General relativity for sophomores

James B. Hartle[1] has presented a strong case for teaching general relativity to undergraduates. General relativity, he says, "underlies our contemporary understanding of the big bang, black holes, pulsars, quasars, x-ray sources, the final destiny of stars, gravitational waves, and the evolution of the universe. … Paradoxically, general relativity, which is so well established, so important, and so simple in its basic conception, is often not represented in the undergraduate physics curriculum. … The subject excites interest in students. Warped spacetime, black holes, and the big bang are the focus both of contemporary research and of popular scientific fascination. Students specializing in physics naturally want to know more."

Hartle proposes an approach in which "the simplest physically relevant solutions to the Einstein equations are introduced first, without derivation, as curved spacetimes whose properties and observable consequences can be explored." This proposal and those of others led to an AAPT workshop at Syracuse University in July 2006.[2] Participants reported their experience teaching such courses, described a variety of teaching methods, and discussed current applications of general relativity in astrophysics and cosmology.

How far can we drive Hartle's physics-first strategy? Can sophomores study general relativity without exceeding either their physics background or their mathematical skills and without distorting the subject? We have taught such a course for years using no tensors, differential geometry, or linear algebra and with no required mathematics beyond single-variable calculus. Our students include sophomores through seniors majoring in biology, engineering, and mathematics as well as physics. Here we report some strategies derived from our experience, paying special attention to John Archibald Wheeler's first two *Rules of Writing*:[3] "Motivate! Motivate!" and "Simplify! Simplify!"

Simplicity and motivation combine in studying the black hole. The isolated nonspinning black hole is nature's simplest structure; all of its properties are completely determined by a single parameter: its mass. The endlessly fascinating properties of the *spinning* black hole flow from one additional parameter: its angular momentum. Black holes may be the only structures in the history of science whose basic characteristics were deeply understood before there was general agreement that they exist. It gets better: Many key ideas and observable consequences of general relativity flow from the study of black holes: the global positioning system,[4] gravitational redshift,[5] bending of light by the Sun,[6] precession of Mercury's perihelion,[7] Shapiro time delay of light,[8] gravitational lensing,[9] frame-dragging by a rotating body,[10] spin and orbit precessions,[11] and the physics of spacetime external to neutron stars.[12] The general public is fascinated by black holes and so are students. To motivate and simplify general relativity, begin with the black hole.

The metric is the solution of Einstein's equations. The metric for the black hole completely specifies the spacetime geometry and all gravitational effects in its vicinity. However, the metric alone tells us nothing about particle orbits. To track these we need the principle of maximal aging, defined in the summary at the end of this editorial.

After employing the metric and the principle of maximal aging to analyze black holes, students can apply the same tools to time-dependent metrics for gravitational waves and for the universe as a whole, bringing the beginner close to leading-edge observations and research.

Several additional major and minor strategies increase simplicity and motivation:

- *Simplified units.* The first paragraph of most relativity research papers contain a sentence such as "In the following we set $G=c=1$." This statement bewilders beginning students, because neither G nor c have the value unity. Shorthand requires an introduction: In special relativity we measure time in meters of light-travel time, leading to the speed v as a unitless fraction of the speed of light. The black hole's mass can be expressed in meters using the conversion $M_{\text{meters}} \equiv GM_{\text{kilograms}}/c^2$. Expressing time and mass in meters simplifies equations.
- *Metric in a plane.* A stone moving in a plane through the center of a nonspinning black hole or in the equatorial plane of a spinning black hole will continue to move in that plane. Many important consequences of general relativity derive from the metric in a spatial plane, leading to further simplification of equations.
- *Thought experiments:* Part of Einstein's genius was his use of thought experiments as an aid to thinking and visualization. Thought experiments can make theoretical results more concrete. Assemble around each black hole a set of (imaginary) concentric spherical shells and ask what life is like standing on a shell. Construct concentric rings in the equatorial plane of a spinning black hole; each ring has zero angular momentum and therefore is swept around by rotating space. Ask what a rider on each ring experiences. *Raindrops* are small capsules that fall from rest at a great distance from the black hole. Raindrops cross the horizon without a jounce or jolt. What story is lived out by riders in raindrop capsules?
- *Self-descriptive terminology.* Teachers are responsible for introducing students to the standard terminology of their subject, but may use additional terms to remind the hearer of their meaning. In addition to "proper time" use "wristwatch time;" in addition to "test particle" use "stone;" in addition to "zero angular momentum observer" use "ring rider." A "patch" is a local region of spacetime small

enough to be effectively flat for a given experiment. A local metric on a patch is a stand-in for the simplest aspects of a powerful tool in advanced general relativity called a *tetrad*.[13]

- *Reducing hiccups.* What is the meaning of the symbol m? Is it mass? meters? What is s? A second? A spatial distance? The student must stop to decode each abbreviation—a distracting interruption that we call a hiccup. Reduce hiccups by not using abbreviations at all (except in a few subscripts). Spell out meter and second and all other abbreviated terms—even the title American Journal of Physics in references. Then m always means *mass* and s always means *spatial distance* in the text as well as in equations. Using self-descriptive terminology reduces the hiccups of recalling technical terms.
- *Interactive software.* The principle of maximal aging applied to the metric leads to complicated algebraic expressions for the orbit of a stone or a massless particle in the equatorial plane of a spinning black hole. Rather than display these equations, embody them in an interactive computer program.[14] Turn students loose to create many possible trajectories, and ask them conceptual and quantitative questions about the resulting plots. Intuition is more likely to follow manipulation that has visual consequences.

Even with every possible simplification and motivation, general relativity remains difficult and perplexing, mostly because it violates our unconscious assumptions developed in everyday experience. We try not to submerge these difficulties in a powerful mathematical formalism. The new learner needs constant reminders of the central concepts and tools of the subject and how they form a unified whole. So post in a central location a summary of the key ideas in logical order (rather than alphabetically). An example of such a summary follows:

GENERAL RELATIVITY SUMMARY

EVENT: An occurrence that can be located at a point in space at an instant of time, such as your birth. Events are the elemental nails on which physics hangs. An event exists independent of any method we use to locate it.

SPACETIME: The arena in which events occur. Newton thought that time is universal, the same for all observers. Einstein realized that different observers typically measure different values of time between events, along with different values of their spatial separation. Einstein combined space and time to provide a spacetime measure of separation between events on which all observers agree.

SPACETIME REGION: A volume of space during a period of time, for example, the vicinity of a black hole during one year.

GRAVITY: In Newtonian physics, a universal force arising from mass. For Newton, forces may be real (such as gravity) or fictitious (such as the centrifugal force and the Coriolis force) and gravity can be correctly analyzed in a single global inertial frame ("inertial" is defined later). In general relativity, gravity is always a fictitious force which can be eliminated by changing to a frame that is in free fall (a different free fall frame for each event).

CURVATURE: A measure of those properties of spacetime which prevent us from defining a global inertial frame. Sources of curvature include mass-energy and pressure.

GENERAL RELATIVITY: A theory of curved spacetime and motion.

SINGULARITY: A spacetime location at which curvature is so extreme that general relativity fails.

PATCH: A spacetime region purposely limited in size and duration so that curvature does not noticeably affect a given experiment or motion. A patch can have uniform gravity (which generates zero curvature). Curved spacetime can be approximated to any desired accuracy by a set of patches, like a floor curving down toward a drain can be covered by "sufficiently small" flat tiles.

STONE: A free particle whose location at each instant is described by an event and whose mass warps spacetime too little to be measured. A stone has nonzero mass and moves slower than light.

WORLDLINE: The path of a stone through spacetime. The worldline can be marked by a chain of intermediate events such as the ticks of the stone's wristwatch. The *wristwatch time* along a worldline is the reading on the stone's wristwatch.

GLOBAL COORDINATE SYSTEM: Any system that assigns numbers ("coordinates") to an event in order to locate it in an extended spacetime region. General relativity frees us to use any global coordinate system in a given spacetime region. We usually analyze events that occur on a single spatial plane for which three coordinates suffice such as (t,x,y) or (t,r,ϕ).

FRAME: A patch onto which local coordinates have been installed. Local coordinates are limited to a single patch.

INERTIAL FRAME or FREE-FALL FRAME: A frame in which special relativity is locally valid. In an inertial frame a stone moves straight with constant speed, that is, along a straight worldline. An inertial frame is available near any event except on singularities such as those inside a black hole. In general relativity inertial frames are local; spacetime curvature precludes global inertial frames.

INTERVAL: Any one of three possible alternative measures of the spacetime separation between events, analyzed here for two events recorded in an inertial frame. If a stone can travel directly from one event to the other, the time lapse on that stone's wristwatch is called the *wristwatch time* or *timelike interval*. If a light ray connects two events, we say that there is a *lightlike interval* between them. If nothing is fast enough to move between the two events, then there exists an alternative inertial frame such that the two events are simultaneous in that frame. The distance between the events measured along a ruler at rest in the alternative frame is called the *ruler distance* or *spacelike interval*.

METRIC: A function whose inputs are global coordinate differentials (such as dt, dr, $d\phi$) between an adjacent pair of events and whose output is the square of the interval between the events. The metric completely specifies the spacetime geometry and completely describes all gravitational effects within the global region in which it is expressed. The metric combines with the principle of maximal aging (defined later) to predict the motion of stones and massless particles in curved spacetime.

FRAME METRIC: A metric expressed in the local coordinates of a frame and valid only in that frame.

INVARIANT: A physical quantity that has the same magnitude whether measured or calculated with respect to any possible frame or global coordinate system. Examples include the interval (wristwatch, lightlike, or ruler) between two infinitesimally close events, the wristwatch time along a worldline, the mass of a stone, and the mass of a center of gravitational attraction.

PRINCIPLE OF MAXIMAL AGING: What worldline will a stone follow? Special relativity commands the free stone: *Follow a straight worldline in a local inertial frame*. From the twin "paradox" special relativity derives an equally valid command: *Follow the worldline of maximal wristwatch time (maximal aging).* Every nearby alternative worldline between given initial and final events requires rocket blasts and has a shorter wristwatch time—smaller aging. Special relativity is limited to an inertial frame, a mere patch in curved spacetime. General relativity expands the maximal-aging command to the principle of maximal aging: *Follow the worldline of maximal aging across any two adjoining patches.* Curved spacetime can be completely tiled with adjoining patches, so the stone knows how to move near a black hole.

CONSTANT OF MOTION: A quantity describing the motion of stone or massless particle whose value does not change with time. Constants of motion arise when coefficients in the metric are independent of time or of one spatial coordinate. Energy and angular momentum are constants of motion for a stone in a black hole spacetime.

Every reader, whether expert or novice, is invited to help us correct and clarify draft teaching materials based on the foregoing strategies. No misstatement or difficulty is too small to examine and correct. Our materials are freely available at ⟨www.eftaylor.com/comments⟩.

ACKNOWLEDGMENTS

We gratefully acknowledge advice on the General Relativity Summary from Tevian Dray, James B. Hartle, Kip S. Thorne, and Clifford W. Will.

[a] Electronic mail: eftaylor.mit.edu

[1] James B. Hartle, "General relativity in the undergraduate physics curriculum," Am. J. Phys. **74**(1), 14–21 (2006).

[2] See talks, papers, and posters presented at the AAPT workshop on Teaching General Relativity to Undergraduates, ⟨www.aapt-doorway.org/TGRU/⟩.

[3] John A. Wheeler, "Wheeler's rules of writing," Am. J. Phys. **67**(11), 945 (1999).

[4] James B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Pearson Addison-Wesley, San Francisco, 2003), p. 121.

[5] Reference 4, p. 219.

[6] Reference 4, p. 210.

[7] Reference 4, p. 230.

[8] Reference 4, p. 229.

[9] Reference 4, p. 234.

[10] Reference 4, p. 296.

[11] Reference 4, p. 302.

[12] Reference 4, pp. 244, 250.

[13] Steven Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, New York, 1972), p. 365.

[14] Slavomir Tuleja, "GRorbits: Interactive software for orbits of stones and massless particles in the equatorial plane of a black hole of arbitrary spin," ⟨vk.upjs.sk/~tuleja/grorbits/⟩.

Edmund Bertschinger and Edwin F. Taylor[a]
Department of Physics
Massachusetts Institute of Technology
77 Massachusetts Avenue, Cambridge, Massachusetts 02139