinning black hole. We
 129 shall see that the so-called **Penrose Process** is not practical; it probably plays no
 130 part in natural processes. Nevertheless, it is important in principle and might be
 131 utilized by an advanced civilization to milk energy from the immense reservoir in
 132 black hole rotation.

133 What are the conditions that E/m be negative? From (16) this condition is:
 134 (#EneGC)

$$|v_{\text{ring}, x}| > \left[1 + \frac{(1 - 2M/r)}{(R\omega_{\text{ring}})^2} \right]^{1/2} \quad (\text{negative map energy, } -1 < v_{\text{ring}, x} < 0) \quad (18)$$

Conditions for negative map energy

135 The conditions expressed to the right of this equation are (a) the velocity must be
 136 negative so that E/m can be negative in (16), and (b) special relativity limits the
 137 speed of the stone to less than the speed of light in the inertial ring frame.
 138 Condition (b) tells us that $r < 2M$; that is, negative map energy is possible only
 139 inside the static limit, in the ergoregion. *Negative map energy is possible if, and*
 140 *only if, the stone is inside the ergoregion.*

POSITIVE ENERGY OF A STONE AT A GREAT DISTANCE

141 The first line of equation (16) verifies that it is not possible to have negative
 142 map energy at a great distance from the black hole. At large r the first term
 143 in the bracket goes to the value one, and the second term goes to zero.
 144 E_{ring}/m is always positive, so E/m must also be positive at large r .

146 How can we use negative map energy to extract positive measurable energy
 147 from the spinning black hole? That is the essence of the Penrose process. The
 148 strategy of the Penrose process is similar to the following series of unethical
 149 financial transactions:

- 150 a. You and I decide to share our money. Our combined net worth is positive.
- 151 b. I give you all my money, then borrow money from a bank and give that to
- 152 you as well. My bank debt is a negative entry on my accounting balance
- 153 sheet, so now my net worth is negative.
- 154 c. I declare bankruptcy and the bank is stuck with my debt.

155 The net result is the transfer of money from me and from the bank to you. The
 156 bank provides the mechanism by which I can enter the state of negative net worth.
 157 Your final net worth is a financial fiction with real value far from the bank.

158 The Penrose process is similar:

- 159 a. Starting from a great distance, you and I together descend to a trajectory
- 160 inside the ergosphere in which we move instantaneously along the tangential
- 161 direction of black hole circulation.

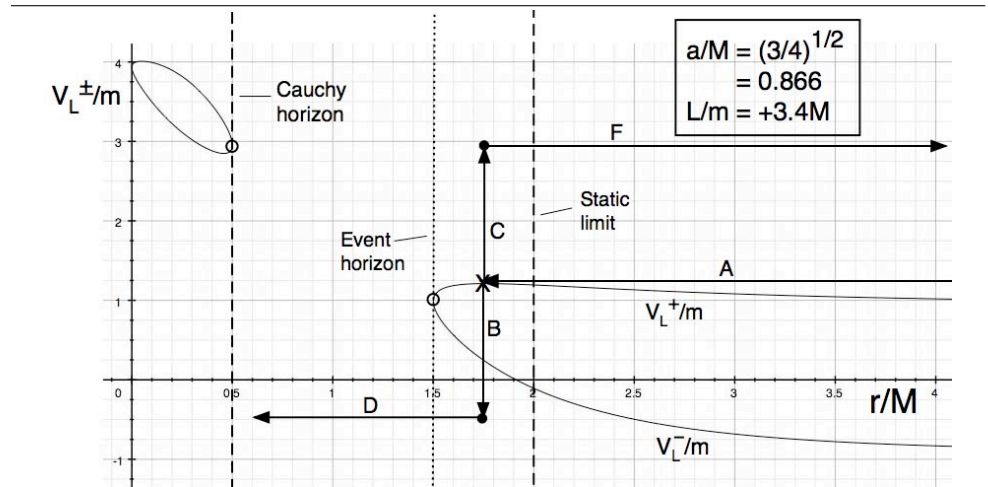


FIGURE 1 ARROWS B THROUGH F NO LONGER RELEVANT; CLEAR UP PENROSE PROCESS BOX BEFORE REDRAWING THIS. Figure 14.6 repeated with expanded scales to illustrate the Penrose process. Open circles show where V^+ and V^- meet. The X indicates the location of the knife-edge orbit for this combination of black hole spin and initial stone angular momentum. Penrose process: Step A: Stone of mass m_0 comes in from infinity to perch on the circular knife edge orbit. Steps B and C: Initial mass m_0 actually half matter, half antimatter. Combined to create pair of gamma rays moving in opposite tangential directions. Step D: Black hole swallows gamma ray with negative map energy. Step E: Gamma ray with larger map energy returns to infinity, carrying off some of the black hole energy of rotation.

- 162 b. You push me away *violently* in a direction opposite to the direction of black
 163 hole circulation. This puts you into a new trajectory and gives me sufficiently
 164 negative $v_{\text{ring } x}$ so that (18) is satisfied and I have negative map energy.
- 165 c. I drop into the black hole, which loses energy equal to my negative map
 166 energy. You continue in your new trajectory with enhanced positive $v_{\text{ring } x}$,
 167 L/m , and E/m . You arrive at a distant radius with augmented map energy,
 168 which is equal to local stationary frame energy at that remote
 169 location—equation(16).

170 The net result is the transfer of energy from me and from the black hole to you.
 171 The spinning black hole provides the mechanism by which I can enter a state of
 172 negative map energy. Your final map energy is a mythical beast with real value far
 173 from the black hole.

174 Figure 1 repeats Figure 14.6 with expanded scales and modifications to
 175 illustrate the Penrose process, as described in the caption. The final result is
 176 reduced rotation energy of the black hole and a flying fragment that carries energy
 177 to infinity.

178

SAMPLE PROBLEM (BOX?): Example of the Penrose Process

179

THIS IS A CAREFULLY-CHECKED CATASTROPHE. SEE ALTERNATIVE BELOW

180

181 Here is an example of the Penrose process. Look at Figure 1: A spinning black hole
 182 has $(a/M)^2 = 3/4$. A stone of mass m_0 and angular momentum $L/m_0 = +3.4M$
 183 approaches from a great distance along horizontal line A, with the energy
 184 $E/m_0 = 1.21$ and circulates in the knife-edge orbit at $r/M = 1.75$. In that orbit
 185 $dr/d\tau = 0$ and $v_{\text{ring } y} = 0$. MANY DIGITS CARRIED ALONG FOR INITIAL
 186 VERIFICATION OF NUMERICAL RESULTS. ALL RESULTS CHECKED, BUT
 187 INCOMING E AND CIRCULAR ORBIT RADIUS r DERIVED BY
 188 INSPECTION OF MAGNIFIED VERSION OF FIGURE 1. ONE PERCENT
 189 ERROR IN EACH CAN THROW OFF RESULT

190

A. Show that the numbers in Part A lead to $R = 2.160936$ meter and
 191 $\omega_{\text{ring}} = 0.211953$ meter $^{-1}$, so that $R\omega_{\text{ring}} = 0.458016$ and therefore from (3),
 192 $rH/R = 0.258692$.

193

B. From (11) and the results of Part A, show that the ring energy of the stone
 194 $E_{\text{ring}}/m_0 = 1.891669$. From (14), show that the ring velocity of the stone
 195 $v_{\text{ring}, x} = 0.831748$.

196

C. In the rest frame of the stone its ring energy is m_0 . Suppose that the stone
 197 is composed of equal masses of matter and antimatter. They combine to
 198 create two gamma ray bursts that travel in opposite tangential directions
 199 along the ring. The ring energy of each of these gamma ray bursts is $m_0/2$ in
 200 the rest frame of the original stone.

201

D. Find the energies $E_{\text{ring}+}$ and $E_{\text{ring}-}$ of the forward-moving and
 202 backward-moving gamma ray bursts, respectively, as measured in the ring
 203 frame. Use the special relativity Doppler shift equations: (#Doppler)

$$E_{\text{ring}+} = \left(\frac{1 + v_{\text{ring}, x}}{1 - v_{\text{ring}, x}} \right)^{1/2} \frac{m_0}{2} \quad (\text{light}) \quad (19)$$

$$E_{\text{ring}-} = \left(\frac{1 - v_{\text{ring}, x}}{1 + v_{\text{ring}, x}} \right)^{1/2} \frac{m_0}{2} \quad (\text{light})$$

204

where $v_{\text{ring}, x}$ is the speed of the original mass in the ring frame. Show the
 205 results are $E_{\text{ring}+} = 1.649768m_0$ and $E_{\text{ring}-} = 0.151536m_0$. The sum is
 206 $1.801304m_0$. PROBLEM APPEARS TO BE READING FROM FIGURE 1
 207 THE MAXIMUM OF V^+ AND r AT THAT MAXIMUM. SEE
 208 COMPLAINT ABOUT THIS ON PAGE 25 OF CHAPTER 14.

209

E. Adapt equation (16) to light: multiply through by m and set $v_{\text{ring}, x} = \pm 1$
 210 for gamma ray pulses. Substitute numerical values from earlier parts of this
 211 box: (#Eoverm)

$$E^{\pm} = (0.258692 \pm 0.458016) E_{\text{ring}}^{\pm} \quad (20)$$

212

F. Find the map energy of each oppositely-moving gamma ray pulse. Answers:
 213 $E^+ = 1.182403m_0$ and $E^- = -0.030204m_0$. The sum is $1.152199m_0$.
 214 EXITING GAMMA RAY DOES NOT EVEN HAVE ENERGY OF
 215 INITIAL STONE, $1.2m_0$.

216

217

218

SAMPLE PROBLEM (BOX?): ALTERNATIVE the Penrose Process

219 A raindrop-stone of mass m_0 drops from rest at a great distance and descends with
 220 zero angular momentum. It passes vertically through every
 221 (zero-angular-momentum) ring frame. Finally it reaches the ring frame at $r = 1.75$.
 222 See Figure 2. An L-shaped tube deflects the stone into a tangential ring direction.
 223 REPEAT parts B through F of preceding Penrose process

224

B. From (11), the ring energy of the descending stone is $E_{\text{ring}} = 3.865601$. From
 225 (10) find the ring speed of the descending mass $v_{\text{ring}, y} = 0.965960$. The
 226 L-shaped tube in Figure 2 deflects the stone into the tangential direction,
 227 along the ring with the same speed: $v_{\text{ring}, x} = 0.965960$

228

C. In the rest frame of the stone its ring energy is m_0 . Suppose that the stone
 229 is composed of equal masses of matter and antimatter. They combine to
 230 create two gamma ray bursts that travel in opposite tangential directions
 231 along the ring. The ring energy of each of these gamma ray bursts is $m_0/2$ in
 232 the rest frame of the original stone.

233

D. Find the energies $E_{\text{ring}+}$ and $E_{\text{ring}-}$ of the forward-moving and
 234 backward-moving gamma ray bursts, respectively, as measured in the ring
 235 frame. Use the special relativity Doppler shift equations: (#Doppler)

$$E_{\text{ring}}^+ = \left(\frac{1 + v_{\text{ring}, x}}{1 - v_{\text{ring}, x}} \right)^{1/2} \frac{m_0}{2} \quad (\text{light}) \quad (21)$$

$$E_{\text{ring}}^- = \left(\frac{1 - v_{\text{ring}, x}}{1 + v_{\text{ring}, x}} \right)^{1/2} \frac{m_0}{2} \quad (\text{light})$$

236

where $v_{\text{ring}, x}$ is the speed of the original mass in the ring frame. Show the
 237 results are $E_{\text{ring}}^+ = 3.799816m_0$ and $E_{\text{ring}}^- = 0.065793m_0$. The sum is
 238 $3.865609m_0$, almost exactly the same as the initial ring energy as it should
 239 be.

240

E. Adapt equation (16) to light: multiply through by m and set $v_{\text{ring}, x} = \pm 1$
 241 for gamma ray pulses. Substitute numerical values from earlier parts of this
 242 box: (#Eoverm)

$$E^\pm = (0.258692 \pm 0.458016) E_{\text{ring}}^\pm \quad (22)$$

243

F. Find the map energy of each oppositely-moving gamma ray pulse. Answers:

244

$E^+ = 2.7705129m_0$ and $E^- = -0.013114m_0$. The sum is 2.757399.

245

OUTPUT ENERGY MINUS INPUT ENERGY

246

$= 2.7705129m_0 - m_0 = 1.7705129m_0$. PROBLEM: SUM OF FINAL MAP

247

ENERGIES MUCH GREATER THAN ORIGINAL MAP ENERGY: m_0 .

4.8 CHANGE THE SPINNING BLACK HOLE?

249

How much energy available?

250

The Penrose process is one method for milking useful energy from a spinning black
 251 hole. How much such energy is available? How long could the monster black hole at

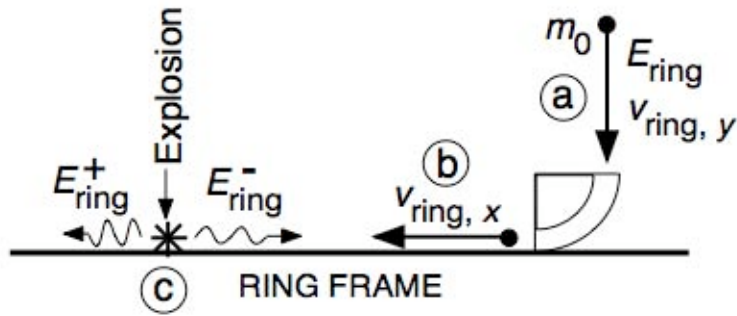


FIGURE 2 Penrose process using ring frame. (a) Raindrop of mass m_0 descends vertically in ring frame with local energy E_{ring} and speed $v_{ring, y}$. (b) Tube changes radial speed to equal horizontal speed. (c) Initial stone is half matter and half antimatter, which combine to emit two oppositely-directed gamma rays.

252 the center of our galaxy supply humanity’s energy needs on Earth? Can we dump
 253 our trash, radioactive waste, and excess global heat into a black hole?

254 Section 7 of Chapter 6, Plunging demonstrated that when distant matter
 255 plunges into a star, the star’s increase in mass is equal to the map energy of the
 256 plunging matter: (#GRGravMass)

$$M_{total} = M_{initial} + E \tag{23}$$

257 The same is true for the mass of the spinning black hole. More: A plunging
 258 stone also simply adds its angular momentum L to that of the spinning black hole:
 259 (#GRaddL)

$$J_{total} = J_{initial} + L \tag{24}$$

260 EB: I RECOMMEND THAT WE EITHER DO THE FOLLOWING
 261 HANDWAVING OR DELETE THE NOTE ENTIRELY. THE ANALYTIC
 262 PROOF REQUIRES SHOWING THAT FOR A MAXIMUM-SPIN BLACK
 263 HOLE, V^+ DOES NOT ACHIEVE ITS MAXIMUM VALUE OUTSIDE THE
 264 EVENT HORIZON BUT HAS A GREATER VALUE AT THE HORIZON THAN
 265 ELSEWHERE OUTSIDE THE HORIZON. THIS IS EASY TO SEE WITH
 266 GRorbits BUT MESSY TO PROVE ANALYTICALLY AND, IN MY OPINION,
 267 DOES NOT HAVE SUFFICIENT PAYOFF TO BE WORTH THE ANALYTIC
 268 EFFORT.

SUPER-MAXIMUM SPIN?

269 Suppose that we fire a stone with positive angular momentum into a
 270 maximum-spin black hole. *Question:* Will the resulting black hole have an
 271 angular momentum J greater than its supposed maximum,
 272 $J_{max} = Ma_{max} = M^2$? *Answer:* No, but the details are a bit complicated.
 273 Here is the overview: If the map energy of the incoming stone is less than the
 274 maximum value of V^+ outside the event horizon, then the stone reflects from
 275 that effective potential and returns to a remote location, leaving the angular
 276 momentum of the black hole unchanged. In contrast, if the the map energy
 277

278 of the incoming stone is greater than the maximum value of V^+ outside the
 279 event horizon, then the stone enters the black hole and increases the black
 280 hole angular momentum by its own angular momentum L , according to (24).
 281 However, in that case the mass of the black hole increases by the value of the
 282 stone's energy E , according to (23). It turns out that the value of this
 283 increased mass is such that the resulting black hole has less than the
 284 maximum spin at its new value of mass, so does not violate the maximum.
 285 As a limiting case, when the map energy of the incoming stone exactly equals
 286 the maximum value of V^+ outside the event horizon, then the resulting black
 287 hole increases both its angular momentum and its mass by just the amounts
 288 needed to yield a maximum-spin black hole at its new value of mass.

289 How much energy can we milk from a spinning black hole using the Penrose
 290 process or some other method? The answer was discovered in 1970 by a 19-year-old
 291 Athenian, Demetrios Christodoulou, who never finished high school. Milking all
 292 possible energy from a spinning black hole with initial angular momentum
 293 parameter a reduces it to a nonspinning black hole of so-called **irreducible mass**
 294 M_{irr} given by the equation (#Mirr)

$$M_{\text{irr}}^2 = \frac{1}{2} \left[M^2 + M (M^2 - a^2)^{1/2} \right] \quad (25)$$

295 EB: YOU SUGGEST A QUERY DERIVING THIS. WAS
 296 CHRISTODOULOU'S CONTRIBUTION SO STRAIGHTFORWARD? IS IT
 297 WORTH THE EFFORT? IF SO, CAN YOU GIVE ME AN INITIAL HINT OR
 298 OUTLINE?

299

QUERY 5. Energy from the monster

- A. Show that the energy that can be milked from a rotating black hole is $E_{\text{milk}} = M - M_{\text{irr}}$ has the value 0 for a nonspinning black hole and the value 0.293 for a maximum-spin black hole.
- B. Suppose that the monster black hole at the center of our galaxy, $M = 3.7 \times 10^6 M_{\text{Sun}}$, is a maximum-spin black hole. How much energy can be milked from it? Express your answer as a multiple of the mass M_{Sun} of our Sun.

306

5.1 LOCAL RAIN FRAME

308 *Drop from rest at a great distance.*

Head toward
the center in
a rain frame.

309 Riding on a ring is fun, but we want to head inward toward the singularity. The
 310 simplest unpowered taxi for that trip is the local inertial rain frame. By definition,
 311 the rain frame is the local rest frame of a freely falling observer which starts from
 312 rest at a great distance from the black hole and thereafter falls while maintaining
 313 the constants of motion.

314 *The term raindrop means an object at rest in the local inertial rain frame.* By
 315 definition, the raindrop has map energy and map angular momentum given by the
 316 expressions: (#rainconsts)

$$\frac{E}{m} = 1 \quad \text{and} \quad \frac{L}{m} = 0 \quad (\text{raindrop}) \quad (26)$$

317 For a *nonspinning* black hole, the trajectory of the raindrop is radially straight
 318 inward in Schwarzschild map coordinates. The same is *not* true for the trajectory of
 319 the raindrop in Kerr map coordinates. As the zero-angular-momentum raindrop
 320 falls, equation (1) tells us that it moves in a spiral with its tangential component of
 321 map velocity given by $R\omega_{\text{ring}} = 2Ma/(rR)$. Figure 14.12 shows the resulting spiral
 322 trajectory in space for a raindrop falling into a maximum-spin black hole ($a = M$).

323

QUERY 6. Raindrop motion in ring map coordinates

A. Substitute (26) into (14.35) and (14.36) and simplify to obtain: (#Astuffdtaurain)

$$\frac{dt}{d\tau} = \left(\frac{R}{rH}\right)^2 \quad (\text{raindrop}) \quad (27)$$

and (#Bstuffdtaurain)

$$\frac{d\phi}{d\tau} = \frac{2Ma}{r^3 H^2} \quad (\text{raindrop}) \quad (28)$$

where $d\tau$ is the wristwatch time differential for the raindrop,

B. Substitute (27) and (28) into the third term on the right side of the modified Kerr metric (2) to show that this third term goes to zero: (#3rdtermzero)

$$d\phi - \omega_{\text{ring}}(r)dt = 0 \quad (\text{raindrop}) \quad (29)$$

C. Use (27) and (29) to eliminate dt from what remains of the Kerr metric (2), solve for $dr/d\tau$, and substitute for R^2 and H^2 from equations (14.4) and (14.6) respectively, to obtain: (#vmapZ)

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \left(1 + \frac{a^2}{r^2}\right)^{1/2} \quad (\text{raindrop}) \quad (30)$$

where the minus sign comes from our knowledge that the raindrop is descending.

333

334 Figure 1 plots (30) for several values of a/M . Compare (30) with the
 335 corresponding rain equation (7.8)—that is, equation (8) of Chapter 7, Inside the
 336 [nonspinning] Black Hole.

337

QUERY 7. Time to the end

You ride on a raindrop. From (30) and Figure 1, show that the total time lapse on your wristwatch between passing any radius r_0 and arrival at the singularity will be shorter when you fall into a spinning black hole than when you fall into a nonspinning black hole. Use no integration!

342

“Rain falls vertically”
in ring frame.

343 By definition, the raindrop descends with zero angular momentum; by
 344 definition, the ring frame also moves with zero angular momentum. Result: The
 345 ring observer at every map radius records that the rain frame moves vertically
 346 downward with respect to his local frame—“rain” falls vertically for the ring rider.

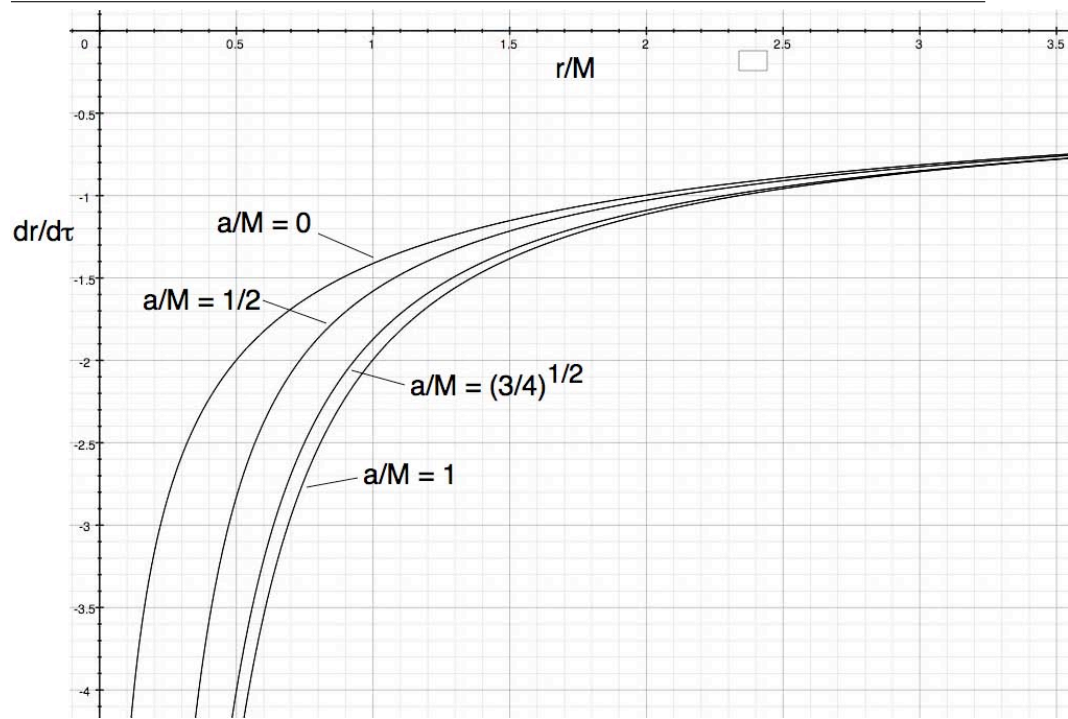


FIGURE 3 Negative radial map motion $dr/d\tau$ for a raindrop for the same values of a/M as in the plot of $H^2(r)$ in Figure 14.1.

347 To find expressions for rain frame coordinates of an event, carry out a special
 348 relativity Lorentz transformation from the ring frame. What is the relative speed of
 349 the ring frame measured by the rain observer? Combine (16) for $E/m = 1$ and
 350 $v_{\text{ring } x} = 0$ with equation (10) to obtain: (#endrainyvel)

$$v_{\text{relrain}} = \left[1 - \left(\frac{rH}{R} \right)^2 \right]^{1/2} \quad (\text{ring frame speed measured in rain frame}) \quad (31)$$

351 Equations (14.4) and (14.6) and Figure 14.1 tell us that R^2 is everywhere
 352 positive, but H^2 is negative between the Cauchy and event horizons, that is for
 353 $r_{\text{CH}} < r < r_{\text{EH}}$. In this region $v_{\text{relrain}} > 1$. This does not violate any direct
 354 observation, because no ring frame exists between the two horizons; in that region
 355 v_{relrain} remains a perfectly acceptable real numerical parameter. This is similar to
 356 the case of raindrops with shell velocity $|(2M/r)^{1/2}| > 1$ inside the horizon of a
 357 nonspinning black hole.

358 Now carry out the special relativity Lorentz transformation between ring and
 359 rain frame coordinates, equations (1.37), treating the rain frame as the laboratory
 360 frame and the ring frame as the rocket frame, using the positive value of v_{relrain}
 361 from (31). This time the x -coordinate is transverse to the direction of relative
 362 motion and therefore has the same value in both frames: (#frame17)

$$\Delta t_{\text{rain}} = \frac{\Delta t_{\text{ring}} + \bar{v}_{\text{relrain}} \Delta y_{\text{ring}}}{(1 - \bar{v}_{\text{relrain}}^2)^{1/2}} \quad (32)$$

Local rain
coordinates

$$\Delta y_{\text{rain}} = \frac{\Delta y_{\text{ring}} + \bar{v}_{\text{relrain}} \Delta t_{\text{ring}}}{(1 - \bar{v}_{\text{relrain}}^2)^{1/2}}$$

$$\Delta x_{\text{rain}} = \Delta x_{\text{ring}}$$

363 where the average value \bar{v}_{relrain} defines itself using average values of the variables in
 364 (31). Special relativity guarantees that this transformation yields a local inertial
 365 rain metric: (#ringflat)

$$\Delta\tau^2 \approx \Delta t_{\text{rain}}^2 - \Delta y_{\text{rain}}^2 - \Delta x_{\text{rain}}^2 \quad (33)$$

366 Substitute equations (31) and (8) into (32) to find rain coordinates in terms of
 367 map coordinate increments: (#frame18)

$$\Delta t_{\text{rain}} = \Delta t + \frac{\bar{v}_{\text{relrain}} \bar{R}}{\bar{r} \bar{H}^2} \Delta r \quad (34)$$

$$\Delta y_{\text{rain}} = \frac{\bar{R}}{\bar{r} \bar{H}^2} \Delta r + \bar{v}_{\text{relrain}}(\bar{r}) \Delta t$$

$$\Delta x_{\text{rain}} = \bar{R} \Delta\phi - \frac{2Ma}{\bar{r} \bar{R}} \Delta t$$

368 LOCAL RAIN FRAMES EVERYWHERE!

Local rain frames
exist everywhere.

369 Transformations (34) from global map coordinates to the rain frame are
 370 well-defined everywhere: R , H^2 , and v_{relrain} are everywhere real. Rain frames
 371 exist for all $r > 0$, which has big consequences: First, a cloudburst of rain
 372 frames can cover all spacetime in the vicinity of the black hole, which means
 373 that we can define a global rain frame (Section 6). Second, you can cheer up:
 374 General relativity permits you to fall onto the singularity of a spinning black
 375 hole, as described in Chapter 16, Explorations.

Rain frame
descriptions
of motion

376 The rain frame is a local inertial frame. In principle, a stone can change rain
 377 speed in the rain frame and thereby change its energy measured in that frame. Take
 378 the calculus limit of wristwatch time lapse $\Delta\tau$ of the stone to obtain (#rainpreview)

$$\frac{E_{\text{rain}}}{m} = \lim_{\Delta\tau \rightarrow 0} \frac{\Delta t_{\text{rain}}}{\Delta\tau} = \frac{1}{(1 - v_{\text{rain}}^2)^{1/2}} \quad (35)$$

379 The energy of a stone expressed in map quantities E/m and L/m is more
 380 complicated. In this case the relationship between Δt_{rain} and proper time $\Delta\tau$ is
 381 obtained by combining the first of equations (34) with equation (14.41). (#frame19)

$$\begin{aligned} \frac{E_{\text{rain}}}{m} &= \left(\frac{R}{rH} \right)^2 \left(\frac{E}{m} - \frac{2Ma}{rR^2} \frac{L}{m} \right) \\ &\pm v_{\text{relrain}} \left(\frac{R}{rH} \right)^2 \left[\left(\frac{E}{m} - \frac{V_L^+}{m} \right) \left(\frac{E}{m} - \frac{V_L^-}{m} \right) \right]^{1/2} \end{aligned} \quad (36)$$

382 The plus sign is for outgoing trajectories, the minus sign for infalling ones.

383 Another way to obtain energy and (linear) momentum in the rain frame is to
 384 Lorentz-transform these quantities from the ring frame. (#frame108)

$$E_{\text{rain}} = \frac{E_{\text{ring}} + v_{\text{relrain}} p_{\text{ring}, y}}{(1 - v_{\text{relrain}}^2)^{1/2}} \tag{37}$$

$$p_{\text{rain}, x} = p_{\text{ring}, x}$$

$$p_{\text{rain}, y} = \frac{p_{\text{ring}, y} + v_{\text{relrain}} E_{\text{ring}}}{(1 - v_{\text{relrain}}^2)^{1/2}}$$

385 We also have the expressions from special relativity: (#vrain)

$$v_{\text{rain}, x} = \lim_{\Delta t_{\text{rain}} \rightarrow 0} \left(\frac{\Delta x_{\text{rain}} / \Delta \tau}{\Delta t_{\text{rain}} / \Delta \tau} \right) = \frac{p_{\text{rain}, x}}{E_{\text{rain}}} \tag{38}$$

$$v_{\text{rain}, y} = \lim_{\Delta t_{\text{rain}} \rightarrow 0} \left(\frac{\Delta y_{\text{rain}} / \Delta \tau}{\Delta t_{\text{rain}} / \Delta \tau} \right) = \frac{p_{\text{rain}, y}}{E_{\text{rain}}}$$

Map constants
of motion

386 After a lot of algebra, we obtain the following simple results: (#rainsimples)

$$\frac{E}{m} = \left(1 - v_{\text{relrain}} v_{\text{rain}, y} + \frac{2Ma}{rR} v_{\text{rain}, x} \right) \frac{E_{\text{rain}}}{m} \tag{39}$$

$$\frac{L}{m} = R \frac{p_{\text{rain}, x}}{m} = R \frac{p_{\text{ring}, x}}{m}$$

$$\frac{dr}{d\tau} = \frac{R}{r} (v_{\text{rain}, y} - v_{\text{relrain}}) \frac{E_{\text{rain}}}{m}$$

387 All quantities in (39) are well-defined for all $r > 0$, an additional demonstration
388 that rain frames exist everywhere $r > 0$ for a spinning black hole.

389 MORE FIREWORKS HERE: PAYOFFS FROM RAIN FRAME EQUATIONS.

6. ■ STATIONARY LOCAL SHELL FRAME

391 *Stand still outside the horizon*

392 Now think of a **shell observer**, defined as someone who inhabits a local inertial
393 frame on a shell concentric to the black hole, that is *stationary* with respect to the
394 distant stars. To distinguish the stationary frame from the rotating ring frame, we
395 continue to use the term “shell,” even though the shell is now confined to a ring in
396 the equatorial plane. With respect to the rotating ring, the shell observer moves
397 backward, opposite to the direction of rotation of the black hole. With what speed
398 does the ring observer measure the shell observer to move past him? Find this ring
399 velocity from equations (34), then go to the differential limit. (#shellbackward)

Stationary
shell frames

$$v_{\text{ring}, x} = \lim_{\Delta t_{\text{ring}} \rightarrow 0} \frac{\Delta x_{\text{ring}}}{\Delta t_{\text{ring}}} = \frac{rd\phi - \frac{2Ma}{rR}}{\frac{rH}{R} dt} = \frac{R}{rH} \frac{d\phi}{dt} - \frac{2Ma}{r^2 H} \tag{40}$$

400 Now, the shell observer is, by definition, at rest in Kerr map coordinates. Therefore
401 $d\phi/dt = 0$ for the shell observer. Hence the ring observer measures the shell to move
402 backwards with speed (magnitude of velocity):(#shellrel)

$$v_{\text{relshell}} = -\frac{2Ma}{r^2 H} \quad (\text{speed of shell measured by ring observer}) \quad (41)$$

403 Treat local shell and ring frames as overlapping inertial frames. The rain
404 inertial metric is (9); the shell inertial metric is (#frame8Z)

$$\Delta\tau^2 \approx \Delta t_{\text{shell}}^2 - \Delta y_{\text{shell}}^2 - \Delta x_{\text{shell}}^2 \quad (42)$$

Local shell
coordinates

405 The shell and ring coordinates, energies, and linear momenta are related by the
406 special relativity Lorentz transformations (Chapter 1). Consider the ringrider as
407 rocket observer, shell occupant as laboratory observer, and use the positive value of
408 $\bar{v}_{\text{relshell}}$ to derive the transformations: (#shelltransform)

$$\begin{aligned} \Delta t_{\text{shell}} &= \frac{\Delta t_{\text{ring}} + \bar{v}_{\text{relshell}} \Delta x_{\text{ring}}}{(1 - \bar{v}_{\text{relshell}}^2)^{1/2}} \quad (43) \\ &= \left(1 - \frac{2M}{\bar{r}}\right)^{1/2} \Delta t + \frac{2Ma}{\bar{r}} \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \Delta\phi \\ \Delta y_{\text{shell}} &= \Delta y_{\text{ring}} = \frac{\Delta r}{H} \\ \Delta x_{\text{shell}} &= \frac{\Delta x_{\text{ring}} + \bar{v}_{\text{relshell}} \Delta t_{\text{ring}}}{(1 - \bar{v}_{\text{relshell}}^2)^{1/2}} \\ &= \bar{r} \bar{H} \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \Delta\phi \end{aligned}$$

409 **NO LOCAL SHELL FRAMES INSIDE THE STATIC LIMIT.**

410 Shell frames exist only where all the coefficients in (43) are real, namely for
411 $\bar{r} > 2M$, that is outside the static limit, as we already know.

Shell frame
description
of motion

412 Special relativity Lorentz momentum/energy transformations yield expressions
413 for shell energy and shell momentum components of a stone:(#frame5) LUC
414 LONGTIN SAYS HAVE E/m ON THE LEFT, SIMILAR TO CASES FOR RING
415 AND RAIN.

$$\begin{aligned} \frac{E_{\text{shell}}}{m} &= \left(1 - \frac{2M}{r}\right)^{-1/2} \frac{E}{m} \quad (44) \\ \frac{p_{\text{shell},x}}{m} &= \left(1 - \frac{2M}{r}\right)^{1/2} \frac{L}{mrH} + \left(1 - \frac{2M}{r}\right)^{-1/2} \frac{2Ma}{r^2 H} \frac{E}{m} \\ \frac{p_{\text{shell},y}}{m} &= \frac{1}{H} \frac{dr}{d\tau} \end{aligned}$$

416 where $dr/d\tau$ comes from (14.41).

417

QUERY 8. Map quantities from shell motion

Solve equations (44) for map quantities E/m , L/m , and $dr/d\tau$. Compare your results with entries in Table 1

420

421 Find expressions for $v_{\text{shell},x}$ and $v_{\text{shell},y}$, using a method similar to that for
 422 deriving components of ring velocity: (#vshell)

$$\begin{aligned}
 v_{\text{shell},x} &= \lim_{\Delta t_{\text{shell}} \rightarrow 0} \frac{\Delta x_{\text{shell}}}{\Delta t_{\text{shell}}} = \frac{p_{\text{shell},x}}{E_{\text{shell}}} & (45) \\
 &= \left(1 - \frac{2M}{r}\right) \frac{L}{rHE} + \left(1 - \frac{2M}{r}\right)^{-1} \frac{2Ma}{r^2H} \\
 v_{\text{shell},y} &= \lim_{\Delta t_{\text{shell}} \rightarrow 0} \frac{\Delta y_{\text{shell}}}{\Delta t_{\text{shell}}} = \frac{p_{\text{shell},y}}{E_{\text{shell}}} \\
 &= \left(1 - \frac{2M}{r}\right)^{1/2} \frac{m}{HE} \frac{dr}{d\tau}
 \end{aligned}$$

7.2 ■ SUMMARY OF FRAMES

424 *Multiple platforms for observation*

425 Table 1 summarizes results for shell, ring, and rain frames. The following Query
 426 concerns results of that table.

427

QUERY 9. Exploring results of Table 1

- 428 A. Show that the limit $a = 0$ reproduces the shell and rain results for a nonspinning black hole.
- 429 B. A ring rider at a map radius r inside the ergosphere shines light that he measures to have wavelength
 430 λ_{emit} (a) radially outward, (b) tangentially forward, and (c) tangentially backward. Do all light
 431 flashes escape to infinity? What is the wavelength λ_{remote} of those that escape, as measured by a
 432 stationary remote observer at $r \gg M$? LONGTIN: FOR B AND C NEED TABLE FOR LIGHT
 433 MOTION. TABLE 2 FOR LIGHT?
- 434 C. Same questions as item B for light emitted by a rain observer in the ergosphere in directions radially
 435 outward and tangentially forward and backward.
- 436 D. A shell observer located at $r > 2M$ is struck by a raindrop of mass m . What is the kinetic energy of
 437 the raindrop measured by the shell observer?
- 438 E. Same questions as item D for a ring rider located at $r > r_{\text{EH}}$ who is struck by a raindrop of mass m .
- 439 F. MORE SIMILAR QUESTIONS.

441

442 Riders in ring, rain, and shell frames are all subject to killer tides. The
 443 Appendix describes these tidal accelerations. You can use expressions in this
 444 Appendix to plan your explorations of and plunge into the spinning black hole.

445 EB: YOU SUGGESTED MODIFICATION OF THE E COLUMN FOR THE
 446 RING FRAME IN THE TABLE AS FOLLOWS: $\frac{rH}{R} (1 - v_{\text{relshell}}v_{\text{ring},x}) E_{\text{ring}}$
 447 WILL THIS CAUSE A STUDENT HICCUP: v_{relshell} HAS NO OBSERVATIONAL
 448 MEANING INSIDE THE ERGOREGION, WHERE SHELLS ARE NOT
 449 POSSIBLE? OF COURSE, THE EXPRESSION FOR v_{relshell} IS JUST AN
 450 EXPRESSION, WITHOUT ITS USUAL MEANING IN THAT REGION. IF WE
 451 USE IT, WE NEED TO COMMENT ON THAT FACT.

TABLE 1 Shell, ring, and rain energies, angular momenta and radial momentum as functions of the map E , L , and $m \frac{dr}{d\tau}$.

FRAME	Validity	$E =$	$L =$	$m \frac{dr}{d\tau} =$
Shell	$r > r_S$	$(1 - \frac{2M}{r})^{1/2} E_{\text{shell}}$	$rH (1 - \frac{2M}{r})^{-1/2} (v_{\text{shell},x} - v_{\text{relshell}}) E_{\text{shell}}$	$H p_{\text{shell},y}$ (44)
Ring	$r > r_{\text{EH}}$	$(\frac{rH}{R} + \frac{2M\alpha}{rR} v_{\text{ring},x}) E_{\text{ring}}$	$R p_{\text{ring},x}$	$H p_{\text{ring},y}$ (13)
Rain	$r > 0$	$(1 - v_{\text{relrain}} v_{\text{rain},y} + \frac{2M\alpha}{rR} v_{\text{rain},x}) E_{\text{rain}}$	$R p_{\text{rain},x}$	$\frac{E}{r} (v_{\text{rain},y} - v_{\text{relrain}}) E_{\text{rain}}$ (39)

8.3 GLOBAL RAIN (DORAN) COORDINATES

454 *A cloudburst of rain observers*

455 Local metrics like local ring metric (9), local rain metric (33), and the
 456 corresponding local shell metric cannot be extended to cover all spacetime because
 457 the coordinate transformations (8), (34), and (43) are all purely local. For example,
 458 one cannot convert $\Delta t_{\text{ring}} = (\bar{r}\bar{H}/\bar{R})\Delta t$ to a differential and integrate it, because
 459 the resulting integrand changes with \bar{r} .

460 However, the first of equations (34) *does* make sense in differential form, and it
 461 *can* be integrated to define a new time coordinate. The procedure parallels that for
 462 the rain frame describing the nonspinning black hole (Section 4 of Chapter 7, Inside
 463 the Black Hole): (#dtrain)

$$\lim_{\Delta t_{\text{rain}} \rightarrow 0} \Delta t_{\text{rain}} = dT = dt + \frac{v_{\text{relrain}} R}{r H^2} dr \quad (46)$$

464 where every symbol in the coefficient of dr is a function of r , and going to the
 465 calculus limit removes average values. Integrate the result to find global rain time:
 466 (#frame20)

$$T = t + \int \frac{v_{\text{relrain}} R}{r H^2} dr \quad (47)$$

467 [The square of the horizon function H^2 in the denominator of the integral goes to
 468 zero at the two horizons, because of the pathology of Schwarzschild time there, but
 469 the integral continues across these local infinities. The same comment applies to the
 470 integral for Φ in the later equation (52).] In the limit $a = 0$, this reproduces
 471 equations (7.19) and (7.20) of Chapter 7, Inside the Black Hole. Even for a spinning
 472 black hole, there exists a well-defined global time coordinate T that equals the
 473 wristwatch time of raindrops. The raindrop rider passes to the singularity in finite
 474 time from any given greater radius.

475 We might think that to get coordinates that are well-behaved everywhere we
 476 simply replace Kerr (Boyer-Lindquist) time t with rain time T and retain the other
 477 coordinates r and ϕ . However, equation (14.36) shows that $d\phi/d\tau$ has a problem
 478 that prevents its use as a global coordinate valid everywhere: H appears in the
 479 denominator, and it goes to zero at the event horizon. Hence $d\phi/d\tau$ blows up for
 480 any particle that approaches the event horizon: $\phi \rightarrow \infty$ there. Figure 14.12 shows
 481 the result for a raindrop falling into a maximum-spin black hole: it circulates
 482 forever at the event horizon, even when clocked in coordinate time T .

483 To solve this problem use the third equation of (8) to define a global angle
 484 increment Φ : (#frame18B)

$$\Delta x_{\text{ring}} \approx \bar{R} (\Delta\phi - \omega_{\text{ring}} \Delta t) \equiv \bar{R} \Delta\Phi \quad (48)$$

485 Take (48) to the calculus limit: (#defPhi)

$$d\Phi \equiv d\phi - \omega_{\text{ring}} dt \quad (49)$$

486 Now we can construct the the global rain metric by substituting into the Kerr
 487 metric differentials for dt and $d\phi$ from (46) and (49). Before that step, however, we
 488 need to show that Φ is a global metric in the sense that we can calculate the global
 489 rain angle of any event directly from Kerr coordinates of that event. To show this
 490 capability, substitute dt from (46) into (49): (#frame22)

$$d\Phi = d\phi + \frac{R\omega_{\text{ring}}v_{\text{relrain}}}{rH^2}dr - \omega_{\text{ring}}dT \tag{50}$$

491 Now, by definition dT is the wristwatch time $d\tau$ of a raindrop, for which we
 492 know the relation between dr and $d\tau$ from (30): (#frame23)

$$dT \equiv d\tau_{\text{raindrop}} = - \left[\left(\frac{2M}{r} \right)^{1/2} \left(1 + \frac{a^2}{r^2} \right)^{1/2} \right]^{-1} dr \quad (\text{raindrop}), \tag{51}$$

493 Using (50) and (51), we can write down an explicit integral expression for the
 494 rain angle Φ : (#IntegralPhi)

$$\Phi = \phi + \int \frac{R\omega_{\text{ring}}v_{\text{relrain}}}{rH^2}dr + \int \left[\left(\frac{2M}{r} \right)^{1/2} \left(1 + \frac{a^2}{r^2} \right)^{1/2} \right]^{-1} dr \tag{52}$$

495 where, remember, ω_{ring} and v_{relrain} , along with R and H , are functions of r .
 496 Integral equations (47) and (52), though messy to integrate, verify that global rain
 497 coordinates are legal, since they tie the rain coordinates of a given event to legal
 498 global Kerr coordinates for that same event. The continuity of Doran coordinates
 499 allow us to follow a stone or light flash through event and Cauchy horizons.

500 Rain coordinates (T, r, Φ) carry the formal name **Doran coordinates** after
 501 their inventor, C. Doran. Doran coordinates generalize Painlevé-Gullstrand
 502 coordinates, global rain coordinates of the non-spinning black hole (Section 4 of
 503 Chapter 7, Inside the Black Hole).

504 With verification that rain coordinates are valid global coordinates, we are free
 505 to substitute dt from (46) and $d\phi$ from (49) into the Kerr metric (2). After
 506 manipulation, the result is the global metric in rain (Doran) coordinates:
 507 (#frame25B)

$$d\tau^2 = dT^2 - \left[\frac{r dr}{(r^2 + a^2)^{1/2}} + \left(\frac{2M}{r} \right)^{1/2} (dT - a d\Phi) \right]^2 - (r^2 + a^2)d\Phi^2 \quad (\text{rain, Doran}) \tag{53}$$

508
 509 With $a = 0$, this reduces to equation (20) of Project 2, Inside the Black Hole.
 510 Metric (53) may be more complicated than the Boyer-Lindquist form, but it
 511 provides a rigorous foundation for probing inside a spinning black hole, as we do in
 512 Chapter 16, Spinning: Explorations.

513 *****

514 EB: YOU SAY: “Also show that E/m and L/m are the same for Kerr and
 515 Doran global coordinates. This must be done in Chapter 15.” DO YOU MEAN (A)
 516 THAT THE ALGEBRAIC EXPRESSIONS FOR E/m and L/m ARE
 517 IDENTICAL IF WE SIMPLY SUBSTITUTE $d\Phi$ FOR $d\phi$ AND dT FOR dt (A
 518 MESS TO SHOW), OR (B) THAT THEIR NUMERICAL VALUES ARE THE
 519 SAME, RESPECTIVELY, IN THE TWO COORDINATE SYSTEMS (EASY TO
 520 SHOW, SINCE FAR FROM THE BLACK HOLE $dT \rightarrow dt$ AND $d\Phi \rightarrow d\phi$ AND
 521 THE TWO QUANTITIES ARE CONSTANTS OF MOTION)?

9.2 ■ REFERENCES

523 Chris Doran, “New form of the Kerr solution,” Physical Review D, Volume 61,
524 2000, pages 067503-1 to 067503-4.

525

IN THE FOLLOWING CHAPTER 16, SPINNING: INSIDE THE KERR BLACK HOLE, AMONG OTHER SUBJECTS:

526
527 A. MOTION INSIDE THE SPINNING BLACK HOLE: INFINITE BLUE
528 SHIFTS AND MASS INFLATION AT THE CAUCHY HORIZON.

529
530 B. DOES THE KERR METRIC BREAK DOWN AT AND INSIDE CAUCHY
531 HORIZON?

532
533 C. DO WE BOUNCE RADially BACK OUTWARD FROM THE CAUCHY
534 HORIZON, THEN ENTER AN ENTIRELY DIFFERENT UNIVERSE?

535
536 D. DESCRIPTIONS OF VISUAL PANORAMAS AS WE FALL AS A
RAINDROP TO THE SINGULARITY OF BLACK HOLES WITH DIFFERENT
VALUES OF THE SPIN PARAMETER a .

10.3 ■ APPENDIX: TIDES NEAR THE SPINNING BLACK HOLE

538 ADD FIGURES, ESPECIALLY THOSE DUE TO OFF-DIAGONAL
539 COMPONENTS AND—IN THE ORBITER CASE—THE RESULTING RIBBON
540 DUE TO HIGH ORBITER SPEED.

541 ONCE EB APPROVES THIS APPENDIX, REVISE TIDE INTRODUCTION IN
542 SECTION 2 OF CHAPTER 19, ORBITING, TO SUPPRESS PHRASE “TIDAL
543 ACCELERATION” AND USE “TIDAL COMPONENT” INSTEAD. SEE
544 ATTACHED.

545 Expressions for tides in the Kerr metric are ugly, much worse than the expressions
546 for tides near a Schwarzschild black hole given in Section 2 of Chapter 10, Orbiting.
547 As in that earlier case, we cannot derive tides near a spinning black hole using only
548 the tools developed in this book. So we simply quote the resulting expressions.

549 The component of a tide in a given direction is the change in gravitational
550 acceleration in that direction proportional to change in position in that direction or
551 in some other direction. Here are expressions for the tidal components for someone
552 at rest in the ring frame. (#ringtidesB)

$$\Delta g_{\text{ring}, x} = -\frac{M}{\bar{r}^3} \Delta x_{\text{ring}} \quad (54)$$

$$\Delta g_{\text{ring}, y} = \frac{M}{\bar{r}^3} \frac{(2+q)}{(1-q)} \Delta y_{\text{ring}}$$

$$\Delta g_{\text{ring}, z} = -\frac{M}{\bar{r}^3} \frac{(1+2q)}{(1-q)} \Delta z_{\text{ring}}$$

553 where (#TideDefqB)

$$q \equiv \left(\frac{a\bar{r}\bar{H}}{\bar{r}^2 + a^2} \right)^2 \quad (55)$$

554

QUERY 10. Ring tidal components do not differ much from Newton's

Section 2 of Chapter 10, Orbiting, showed that the tidal components for someone at rest on a stationary shell are the same as those predicted by Newton. In this Query you show that the ring tidal components near a spinning black hole do not differ much from those of that nonspinning case.

- A. Show that q is dimensionless, is zero for $a = 0$, and increases monotonically with the value of a for any fixed map radius r .
- B. Take the calculus limit of q with $a = M$, take its derivative with respect to r , and set the resulting derivative equal to zero. Show that the maximum of q occurs at $r = 2.1414M$.
- C. Show that the maximum value of q is 0.043, so that tidal components (54) do not differ much from those for someone at rest on a shell outside a nonspinning black hole.

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566

QUERY 11. Volume of dust cloud conserved I

Show that the sum of the tidal components per unit separation, each in the same direction ($\Delta g_{\text{ring},x}/\Delta x_{\text{ring}} + \Delta g_{\text{ring},y}/\Delta y_{\text{ring}} + \Delta g_{\text{ring},z}/\Delta z_{\text{ring}}$) add up to zero. This means that the volume of a cloud of free dust particles is not changed by tides. This is a general result that we shall check in each case.

571

572 Next are tidal components for a stone moving in the positive x -direction along
 573 the ring with ring speed $v_{\text{ring},x}$. In this case there is an added tidal component in
 574 the x ring direction due to extension in the y -direction. This is due to the fact that
 575 adjacent rings at different radii move at different speeds. (#orbittidesB)

$$\begin{aligned} \Delta g_{\text{orbiter},x} &= -\frac{M}{\bar{r}^3} \left[1 - \left(\frac{3q}{1-q} \right) \left(\frac{v_{\text{ring},x}^2}{1-v_{\text{ring},x}^2} \right) \right] \Delta x_{\text{orbiter}} & (56) \\ \Delta g_{\text{orbiter},x} &= -\frac{3M}{\bar{r}^3} \left[\frac{\bar{H}a(\bar{r}^2 + a^2)}{\bar{r}\bar{R}^2} \right] \left[\frac{v_{\text{ring},x}^2}{(1-v_{\text{ring},x}^2)^{1/2}} \right] \Delta y_{\text{orbiter}} \\ \Delta g_{\text{orbiter},y} &= \frac{M(2+q)}{\bar{r}^3(1-q)} \Delta y_{\text{orbiter}} \\ \Delta g_{\text{orbiter},z} &= -\frac{M}{\bar{r}^3} \left[1 + \left(\frac{3q}{1-q} \right) \frac{1}{(1-v_{\text{ring},x}^2)} \right] \Delta z_{\text{orbiter}} \end{aligned}$$

576

MEANING OF "ORBITER"

577

The label *orbiter* also applies to a stone that move temporarily in the tangential Δx_{ring} direction along the ring, for example a spaceship that moves tangentially for an instant between a swoop in from a great distance and a retreat back outward.

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QUERY 12. Volume of dust cloud conserved II

Show that the sum of the tidal components per unit separation, each in the same direction ($\Delta g_{\text{orbiter},x}/\Delta x_{\text{orbiter}} + \Delta g_{\text{orbiter},y}/\Delta y_{\text{orbiter}} + \Delta g_{\text{orbiter},z}/\Delta z_{\text{orbiter}}$) add up to zero. (Ignore the tidal

component in the x_{orbiter} direction due to extension in the y_{orbiter} direction.) This means that the volume of a cloud of free dust particles is not changed by tides.

587

588 Here are the tidal components for someone at rest in the rain frame:
 589 (#raintidesB)

$$\Delta g_{\text{rain}, x} = -\frac{M}{\bar{r}^3} \left[1 - \frac{3a^2 \beta^2}{\bar{r}^2} \right] \Delta x_{\text{rain}} \quad (57)$$

$$\Delta g_{\text{rain}, x} = -\frac{3M}{\bar{r}^3} \frac{a\beta(\bar{r}^2 + a^2)}{R\bar{r}^2} \Delta y_{\text{rain}}$$

$$\Delta g_{\text{rain}, y} = \frac{M}{\bar{r}^3} \frac{(2+q)}{(1-q)} \Delta y_{\text{rain}}$$

$$\Delta g_{\text{rain}, z} = -\frac{M}{\bar{r}^3} \left[1 + \frac{3a^2}{\bar{r}^2} \right] \Delta z_{\text{rain}}$$

590 where (#TideDefqB)

$$\beta \equiv \frac{(\bar{r}^2 + a^2)^{1/2}}{\bar{R}} \left(\frac{2M}{\bar{r}} \right)^{1/2} \quad (58)$$

591

QUERY 13. Volume of dust cloud conserved III

Show that the sum of the tidal components per unit separation, each in the same direction ($\Delta g_{\text{rain}, x}/\Delta x_{\text{rain}} + \Delta g_{\text{rain}, y}/\Delta y_{\text{rain}} + \Delta g_{\text{rain}, z}/\Delta z_{\text{rain}}$) add up to zero. (Ignore the tidal component in the x_{rain} direction due to extension in the y_{rain} direction.) This means that the volume of a cloud of free dust particles is not changed by tides.

597

598 PROBLEMS

599 THOUGHT QUESTIONS

600 1. Overhead stars for rain observer

601 Thomas says, "By definition, the rain frame has zero map angular momentum. Now
 602 think of a rain frame viewer far from the spinning black hole who sees a star
 603 radially overhead. I claim that as the rain frame descends, its occupant will
 604 continue to see that same star overhead as she falls all the way through the event
 605 horizon." Is Thomas correct?

606 PROBLEMS

607 2. Penrose process for nonspinning black hole?

608 A. From Table 1, show that for a nonspinning black hole the relation between map
 609 energy of a stone and its energy measured in the rain frame is: (#maprainenergy)

$$E = \left[1 - \left(\frac{2M}{r} \right)^{1/2} v_{\text{rain}, y} \right] E_{\text{rain}} \quad (a = 0) \quad (59)$$

- 610 **B.** Show that it is impossible for the map energy of the stone to be negative outside the
 611 event horizon. Show that inside the event horizon, the stone will have negative map
 612 energy, provided that the following condition is satisfied: (#stoneneg)

$$v_{\text{rain}, y} > \left(\frac{r}{2M}\right)^{1/2} \quad (\text{for } E < 0, a = 0) \quad (60)$$

613 Equation (60) reinforces the conclusion that negative map energy is impossible
 614 outside the horizon, where it would require a rain-frame speed greater than that of
 615 light.

- 616 **C.** Do equations (59) and (60) allow a plunging stone to *decrease* the mass of a
 617 nonspinning black hole? (Refer to Section 7 Gravitational Mass, of Chapter 6,
 618 Plunging.) For example, suppose that inside the horizon a powerful rocket increases
 619 the rain speed $v_{\text{rain}, y}$ of the stone to satisfy (60) inside the horizon. Will this
 620 maneuver decrease the mass of the black hole? Hint: Think about the rocket
 621 propellant blasted inward by the rocket.

- 622 **D.** Describe sequential changes in black hole mass as, first, the rocket propellant, then
 623 the rain frame with its moving stone, reach the singularity. What is the final mass of
 624 the black hole measured by the distant exterior observer?

625 [NOTE: If we treat the stone and its rocket propellant as spherical shells, then the
 626 exterior gravitational field never deviates from that of a black hole with mass $M + E$
 627 (using the total positive E), by Birkhoff's theorem. When spherical symmetry is
 628 broken (for example, when *any* stone plunges into the black hole), there will be some
 629 emission of gravitational waves that decreases the black hole mass by a small amount
 630 proportional to the original positive E of the stone. ARE THESE GRAVITATIONAL
 631 WAVES GENERATED BEFORE OR AFTER THE STONE CROSSES THE
 632 EVENT HORIZON? IF AFTER, IS THERE A PROBLEM, FOR EXAMPLE A
 633 NECESSARY DEPARTURE FROM SCHWARZSCHILD SPACETIME?]