# Deriving the Nonrelativistic Principle of Least Action from the Schwarzschild Metric and the Principle of Maximal Aging[1]

(Reference 12 in the accompanying AJP Guest Comment "A Call to Action")

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The *Schwarzschild metric* describes spacetime external to any spherically symmetric (and nonrotating) center of gravitational attraction. It tells us the time lapse $d\tau$ on the wristwatch of a particle that travels between two nearby events separated by the coordinates $dr$, $d\phi$, and $dt$ along its worldline.

$$\left(cd\tau\right)^2 = \left(1 - \frac{2GM}{c^2 r}\right)\left(cdt\right)^2 - \frac{dr^2}{\left(1 - \dfrac{2GM}{c^2 r}\right)} - r^2 d\phi^2 \qquad (1)$$

We apply the Schwarzschild metric to the region above Earth' surface. The coordinate $r$ is the *reduced circumference* of the particle with respect to the center of Earth, the circumference of a great circle at that radius divided by $2\pi$, and $t$ is the time recorded on clocks far from Earth and far from any center of gravitational attraction. For the radius $R$ of Earth,

$$\frac{2GM}{c^2 R} = \frac{0.89 \text{ centimeter}}{R} \qquad (2)$$

At Earth's surface and at greater radii $r > R$, one can neglect this term compared with unity in the denominator of the second term on the right side of (1). (We will see later why this quantity cannot be neglected in the first term.) Divide equation (1) through by $c^2$, take the square root of both sides, and substitute the relations

$$v_\mathrm{r} = \frac{dr}{dt} \quad \text{and} \quad v_\phi = r\frac{d\phi}{dt} \qquad (3)$$

The result is

$$d\tau = \left[1 - \frac{2GM}{c^2 r} - \frac{v_\mathrm{r}^2}{c^2} - \frac{v_\phi^2}{c^2}\right]^{1/2} dt = \left[1 - \frac{2GM}{c^2 r} - \frac{v^2}{c^2}\right]^{1/2} dt \qquad (4)$$

We cannot neglect the second term in the parenthesis on the right, because we want to examine the low-velocity limit $v << c$, for which the third term is also small. We now use the approximation

$$\left(1-\varepsilon\right)^n \approx 1 - n\varepsilon \qquad \text{which is true when} \quad \varepsilon << 1 \quad \text{and} \quad \left|n\varepsilon\right| << 1 \qquad (5)$$

and then integrate the result to determine the wristwatch time $\tau$ along the worldline:

$$\tau \approx \int_{\substack{\text{along the}\\\text{worldline}}} \left[1 - \frac{1}{c^2}\left(\frac{1}{2}v^2 + \frac{GM}{r}\right)\right] dt = \text{constant } - \frac{S}{mc^2} \qquad (6)$$

where $S$ is the Newtonian action:

$$S = \int_{\substack{\text{along the}\\\text{worldline}}} \left(\frac{1}{2}mv^2 + \frac{GMm}{r}\right) dt = \int_{\substack{\text{along the}\\\text{worldline}}} \left[\frac{1}{2}mv^2 - \left(-\frac{GMm}{r}\right)\right] dt = \int_{\substack{\text{along the}\\\text{worldline}}} \left(T - V\right) dt \qquad (7)$$

The Principle of Maximal Aging says that wristwatch time $\tau$ along the actual worldline of a free particle is a *maximum* compared with nearby worldlines. Equation (6) then tells us that in the low-velocity limit the Newtonian action $S$ is a *minimum*. This completes the derivation of the Principle of Least Action from the Principle of Maximal Aging for the special case of low-velocity motion of a free particle above the surface of Earth.

[1]Clifford M. Will , *Theory and experiment in gravitational physics*, revised edition, Cambridge University Press, 1993, page 89.