Derivation of Lagrange's Equation from *F* = *ma*

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Here is a quick derivation of Lagrange's equation from Newton's second law for motion in one dimension, adapted from a similar derivation by Zeldovich and Myskis.[1]

For one-dimensional motion in a conservative potential $V(x)$, the force can be written:

$$F = -\frac{dV}{dx} \tag{1}$$

Then Newton's second law $F = ma$ becomes

$$-\frac{dV}{dx} = m\frac{d^2x}{dt^2} \tag{2}$$

or, after a simple rearrangement:

$$\frac{d(-V)}{dx} - \frac{d}{dt}\left(m\dot{x}\right) = 0 \tag{3}$$

where $\dot{x}$ is the time derivative of x. After further arrangement we have

$$\frac{d(-V)}{dx} - \frac{d}{dt}\left[\frac{d}{d\dot{x}}\left(\frac{m\dot{x}^2}{2}\right)\right] = 0 \tag{4}$$

Now add, under the derivative signs, terms whose derivatives are equal to zero, converting to partial derivatives in the process:

$$\frac{\partial}{\partial x}\left[\frac{m\dot{x}^2}{2} - V(x)\right] - \frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}\left(\frac{m\dot{x}^2}{2} - V(x)\right)\right] = 0 \tag{5}$$

which is the same as the Lagrange equations:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] = 0 \tag{6}$$

where

$$L \equiv \frac{m\dot{x}^2}{2} - V(x) = T - V \tag{7}$$

Equation (6) is Lagrange's equation in one dimension.

Of course, the sequence of steps above that lead from $F = ma$ to Lagrange's equation can be reversed to lead from Lagrange's equation to $F = ma$.

[1] *Elements of Applied Mathematics* by Ya. B. Zeldovich and A. D. Myskis, translated from the Russian by George Yankovsky, MIR Publishers Moscow,1976, pages 499-500.