Kerr orbits

Slavomir Tuleja and Jozef Hanc

July 18, 2006

1 Motion of stones

1.1 Equations of motion using Lagrange formalism

We start with the metric for a Kerr black hole in Boyer-Lindquist coordinates:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 + \frac{4Ma}{r} dt d\phi - \frac{dr^2}{1 - \frac{2M}{r} + \frac{a^2}{r^2}} - \left(1 + \frac{a^2}{r^2} + \frac{2Ma^2}{r^3}\right) r^2 d\phi^2 \quad (1)$$

Now consider the Principle of stationary aging that can be written as follows

$$\int_1^2 d\tau \text{ is stationary} \quad (2)$$

Consider a quantity $L\left(r, \phi, \dot{r}, \dot{\phi}\right)$ defined by the equality

$$d\tau = L\left(r, \phi, \dot{r}, \dot{\phi}\right) dt \quad (3)$$

where

$$\dot{r} = \frac{dr}{dt} \text{ and } \dot{\phi} = \frac{d\phi}{dt} \quad (4)$$

It takes the following form:

$$L\left(r, \phi, \dot{r}, \dot{\phi}\right) = \left[\left(1 - \frac{2M}{r}\right) + \frac{4Ma}{r}\dot{\phi} - \frac{\dot{r}^2}{1 - \frac{2M}{r} + \frac{a^2}{r^2}} - \left(1 + \frac{a^2}{r^2} + \frac{2Ma^2}{r^3}\right) r^2 \dot{\phi}^2\right]^{\frac{1}{2}} \quad (5)$$

Now the Principle of stationary aging can be written as

$$\delta \int_{t_1}^{t_2} L dt = 0 \quad (6)$$

This equation is equivalent with a set of two Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \text{ and } \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0 \quad (7)$$

These two equations represent the equations of motion for a stone orbiting a Kerr black hole. They are just a bit general. The hardest thing is substituting L into the equations and simplifying. We didn't do this with pen and paper. Rather we used Mathematica – algebraic manipulation software that did the job for us.

Here are the results:

$$\begin{aligned}
\ddot{r} &= \frac{(a^2-2Mr+r^2)(-M+2aM\dot{\phi}-a^2M\dot{\phi}^2+\dot{\phi}^2r^3)}{r^4} - \\
&\quad -\frac{\dot{r}^2(-2a^2M+2a^3M\dot{\phi}+a^2r-3Mr^2+6aM\dot{\phi}r^2)}{r^2(a^2-2Mr+r^2)} \\
\ddot{\phi} &= -\frac{2\dot{r}(aM-2a^2M\dot{\phi}+a^3M\dot{\phi}^2-3M\dot{\phi}r^2+3aM\dot{\phi}^2r^2+\dot{\phi}r^3)}{r^2(a^2-2Mr+r^2)} \qquad (8)
\end{aligned}$$

What is good is the fact that in the limiting case of $a = 0$ equations take the same form as the equations for the Schwarzschild black hole.

1.2 Equations of motion using the method described in EBH

In this place we will describe our results that we obtained for Kerr black hole using similar methods to those used in the case of Schwarzschild black hole.

Using the same derivation as in the case of Schwarzschild orbits we came to the following expressions for energy and angular momentum as constants of motion:

$$\frac{E}{m} = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} + \frac{2Ma}{r}\frac{d\phi}{d\tau} \qquad (9)$$

$$\frac{L}{m} = \left(1+\frac{a^2}{r^2}+\frac{2Ma^2}{r^3}\right)r^2\frac{d\phi}{d\tau} - \frac{2Ma}{r}\frac{dt}{d\tau} \qquad (10)$$

Using these two equations and the metric, we derived expressions for increments dr, dt and dt when the wristwatch time increases by $d\tau$. Here they are:

$$\begin{aligned}
\frac{dr}{d\tau} &= \pm\left\{\frac{\left(\frac{L}{m}\right)^2(2M-r)+r^2\left[2M-r+\left(\frac{E}{m}\right)^2 r\right]-4a\left(\frac{E}{m}\right)\left(\frac{L}{m}\right)M+a^2\left[\left(\frac{E}{m}\right)^2(2M+r)-r\right]}{r^3}\right\}^{\frac{1}{2}} \\
\frac{d\phi}{d\tau} &= \frac{\left(\frac{L}{m}\right)(r-2M)+2a\left(\frac{E}{m}\right)M}{r(a^2-2Mr+r^2)} \\
\frac{dt}{d\tau} &= \frac{\left(\frac{E}{m}\right)r^3-2a\left(\frac{L}{m}\right)M+a^2\left(\frac{E}{m}\right)(2M+r)}{r(a^2-2Mr+r^2)} \qquad (11)
\end{aligned}$$

As we mentioned above, these three equations are not suitable for Runge-Kutta method simulation because of the ambiguity caused by $\pm$ sign in the expression for $dr/d\tau$. That's why we tried to derive equations (8) that are perfectly suitable for Runge-Kutta method simulation.

1.3 Effective potential(s) for a Kerr black hole

Using the first one of equations (11) one should derive an expression for Effective Potential. It is reasonable to suppose that the effective potential is not a function

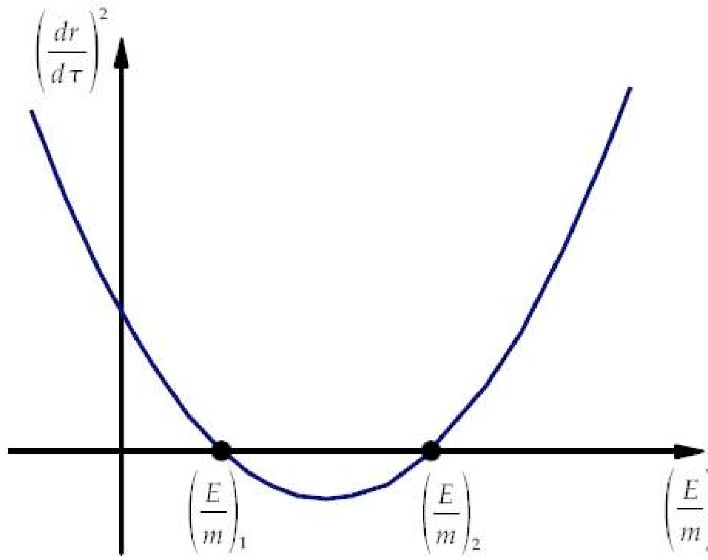

Figure 1: The expression $\left(\frac{dr}{d\tau}\right)^2$ in Eq. (12) can be viewed as a quadratic function of the parameter E/m. Depicted is such a function for some values of a, r, M, and L/m. We solve the inequality in Eq. (12) by finding all values of E/m such that the corresponding parabola points are above the horizontal axis. This gives two intervals of E/m values expressed in Eq. (13).

of (E/m). Raising this equation to the second power and rearranging its terms one obtains:

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{r^3 + a^2(2M+r)}{r^3}\left(\frac{E}{m}\right)^2 - \frac{4a\left(\frac{L}{m}\right)M}{r^3}\left(\frac{E}{m}\right) + \frac{\left(\frac{L}{m}\right)^2(2M-r) - a^2 r + r^2(2M-r)}{r^3} \geq 0 \tag{12}$$

This is a quadratic expression in (E/m) (see Fig. 1). Relation (12) will be valid if and only if

$$\left(\frac{E}{m}\right) \leq \left(\frac{E}{m}\right)_1 \text{ or } \left(\frac{E}{m}\right) \geq \left(\frac{E}{m}\right)_2 \tag{13}$$

Here $(E/m)_1$ is the smaller and $(E/m)_2$ the larger solution of the quadratic equation

$$\frac{r^3 + a^2(2M+r)}{r^3}\left(\frac{E}{m}\right)^2 - \frac{4a\left(\frac{L}{m}\right)M}{r^3}\left(\frac{E}{m}\right) + \frac{\left(\frac{L}{m}\right)^2(2M-r) - a^2 r + r^2(2M-r)}{r^3} = 0 \tag{14}$$

One can thus *define* TWO effective potentials $(V/m)_1 = (E/m)_1$ and $(V/m)_2 = (E/m)_2$. Solving quadratic equation (14) one obtains the following expressions:

$$\left(\frac{V}{m}\right)_{1,2} = \frac{2a\left(\frac{L}{m}\right)M \mp \sqrt{r(a^2 - 2Mr + r^2)\left[r^3 + a^2(r+2M) + r\left(\frac{L}{m}\right)^2\right]}}{r^3 + a^2(r+2M)} \tag{15}$$

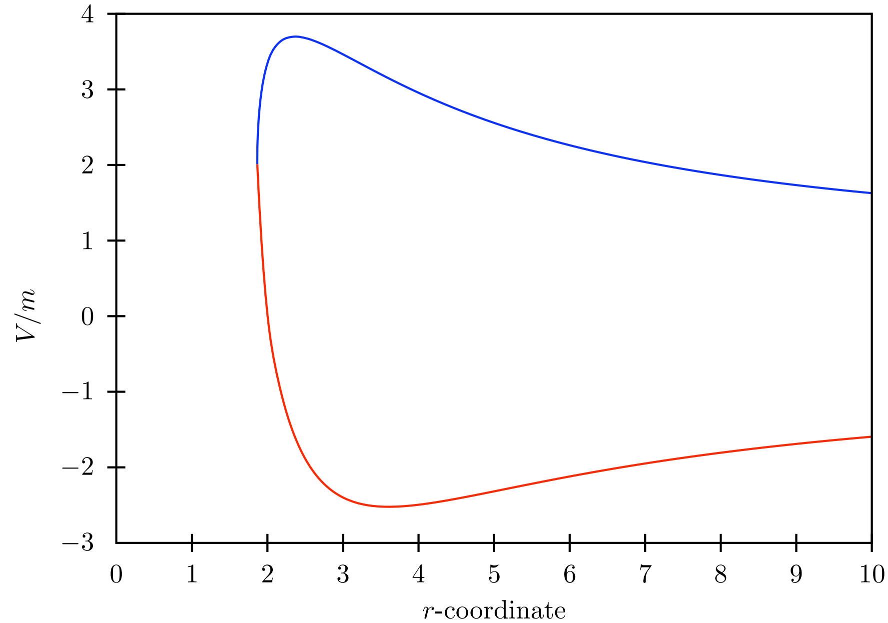

Figure 2: Effective potentials for a Kerr black hole. In this figure one can see both effective potentials in the case $a = 0.5M$ and $L/m = 15M$. The two effective potential curves join on the left at r-coordinate representing the outer horizon of a black hole.

We are quite not sure about one thing however. The above analysis suggests that the dot representing the orbiting stone in this diagram can be either above the upper curve or below the lower curve. Does this mean, that the stone can have negative energy? Or is this just a pure mathematical coincidence? We noticed that in case $a = 0$ (Schwarzschild black hole) both curves are symmetric, with the x axes as an axes of symmetry. Also, we noticed, that when we reverse the sign in L/m in the case of $a > 0$ then the curves change places.

Does that mean that there are two effective potentials, one for orbiting clockwise and the other for orbiting counterclockwise?

1.4 Initial conditions for the simulation

There's yet one thing we need. Suppose we know the initial energy of a stone, its initial angular momentum and its initial r-coordinate. How do we start the simulation using equations (8)? We need the initial value of $\dot{r}$ and $\dot{\phi}$. It is not difficult to derive these quantities from equations (11). We obtain:

$$
\begin{aligned}
\dot{r} &= \pm \frac{r\left[a^2 + r(r-2M)\right] \left\{ \frac{r^2\left[2M - r + \left(\frac{E}{m}\right)^2 r\right] + \left(\frac{L}{m}\right)^2 (2M-r) - 4a\left(\frac{E}{m}\right)\left(\frac{L}{m}\right) M + a^2\left[\left(\frac{E}{m}\right)^2 (2M+r) - r\right]}{r^3} \right\}^{\frac{1}{2}}}{\left(\frac{E}{m}\right) r^3 - 2a\left(\frac{L}{m}\right) M + a^2 \left(\frac{E}{m}\right)(2M+r)} \\
\dot{\phi} &= \frac{\left(\frac{L}{m}\right)(r-2M) + 2a\left(\frac{E}{m}\right) M}{\left(\frac{E}{m}\right) r^3 - 2a\left(\frac{L}{m}\right) M + a^2 \left(\frac{E}{m}\right)(2M+r)}
\end{aligned}
\tag{16}
$$

The plus sign in equation for $\dot{r}$ represents *outward* initial motion and the minus sign represents the *inward* initial motion.

So now, when we know the initial value of energy, angular momentum and r-coordinate, we can compute the initial values of $\dot{r}$ and $\dot{\phi}$ using equations (16). Then we set $\phi = 0$ and start the simulation using equations (8). This should be a straightforward task to program this to the computer.

2 Motion of light

We start from equations (11). Manipulating them, expressing $\left(\frac{dr}{dt}\right)^2 = \left(\frac{dr}{d\tau}\right)^2 \left(\frac{d\tau}{dt}\right)^2$, then using the substitution $b = L/E$ and setting $m = 0$ we obtain the equations of motion for a light particle

$$
\begin{aligned}
\frac{dr}{dt} &= \pm \left\{ \frac{\left[a^2 + r(r-2M)\right]^2 \left[r^3 - 4abM + b^2(2M-r) + a^2(2M+r)\right]}{r\left[r^3 - 2abM + a^2(2M+r)\right]^2} \right\}^{\frac{1}{2}} \\
\frac{d\phi}{dt} &= \frac{b(r-2M) + 2aM}{r^3 - 2abM + a^2(2M+r)}
\end{aligned}
\tag{17}
$$

These equations are not suitable for Runge-Kutta simulation (ambiguity caused by $\pm$ sign) so we use a different equation for change of r-coordinate. (We have applied the same procedure also in the case of Schwarzschild black hole.) We take the first derivative with respect to t of the square of the first equation (17) and then express $\ddot{r}$. The resulting formula is quite long, but is not ambiguous any more. Here it is:

$$
\begin{aligned}
\ddot{r} = & -\left\{(a^2 - 2Mr + r^2)[2a^3(a-b)^3 M^2 + aMr(a-b)^2[3a^3 + 4M^2(a-b)] + a(a-b)(a^4 + a^3 b - 4a^2 M^2 + \right. \\
& + 14abM^2 - 10b^2 M^2)r^2 - 2abMr^3(3a^2 - 5ab + 2b^2) + 2r^4[a^4 + 6abM^2 - 5b^2 M^2 - a^2(b^2 + M^2)] + \\
& \left. + r^5 M(7b^2 - 5a^2) + (a^2 - b^2)r^6 - 2Mr^7\right\} / \left\{r^2[2aM(a-b) + a^2 r + r^3]^3\right\}
\end{aligned}
\tag{18}
$$

2.1 Effective potential for light

The effective potential for light can be found analogously to the case of no rotating black hole. We start with the square of the first of equations (17) for light:

$$
\left(\frac{dr}{dt}\right)^2 = \frac{\left[a^2 + r(r-2M)\right]^2 \left[r^3 - 4abM + b^2(2M-r) + a^2(2M+r)\right]}{r\left[r^3 - 2abM + a^2(2M+r)\right]^2} \geq 0 \tag{19}
$$

The right-side expression must be nonnegative. Multiplying the inequality by the denominator and taking account of the nonnegative sign of the first bracket in the nominator $\left[a^2 + r(r - 2M)\right]^2$ it can be easily seen that the previous inequality is equivalent to this:

$$r^3 - 4abM + b^2(2M - r) + a^2(2M + r) \geq 0 \tag{20}$$

We multiply this inequality by a positive factor $(1/b)^2$ and rearrange:

$$\left(\frac{1}{b}\right)^2 \left[a^2(r + 2M) + r^3\right] - 4aM\left(\frac{1}{b}\right) - (r - 2M) \geq 0 \tag{21}$$

The quadratic term coefficient is positive. It is not difficult to solve this quadratic inequality. Solving it, one obtains the following results:

$$\left(\frac{1}{b}\right) \geq \frac{2aM + \sqrt{r^2(a^2 + r^2 - 2Mr)}}{r^3 + a^2(r + 2M)} \quad \text{or} \quad \left(\frac{1}{b}\right) \leq \frac{2aM - \sqrt{r^2(a^2 + r^2 - 2Mr)}}{r^3 + a^2(r + 2M)} \tag{22}$$

The right-hand expressions are the TWO EFFECTIVE POTENTIALS for light.

The special case of (22), $a = 0$ gives the effective potential for light in the case of Schwarzschild black hole.

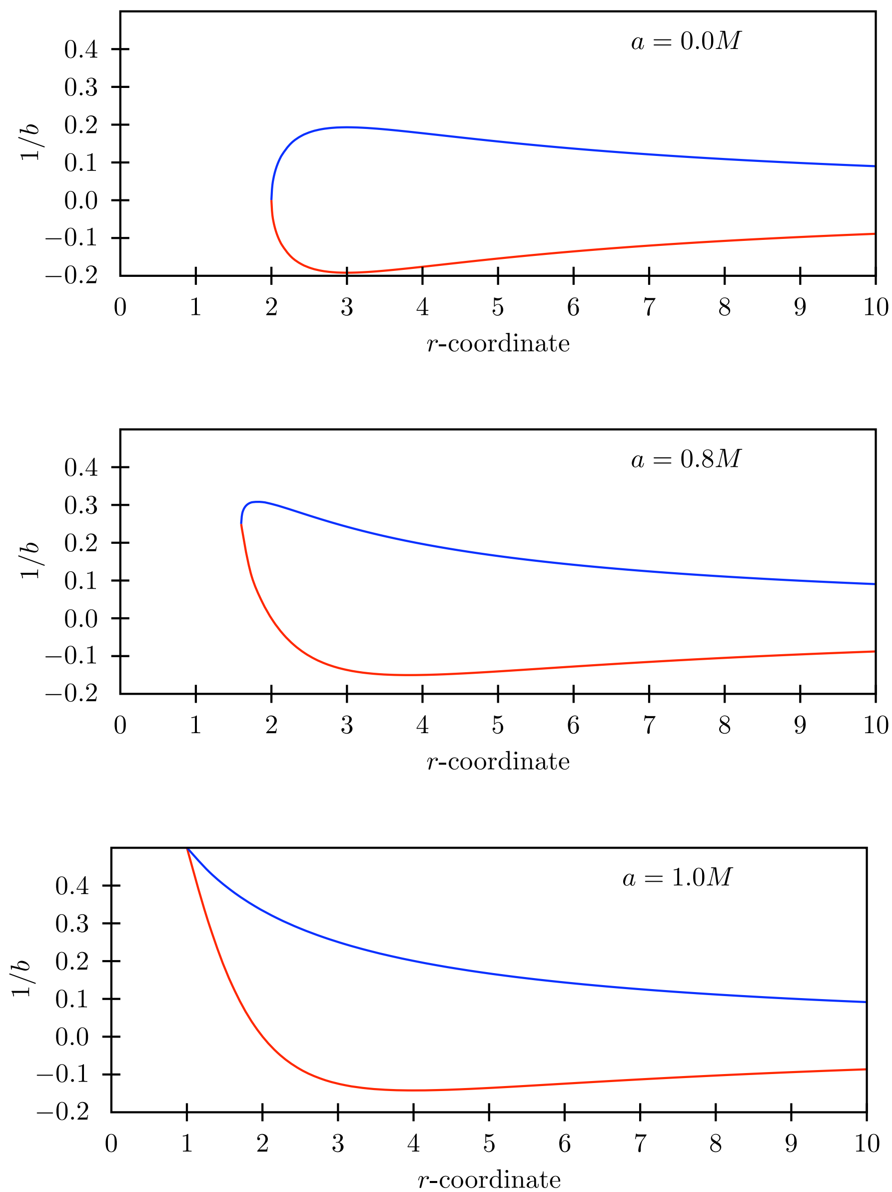

Figure 3: Effective potentials for LIGHT. In this figure one can see both effective potentials in three different cases for value of parameter a. The two effective potential curves join on the left at r-coordinate representing the outer horizon of a black hole. The two potentials are in color.