ANNOTATED BIBLIOGRAPHY ON THE PRINCIPLE OF LEAST ACTION

Edwin F. Taylor

All entries after the first two are listed in alphabetical order by last name of initial author.

Richard P. Feynman, *The Feynman Lectures on Physics*, Volume II, Addison Wesley, 1964, Chapter 19, ISBN 0-201-02117-X-P. To my mind the best analytic introduction to the principle of least action.

> *When I was in high school, my physics teacher—whose name was Mr. Bader—called me down one day after physics class and said, "You look bored; I want to tell you something interesting." Then he told me something which I found absolutely fascinating, and have, since then, always found fascinating. . . . the principle of least action.* page 19-1

> *I have been saying that we get Newton's law [from the principle of least action]. That is not quite true, because Newton's law includes nonconservative forces like friction. Newton said that* ma *is equal to any* F. *But the principle of least action only works for conservative systems—where all forces can be gotten from a potential function. You know, however, that on a microscopic level—on the deepest level of physics—there are no nonconservative forces. Nonconservative forces, like friction, appear only because we neglect microscopic complications—there are just too many particles to analyze. But the* fundamental *laws* can *be put in the form of a principle of least action.* page 19-7

> *There is quite a difference in the characteristic of a law which says a certain integral from one place to another is minimum—which tells something about the whole path—and of a law which says that as you go along, there is a force that makes it accelerate. The second way tells how you inch your way along the path, and the other is a grand statement about the whole path. ... Now if the entire integral from t_1 to t_2 is a minimum, it is also necessary that the integral along the little section from a to b is also minimum. It can't be that the part from a to b is a little bit more. Otherwise you could just fiddle with just that piece of the path and make the whole integral a little lower. ... So the statement about the gross property of the whole path becomes a statement of what happens for a short section of the path—a differential statement. And this differential statement only involves ... the force at a point. That's the qualitative explanation of the relation been he gross law and the differential law.* page 19-8

> *Is it true that the particle doesn't just "take the right path" but that it looks at all the other possible trajectories? ... The miracle of it all is, of course, that it does just that. ... It isn't that a particle takes the path of least action but that it smells all the paths in the neighborhood and chooses the one that has the least action ... That's the relation between the principle of least action and quantum mechanics. The fact that quantum mechanics can be formulated in this way was discovered in 1942 by a student of that same teacher, Bader, I spoke of at the at the beginning of this lecture [namely by Feynman himself].* page 19-9

Richard P. Feynman, *The Character of Physical Law*, (Random House, Inc. New York 1994, ISBN 0-679-60127-9), Chapter 4, pages 97 to 100. Elementary, but very thoughtful introduction to symmetries and their connection to the principle of least action and conservation laws.

> *If we assume that the laws of physics are describable by a minimum principle, then we can show that if a law is such that you can move all the equipment to one side, in other words if it is translatable in space, then there must be conservation of momentum. There is a deep connection between the symmetry principles and the conservation laws, but that connection requires that the minimum principle be assumed. In the second lecture we discussed one way of describing physical laws by saying that a particle goes from one place to another in a given length of time by trying different paths. There is a certain quantity which, perhaps misleadingly, happens to be called the action. When you calculate the action on the various paths you will find that for the actual path taken this quantity is always smaller than for any other. That way of describing the laws of nature is to say that the action of certain mathematical formulae is least for the actual path of all the possible paths. Another way of saying a thing is least is to say that if you move the path a little bit at first it does not make any difference. Suppose you were walking around on hills – but smooth hills, since the mathematical things involved correspond to smooth things – and you come to a place where you are lowest, then I say that if you take a small step forward you will not change your height. When you are at the lowest or at the highest point, a step does not make any difference in the altitude in first approximation, whereas if you are on a slope you can walk down the slope with a step and then if you take the step in the opposite direction you walk up. That is the key to the reason why, when you are at the lowest place, taking a step does not make much difference, because if it did make any difference then if you took a step in the opposite direction you would go down. Since this is the lowest point and you cannot go down, your first approximation is that the step does not make any difference. We therefore know that if we move a path a little bit it does not make any difference to the action on a first approximation.*

Figure 25

> *We draw a path, A to B (fig. 25),), and now I want you to consider the following possible other path. First we jump immediately over to another place near by, C, then we move on exactly the corresponding path to another point, which we will call D, which is displaced the same amount, of course, because it is the corresponding path. Now we have just discovered that the laws of nature are such that the total amount of action going on the ACDB path is the same in the first approximation to that original path AB – that is from the minimum principle, when it is the real motion. I will tell you something else. The action on the original path, A to B, is the same as the action from C to D if the world is the same when you move everything over, because the difference of these two is only that you have moved everything over. So if the symmetry principle of translation in space is right, then the action on the direct path between A and B is the same as that on the direct path between C and D. However for the true motion the total action on the indirect path ACDB is very nearly the same as on the direct path AB, and therefore the same as just the part C to D. This indirect action is*

the sum of three parts – the action going A to C, that of C to D, plus that from D to B. So, subtracting equals from equals, you can probably see that the contribution from A to C and that from D to B must add up to zero. But in the motion for one of these sections we are going one way, and for the other the opposite way. If we take the contribution of A to C, thinking of it as an effect of moving one way, and the contribution of D to B as B to D, taking the opposite sign because it is the other way, we see that there is a quantity A to C which has to match the quantity B to D to cancel off. This is the effect on the action of a tiny step in the B to D direction. That quantity, the effect on the action of a small step to the right, is the same at the beginning (A to C) as at the end (B to D). There is a quantity, therefore, that does not change as time goes on, provided the minimum principle works, and the symmetry principle of displacement in space is right. This quantity which does not change (the effect on the action of a small step to one side) is in fact exactly the momentum that we discussed in the last lecture. This shows the relation of symmetry laws to conservation laws, assuming the laws obey a principle of least action. They satisfy a principle of least action, it turns out, because they come from quantum mechanics. That is why I said that in the last analysis the connection of symmetry laws to conservation laws comes from quantum mechanics.

Jorge Dias de Deus, Mário Pimenta, Ana Noronha, Teresa Peña, and Pedro Brogueira, Introducción al la Física (Spanish) (McGraw-Hill, 2001, ISBN: 84-481-3190-8) and Introdução à Física (Portuguese) (McGraw-Hill, 2000, ISBN: 972-773-035-3), 2nd ed. Introductory university physics text that begins the study of Newtonian mechanics using symmetry and the principle of least action.

When, in 1992, we wrote "Introduction to Physics" two references were sacred for us. We were inspired by two great physicists who were, at the same time, great innovators in physics education. One was the Soviet Lev Landau (1908-1964) and the other the North American Richard Feynman (1918-1988). From one we wanted to attain rigour, *from the another one we wanted to achieve* clarity. *Our ambition was rigour without austerity, and clarity without superficiality. Neither more nor less!*

Even today we do not know how well we accomplished this aim. The book, when it appeared, caused some astonishment. Naturally those who are more conservative didn't like it. But there were also others, perhaps more open-minded, who liked it. In this second edition of the year 2000, we reaffirm our ambition to work within the two frames of rigour and clarity. From the preface to the second edition.

Symmetry is a familiar notion. It is always associated with invariance. One says that there is a symmetry in a system whenever the system remains invariant when one executes a certain operation on the system. The operation can be to interchange the left side with the right side (such as happens to the image of an object in a mirror), to rotate around an axis, to apply a displacement from one point to another. The system possesses a definite symmetry when it looks the same, that is it is invariant, under such an operation.

A tile, as a rule, has symmetry of reflection, because it is invariant under interchanging its left side with its right side. A bottle (without label!) presents symmetry of rotation around an axis, that is, it looks the same before and after the operation of rotation. A straight road in a plain and uniform landscape has a translation symmetry, that is, it looks the same when moved along its length.

In science, as in art, symmetries have enormous importance. They function as factors of structure, organization and representation of the systems that are intended to be described. There is symmetry of reflection in pictures and sculptures, and there are invariances for time translations, that is repetitions, in musical works. All these symmetries are equally present in various branches of *science, in particular in Physics.* Beginning of Chapter 3: From symmetries in space-time to Mechanics, page 87

Historically, the idea that physical laws could be obtained through optimization, using variational calculus, has its roots in Fermat's principle, which says that the path of a light ray in a medium is always the one that corresponds to the minimization of time. Accordingly, in an homogeneous medium the light rays follow a straight line, because this is the shortest path between two points in that medium! When it encounters a border between two different media, the light, in accordance with Fermat's principle, follows a path given by the laws of reflection and refraction. What is the relation between these laws and the minimization of time? page 91

Translations by Teresa Peña

Herbert Goldstein, Charles Poole, and John Safko, *Classical Mechanics* (Addison Wesley, 2002), 3rd ed. ISBN 0-201-65702-3. The second chapter is a very nice introduction to the principle of least action and the derivation of Lagrange's equations. Errors and typos in the text are listed at the website http://astro.physics.sc.edu/Goldstein . Solutions to exercises can be found at http://electron.mit.edu/~homer/physics/goldstein/index.shtml

Another advantage is that the Lagrangian formulation can be easily extended to describe systems that are not normally considered in dynamics—such as the elastic field, the electromagnetic field, and field properties of elementary particles. page 51

In this chapter [Chapter 8] we resume the formal development of mechanics, turning our attention to an alternative statement of the structure of the theory known as the Hamiltonian formulation. Nothing new is added to the physics involved; we simply gain another (and more powerful) method of working with the physical principles already established. The Hamiltonian methods are not particularly superior to Lagrangian techniques for the direct solution of mechanical problems. Rather, the usefulness of the Hamiltonian viewpoint lies in providing a framework for theoretical extensions in many areas of physics. Within classical mechanics it forms the basis for further developments, such as Hamilton-Jacobi theory, perturbation approaches and chaos. Outside classical mechanics, the Hamiltonian formulation provides much of the language with which present-day statistical mechanics and quantum mechanics is constructed. Page 334

A recurring theme throughout this text has been that symmetry properties of the Lagrangian (or Hamiltonian) imply the existence of conserved quantities. Thus, if the Lagrangian does not contain explicitly a particular coordinate of displacement, then the corresponding canonical momentum is conserved. The absence of explicit dependence on the coordinate means the Lagrangian is unaffected by a transformation that alters the value of that coordinate; it is said to be invariant, or symmetric, under the given transformation. Similarly, invariance of the Lagrangian under time displacement implies conservation of energy. The formal description of the connection between invariance or symmetry properties and conserved quantities is contained in Noether's theorem. page 589

Herman H. Goldstine, *A History of the Calculus of Variations from the 17th through the 19th Century*, Springer-Verlag, New York, 1980, ISBN 0-387-90521-9. A full, exhaustive treatment of the history of the subject, using the mathematical formalism of several eras. Lengthy bibliography.

> *It is not quite certain when Euler first became seriously interested in the calculus of variations. Caratheodory, who edited Euler's magnificent 1744 opus,* The Method of Finding Plane Curves that Show Some Property of Maximum or Minimum ...*, believed it unlikely that it occurred during his period in Basel with John Bernoulli [when?]. ...*
> *The introduction to Euler's great work of 1744 ... contains a listing of the 100 special problems taken up by Euler to illustrate his methods; these were arranged in 11 convenient categories by Caratheodory ... No interested reader should overlook this "Complete listing of Euler's example in the calculus of variations."* [I can find no listing of these 100 problems in English.]
> *Perhaps the greatest things that Euler did in his 1744 book were to set up a general apparatus or procedure for writing down the so-called Euler differential equation or first necessary condition and to enunciate and discuss the principle of least action, which he first discovered prior to 15 April 1743. In a letter of that date Daniel Bernoulli congratulated him on this work, and Euler wrote it down in the latter part of 1743.* page 67

> *On 12 August 1755 a 19-year-old, one Ludovico de la Grange Tournier of Turin, wrote [Leonard] Euler a brief letter to which was attached an appendix containing mathematical details of a very beautiful and revolutionary idea ... He saw how to eliminate from Euler's methods of 1744 the tedium and need for geometrical insight and to reduce the entire process to a quite analytic machine or apparatus, which could turn out the necessary condition of Euler and more, almost automatically. This basic idea of Lagrange ushered in a new epoch in the calculus of variations. Indeed after seeing Lagrange's work, Euler dropped his own method, espoused that of Lagrange, and renamed the subject the* calculus of variations.
> *In the summary of his first paper using variations, Euler says "Even though the author of this [Euler himself] had meditated a long time and had revealed to friends his desire yet the glory of first discovery was reserved to the very penetrating geometer of Turin LA GRANGE, who having used analysis alone, has clearly attained the very same solution which the author had deduced by geometrical considerations."* page 110

WILLIAM ROWAN HAMILTON (1805 - 1865)
Here is a website from which you can download "the mathematical papers of Sir William Rowan Hamilton published during his lifetime, transcribed and edited by David R. Wilkins" in TeX, DVI, PDF, or PostScript.
http://www.emis.de/classics/Hamilton/

On the above list is a set of papers in which, I believe, he developed what we now call the principle of least action. These papers are separately available at the website:
http://www.maths.tcd.ie/pub/HistMath/People/Hamilton/Dynamics/

Here is a complete bibliography: "List of Papers, Memoirs, Addresses, and Book published by Sir William Rowan Hamilton, and of Notices of Communications. Robert Perceval Graves [Life of Sir William Rowan Hamilton, Vol. III, pp.645-658.]
http://www.maths.tcd.ie/pub/HistMath/People/Hamilton/GBiblio/GBiblio.html

J. L. Lagrange, *Analytical Mechanics*, Victor N. Vagliente, ed., Auguste Boissonnade, tr. (Kluwer Academic Publishers, 1997) ISBN 0792343492. First published in 1788, during the French Revolution and 101 years after Newton's *Principia*, this book bases mechanics on statics and the principle of virtual work.

Preface to the First Edition

There already exist several treatises on mechanics, but the purpose of this one is entirely new. I propose to condense the theory of this science and the method of solving the related problems to general formulas whose simple application produces all the necessary equations for the solution of each problem. I hope that my presentation achieves this purpose and leaves nothing lacking.

In addition, this work will have another use. The various principles presently available will be assembled and presented from a single point of view in order to facilitate the solution of the problems of mechanics. Moreover, it will also show their interdependence and mutual dependence and will permit the evaluation of their validity and scope.

I have divided this work into two parts: Statics or the Theory of Equlibrium, and Dynamics or the Theory of Motion. In each part, I treat solid bodies and fluids separately.

No figures will be found in this work. The methods I present require neither constructions nor geometrical or mechanical arguments, but solely algebraic operations subject to a regular and uniform procedure. Those who appreciate mathematical analysis will see with pleasure mechanics becoming a new branch of it and hence, will recognize that I have enlarged its domain. page 7.

Cornelius Lanczos, *The Variational Principles of Mechanics* (Dover Publications, New York, 1986). ISBN 0-486-65067-7. THE reference most quoted by all those who explore this field.

We are allowed sovereign freedom in choosing our coordinates, since our processes and resulting equations remain valid for an arbitrary choice of coordinates. page xxv

The Principle of Relativity requires that the laws of nature shall be formulated in an 'invariant' fashion, i.e. independently of any special frame of reference. The methods of the calculus of variations automatically satisfy this principle, because the minimum of a scalar quantity does not depend on the coordinates in which that quantity is measured. page xxviii

In the realm of point mechanics we have encountered . . . the ingenious idea of E. Noether (1918) that any infinitesimal transformation of either the action variables, or the independent variable . . . which leaves the Lagrangian unchanged, leads automatically to a certain conservation law. We deduced on this basis the conservation laws of momentum, energy, and angular momentum. In the realm of partial operators Noether's principle is equally valid and widely employed in contemporary physics. We shall demonstrate the application of this principle in a few characteristic and particular important examples, page 386. [They then apply Noether's principle first to Maxwell's equations to derive the conservation of electric charge and the conservation of energy of the (sourceless) electromagnetic field. Then to The Schroedinger equation to show the conservation of the probabilistic electric charge. pages 387-388.]

L. D. Landau and E. M. Lifshitz, *Mechanics, Course of Theoretical Physics* (Butterworth-Heinenann, 1976), 3rd ed., Vol. 1. ISBN 0 7506 2896 0. Begins with the principle of least action.

> [NOTE: The following quotes do not do justice to Landau and Lifshitz. They spend little time summarizing or providing an overview. On the other hand, no work on the subject is more concise, crystal clear, and consistent.]
>
> *The most general formulation of the law governing the motion of mechanical systems is the principle of least action or Hamilton's principle, ...* page 2
>
> *It should be mentioned that this formulation of the principle of least action is not always valid for the entire path of the system, but only for any sufficiently short segment of the path. The integral (2.1) for the entire path must have an extremum, but not necessarily a minimum. This fact, however, is of no importance as regards the derivation of the equations of motion, since only the extremum condition is used.* page 2
>
> *In reality, when a body moves in a medium, the latter exerts a resistance which tends to retard the motion. The energy of the moving body is finally dissipated by being converted to heat. Motion under these conditions is no longer a purely mechanical process, and allowance must be made for the motion of the medium itself and for the internal thermal state of both the medium and the body. In particular, we cannot in general assert that the acceleration of a moving body is a function only of its co-ordinates and velocity at the instant considered; that is, there are no equations of motion in the mechanical sense. Thus the problem of the motion of a body in a medium is not one of mechanics.* page 74.
>
> *The motion of a mechanical system is entirely determined by the principle of least action: by solving the equations of motion [Lagrange's equations] which follow from that principle, we can find both the form of the path and the position on the path as a function of time.* page 140

Thomas A. Moore "Least-Action Principle" in *Macmillan Encyclopedia of Physics* (Simon & Schuster Macmillan, 1996), Volume 2, ISBN 0-0286457-1, pages 840 – 842.

> *The least-action principle is an assertion about the nature of motion that provides an alternative approach to mechanics completely independent of Newton's laws. Not only does the least-action principle offer a means of formulating classical mechanics that is more flexible and powerful than Newtonian mechanics, variations on the least-action principle have proved useful in general relativity theory, quantum field theory, and particle physics. As a result, this principle lies at the core of much of contemporary physics.* page 840
>
> *Technically, the "least action principle" in classical mechanics refers to a somewhat different principle than described here. First proposed in 1747 (rather vaguely) by Pierre-Louise Moreau de Maupertuis, the original least-action principle was described in careful mathematical language by Joseph Lagrange in 1760. In 1834 William Rowan Hamilton first proposed the principle described in this article, which is called Hamilton's principle and can be derived from it. Feynman's approach to quantum field theory actually generalizes Hamilton's principle, not the old least-action principle, but Feynman called it the "least-action principle" and the term has stuck. As a result, the "least-action principle" and Hamilton's principle have become informally synonymous, particularly in the literature of quantum mechanics.* page 842

David Morin introduces Lagrange's equations in Chapter 5 of his honors introductory physics text. Concludes with a wonderful set of 27 problems with solutions. A draft of is available at
http://www.courses.fas.harvard.edu/~phys16/handouts/textbook/ch5.pdf

> *In many (in fact, probably most) physical situations, this new method is far superior to using* F = ma. *You will soon discover this for yourself when you tackle problems for this chapter.* page V-1

> *It is sometimes claimed that nature has a "purpose", in that it seeks to take the path that produces the minimum action. ... this is incorrect. In fact, nature does exactly the opposite. It takes* every *path, treating them all on equal footing. We simply end up seeing the path with a stationary action, due to the way the quantum mechanical phases add. ... Of course, given that classical mechanics is the approximate theory, while quantum mechanics is the (more) correct one, it is quite silly to justify the principle of stationary action by demonstrating its equivalence with* F = ma. page V-8

> *One nice thing about the Lagrangian method is that we are free to impose any given constraints at the beginning of the problem, thereby immediately reducing the number of variables. ... Often we are not concerned with the exact nature of the forces doing the constraining, but only with the resulting motion. ... By imposing the constraints at the outset, we can find this motion ...* page V-9

> *We now present one of the most beautiful and useful theorems in physics. It deals with two fundamental concepts, namely* symmetry *and* conserved quantities*. The theorem (due to Emmy Noether) may be stated as follows. ... For each symmetry of the Lagrangian, there is a conserved quantity.* page V-16

Gerald Jay Sussman and Jack Wisdom, *Structure and Interpretation of Classical Mechanics* (MIT Press, 2001). Begins with the principle of least action, uses modern mathematical notation, and checks the clarity and consistency of procedures by programming them in computer language.

> *The Newtonian formulation of the equations of motion is intrinsically a particle-by-particle description.*
>
> *In the variational formulation the equations of motion are formulated in terms of the difference of the kinetic energy and the potential energy. The potential energy is a number that is characteristic of the arrangement of the particles in the system; the kinetic energy is a number that is determined by the velocities of the particles in the system. Neither the potential energy nor the kinetic energy depends on how those positions and velocities are specified. The difference is characteristic of the system as a whole and does not depend on the details of how the system is specified. So we are free to choose ways of describing the system that are easy to work with; we are liberated from the particle-by-particle description inherent in the Newtonian formulation.*
>
> *. . . If there are positional constraints among the particles of a system the Newtonian formulation requires that we consider the forces maintaining these constraints, whereas in the variational formulation the constraints can be built into the coordinates. The variational formulation reveals the association of conservation laws with symmetries. The variational formulation provides a framework for placing any particular motion of a system in the context of all possible motions of the system. We pursue the variational formulation because of these advantages.* page 3

We have seen that if a system has a symmetry and if a coordinate system can be chosen so that the Lagrangian does not depend on the coordinate associated with the symmetry, then there is a conserved quantity associated with the symmetry. ... More generally, a Lagrangian has a symmetry if there is a coordinate transformation that leaves the Lagrangian symmetries. Noether proved that for any continuous symmetry there is a conserved quantity. page 83

Dare A. Wells, *Lagrangian Dynamics,* Schaum's Outline Series (McGraw-Hill, 1967) ISBN 007-069258-0, A 350 page comprehensive "outline" of the subject, a wonderfully geeky treatment of endless applications and wild examples of mechanical devices. Straight-ahead formalism motivated by muscular usefulness, not effete elegance. Even EFT can understand the mathematics in this book! Do not miss the final chapters on Hamilton's equations and Hamilton's principle (*a.k.a* our principle of least action). Out of print, but available dirt cheap on the used book market.

Lagrange's equations are valid in any coordinates (inertial or a combination of inertial and non-inertial) which are suitable for designating the configuration of the system. They give directly the equations of motion in whatever coordinates may be chosen. It is not a matter of first introducing formal vector methods and then translating to desired coordinates. ... Fortunately the basic ideas involved in the derivation of Lagrange's equations are simple and easy to understand. When presented without academic trimmings and unfamiliar terminology, the only difficulties encountered by the average student usually arise from deficiencies in background training. The application of Lagrange's equations to actual problems is remarkably simple even for systems which may be quite complex. ... The book is directed to seniors and first year graduate students of physics, engineering, chemistry and applied mathematics, and to those practicing scientist and engineers who wish to become familiar with the powerful Lagrangian methods through self-study. It is designed for use either as a textbook for a formal course or as a supplement to all current texts. Preface

Robert Weinstock, *Calculus of Variations, with Applications to Physics and Engineering* (Dover Publications, 1974). ISBN 0-486-63069-2. An oldie but goodie, with the formalism carefully defined before use in physics and engineering.

The principle of Hamilton [called "principle of least action" by Richard Feynman and his followers] reads:

The actual motion of a system whose lagrangian is

$$(T - V) = L(q_1, q_2, \ldots, q_N, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_N)$$

is such as to render the (Hamilton's) integral

$$I = \int_{t_1}^{t_2} (T - V)\,dt = \int_{t_1}^{t_2} L\,dt\ ,$$

where t_1 and t_2 are two arbitrary instants of time, an extremum with respect to continuously twice-differentiable functions $q_1(t), q_2(t), \ldots, q_N(t)$ *for which* $q_i(t_1)$ *and* $q_i(t_2)$ *are prescribed for all*

$i = 1,2, \ldots, N$. *We accept Hamilton's principle as applicable to the motion of any conservative system.*

Although it is in some places stated that Hamilton's principle may be used to replace Newton's laws of motion as the fundamental starting point for mechanical systems possessing a lagrangian, it should be realized that Newton's laws are implicitly employed in the preceding paragraphs in at least two ways: (i) The definition of mass resides in Newton's third law. (ii) In the tacit assumption that our system of coordinates is fixed relative to an inertial frame of reference, we make use of Newton's first law, by means of which an inertial frame *is defined.* pages 74-75 [The text from the beginning "The actual" to "for all $i = 1,2, \ldots, N$." is italicized in the original.]

Wolfgang Yourgrau and Stanley Mandelstam, *Variational Principles in Dynamics and Quantum Theory* (Dover Publications, 1979). A nice treatment that does not avoid the philosophical implications of the theory and lauds the Feynman treatment of quantum mechanics that reduces to the principle of least action in the limit of large mass. Many practical insights, including the following

For the actual solution of problems, the equations of Lagrange are more convenient than those of Hamilton, since the first step in integrating Hamilton's equations would amount to reducing their number by half, an operation which would lead us back to our original Lagrange equations. In purely theoretical inquiries, on the other hand, Hamilton's equations are often more useful. page 43

We now propose to examine recent formulations of the laws of quantum mechanics themselves by means of action principles. Such formulations have been evolved by Feynman and Schwinger, and they are logically preferable to the now traditional canonical equations for several reasons. ... One cannot fail to observe that Feynman's principle in particular—and this is no hyperbole—expresses the laws of quantum mechanics in an exemplary neat and elegant manner ... Feynman's principle transforms gradually into the principle of least action. In this limit, the probability amplitude will differ appreciably from zero only for those values of the coordinates at a given time t_1, which are close to the corresponding values for the classical system ... that is, in the classical limit the wave packet moves in accordance with the principle of least action. pages 127, 128, 136

"*Amid the more or less general laws which mark the achievements of physical science during the course of the last centuries, the principle of least action is perhaps that which, as regards form and content, may claim to come nearest to that ideal final aim of theoretical research." [Planck]*

It may be contested that the principle of the conservation of energy vies with that of lest action for this privileged position. Planck, however, pointed to the well-known fact that the energy principle provides only one equation for the changes of a system, whereas the principle of least action ... furnishes sufficient equations to specify fully these changes and, indeed, contains the law of the conservation of energy as one of its results. In striking contrast to his otherwise calm and balanced judgement, he dubbed the principle of least action the "most comprehensive of all physical laws which governs equally mechanics and electrodynamics." page 164

It follows immediately that once the laws of a physical theory are expressed as differential equations, the possibility of their reduction to a variational principle is evident from purely mathematical reasoning and does not depend on certain attributes intrinsic in the theory. page 175

The principle of least action, according to Weyl the pinnacle of classical mechanics, had within fifty years lost its pre-eminence, and Larmor's optimistic prophecy that it would outlive all the other laws of physics had been shattered. page 177