When action is not least

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ABSTRACT
This paper examines the nature of the stationary character of the Hamilton action S for a space-time trajectory (worldline) of a single particle moving in one dimension with general (time-dependent) potential energy function $U(x,t)$. We show that the action is a local minimum for sufficiently short worldlines in all potentials and for worldlines of any length in some potentials. For long enough worldlines in a majority of time-independent potentials $U(x)$, however, the action is a saddle point, i.e. a minimum with respect to some nearby alternative curves and a maximum with respect to others. The action is never a true maximum, that is never greater along the actual worldline than along *every* nearby alternative curve. We illustrate these results with the harmonic oscillator, two different nonlinear oscillators, and a scattering system. Appendices briefly discuss the Maupertuis action W, two-dimensional examples, and newer forms of action principles.

I. INTRODUCTION

A number of authors[1-12] have simplified and elaborated the action principle and recommended that it be introduced earlier into the physics curriculum. Their work allows us to see in outline how to empower students early in their study with the fundamental yet simple extensions of Newton's principles of motion made by Maupertuis, Euler, Lagrange, Jacobi, Hamilton and others. The simplicity of the action principle derives from its use of scalars energy and time to predict motion. Its transparency comes from the use of computer[11] and analytic[12] procedures to vary a candidate worldline to find a stationary value of the action, skirting not only equations of motion but also the advanced formalism of Lagrange and others characteristic of upper level mechanics texts. We intend the present paper to contribute to this development by providing background for instructors and advanced students on conditions under which the stationary value of the action for an actual worldline is a minimum and those under which it is a saddle point (neither a true maximum nor a true minimum).

For single-particle motion in one dimension (1D), the Hamilton action S is defined as an integral along an actual or trial space-time trajectory (worldline) connecting two given events P and R,

$$S \equiv \int_P^R L(x,\dot{x},t)\,dt, \tag{I-1}$$

where L is the Lagrangian, x the position, t the time, and $P(x_P, t_P)$ and $R(x_R, t_R)$ are fixed initial and final space-time events. A dot over a symbol, as in $\dot{x}$, indicates the time derivative. The Lagrangian $L(x,\dot{x},t)$ depends on t implicitly through $x(t)$ and $\dot{x}(t)$ and may also depend on t explicitly, for example through a time-dependent potential. For simplicity we use Cartesian coordinates throughout, but the methods and conclusions apply using generalized coordinates. The Hamilton action principle compares the numerical value of the action S along the actual worldline to its value along every adjacent curve (trial worldline) anchored to the same fixed initial and final events. We can freely imagine and construct these adjacent curves; they include, but are not limited to, nearby worldlines which the particle can actually follow. The Hamilton action principle says that with respect to all nearby curves the action along the actual worldline is *stationary*, that is it has zero variation to first order; formally we write $\delta S = 0$. Whether or not this stationary value of the action is a local minimum is determined by examining second ($\delta^2 S$) and higher order variations of the action (defined below) with respect to the nearby curves, as we do in this paper.

Errors concerning the stationary nature of the action abound in the literature. Even the great Lagrange says that the value of the action can be maximum,[13] a common error[14] of which the authors of this paper have themselves been guilty[12,15]. Other authors use the terms *extremum* or *extremal*,[16] which incorrectly include a maximum and formally fail to include a saddle point. (Mathematicians often use the (correct) term *critical* instead of *stationary*, but since the former term has other meanings in physics we use the latter term.) A similar error mars treatments of Fermat's Principle of optics, which is erroneously said to allow the travel time of a light ray between two points to be a maximum.[17]

The present paper has three primary purposes: First it describes conditions under which the action is a minimum and different conditions under which it is a saddle point. Some pioneers[18]of this theory are Legendre, Jacobi, Weierstrass, Kelvin and Tait, Mayer, and Culverwell. Although inspired by the early work of Culverwell[19], our derivation of these conditions is new, simpler, and more rigorous; it is also simpler than modern treatments.[23.] Second, this paper "explains" the results with qualitative heuristic descriptions of how a particle responds to space-varying forces derived from the potential in which it moves. (Those who prefer immediate immersion in the formalism can begin with Sec. IV.) Third it clarifies these results and illustrates the variety of their consequences by applying them to the harmonic oscillator, two nonlinear oscillators, and a scattering system. Criteria used to decide the nature of the stationary value of the action are also useful for other purposes in classical and semiclassical mechanics,[27] but are not discussed in the present paper.

Appendix A adapts the results to the important Maupertuis action W. Appendix B gives examples of both Hamilton and Maupertuis action for two dimensional motion. Appendix C discusses open questions on the stationary nature of action for some newer action principles.

II. KINETIC FOCUS

This section introduces the concept of *kinetic focus*, due to Jacobi[21], which plays a central role in determining the nature of the stationary action. We start with an analogous example taken verbatim from Whittaker[35], an analysis of the relative length along different paths. Whittaker employs the Maupertuis action principle (discussed in our Appendix A), which requires fixed total energy along trial paths, not fixed travel time as with the Hamilton action principle. In force-free systems the value of the Maupertuis action is proportional to the path length. The term *kinetic focus* is defined formally later in this section. Figure 1 illustrates this example. Whittaker says:

> A simple example illustrative of the results obtained in this article is furnished by the motion of a particle confined to a smooth sphere under no forces. The trajectories are great-circles on the sphere and the [Maupertuis] action taken along any path (whether actual or trial) is proportional to the length of the path. The kinetic focus of any point A

> is the diametrically opposite point A' on the sphere, since any two great circles through A intersect again (for the first time) at A'. The theorems of this article amount, therefore in this case to the statement that an arc of a great circle joining any two points A and B on the sphere is the shortest distance from A to B when (and only when) the point A' diametrically opposite to A does not lie on the arc, *i.e.* when the arc in question is less than half a great-circle.

The elaboration of this analogy is carried out in the captions of Fig. 1 using equilibrium lengths of a rubber band on a slippery spherical surface.

For a contrasting example, apply a similar analysis to free-particle motion on a flat plane. In this case the straight path connecting two points has minimum length no matter how far apart these endpoints are. A rubber band stretched between endpoints on a slippery surface will always snap back when deflected in any manner and released. An alternative second straight path that deviates slightly in direction at the initial point A continues to diverge and does not cross the original path again. Therefore no kinetic focus of event A exists for the original path.

Finally note that on both the sphere and the flat plane there is no path of true maximum length between any two separated points. The length of any path can be increased by adding wiggles.

How do we find the kinetic focus? In Fig. 1a we place terminal point C at different points along a great circle path between A and A'. When C lies between A and A', every nearby alternative path such as AEC is *not* a true path (a path of minimum length), because it does not lie along a great circle. However, when terminal point C reaches A', there is suddenly more than one alternative great circle path connecting A and A' (in this special case an infinite number of alternative great circle paths connecting A and A'). Any alternative great circle path between A and A' can be moved sideways to coalesce with the original path ABA'. The kinetic focus is defined by the existence of this coalescing alternative true path: As final point C moves away from initial point A, *the kinetic focus A′ is defined as the earliest terminal point at which two true paths can coalesce.*

The term *kinetic focus* in mechanics derives from an analogy[36] to the *focus* in optics, that point A′ at which rays emitted from an initial point A converge under some conditions, such as interception by a converging lens.

The present paper deals with the action principle for Hamilton action *S*, which determines worldlines in space-time by fixing the end-*events* and the travel time—rather than the action principle for Maupertuis action *W* (Appendix A), which determines spatial orbits (as well as space-time worldlines) by fixing the end-*positions* plus the total energy. The kinetic focus for Hamilton action has a use similar to that for the Fig. 1 example of Maupertuis action: We will show later in the paper that a worldline has minimum action *S* if it terminates before reaching the kinetic focus of its initial event. In contrast, a worldline that terminates beyond the kinetic focus of its initial event P has action that is a saddle point.

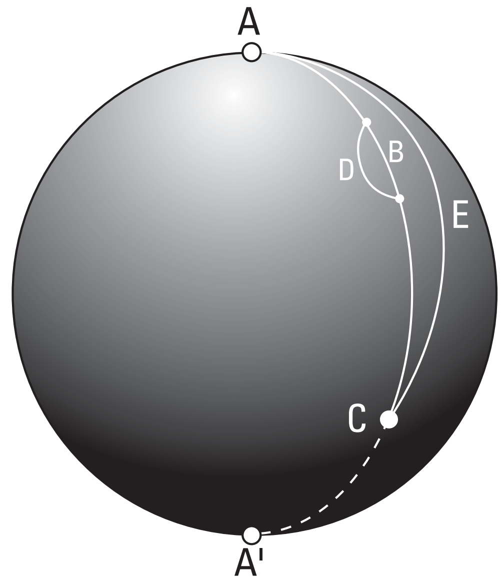

Figure 1a. On a sphere the great circle line ABC starting from the north pole at A is the shortest distance between two points as long as it does not reach the south pole at A'. On a slippery sphere a rubber band stretched between A and C will snap back if displaced either locally, as at D, or by pulling the entire line aside, as along AEC. The point A' is called the *antipode of A* or in general the *kinetic focus of A*. In tech-speak we say: *If a great circle path terminates before the kinetic focus of its initial point, the length of the great circle path is a minimum*.

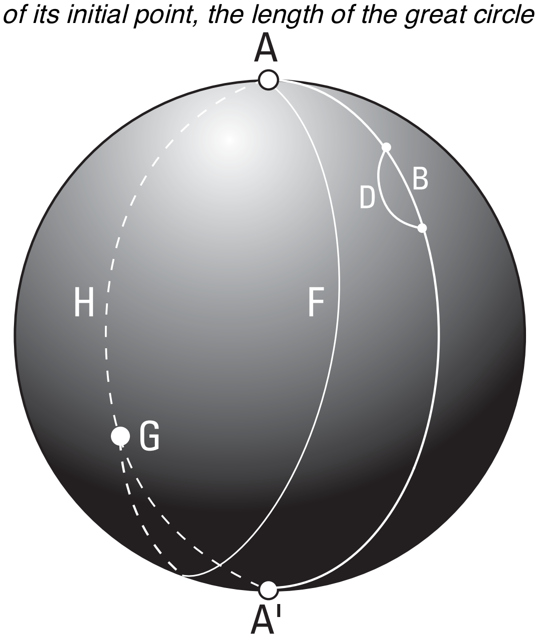

Figure 1b. If the great circle ABA'G passes through antipode A' of the initial point A, then the resulting line has a minimum length only when compared with *some* alternative lines. For example on a slippery sphere the rubber band stretched along this path will still snap back from local distortion, as at D. However if the entire rubber band is pulled to one side, as along AFG, then it will not snap back but rather slide over to the portion AHG of a great circle down the backside of the sphere. With respect to paths like AFG, the length of the great circle line ABA'G is a maximum. With respect to all possible variations we say that the length of path ABA'G is a *saddle point*. In tech-speak: *If a great circle path terminates beyond the kinetic focus of its initial point, the length of the great circle path is a saddle point.*

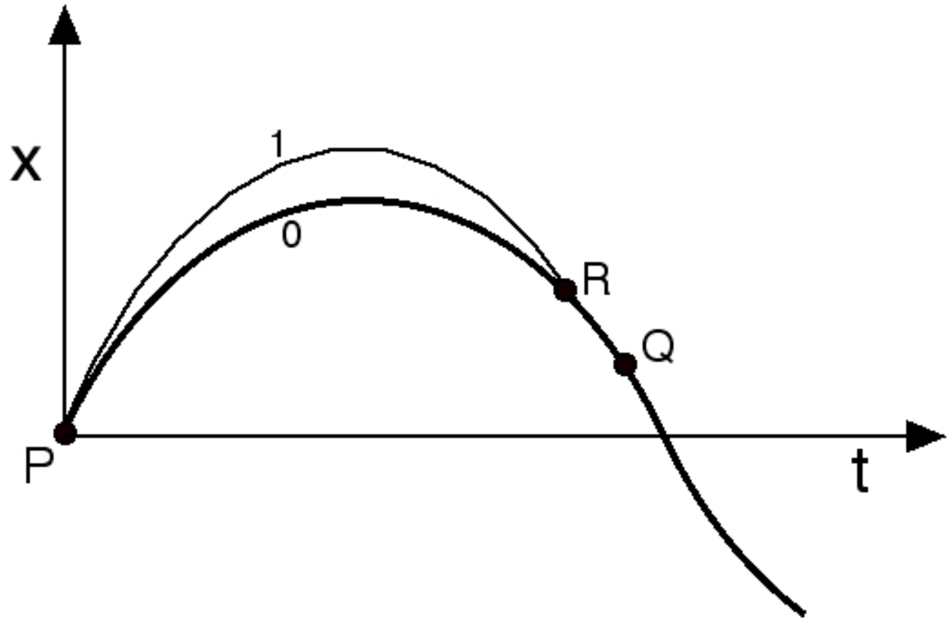

Figure 2. From the common initial event P we draw a true worldline 0 and a second true worldline 1 that terminates at some event R on the original worldline 0. The event nearest to P at which worldline 1 coalesces with worldline 0 is the kinetic focus Q.

We use the label P for the initial event on the worldline 0 (Fig. 2), Q for the kinetic focus of P on the worldline, and R for a fixed but arbitrary event on the worldline that terminates on worldline 0 and also terminates another true worldline (#1 in Fig. 2) connecting P to R. For Hamilton action S our definition of kinetic focus of a worldline is:

> *The kinetic focus Q of an earlier event P on a true worldline is the event closest to P at which a second true worldline, with slightly different velocity at P, intersects the first worldline—in the limit at which the two worldlines coalesce as their initial velocities at P are made equal.*

The kinetic focus is central to an understanding of the stationary nature of action *S*, but its definition may seem a bit obscure. To preview consequences of this definition, look at some later examples in this paper. Figure 8 shows true worldlines of the harmonic oscillator, whose potential energy has the form $U(x) = (1/2)kx^2$. The harmonic oscillator is the single 1D special case of the definition of space-time kinetic focus. Notice that *every* worldline originating at P in Fig. 8 passes through the same crossing point. This is similar to the 2D spatial paths on the sphere; in Fig. 1 *every* great circle path starting at A passes through the antipode at A'. In both cases we can find the kinetic focus without taking the limit in which the velocities at the initial point are equal and the two worldlines coalesce—but we can take that limit. For the harmonic oscillator this occurs when the amplitudes are made equal. The harmonic oscillator will turn out to be the single exception to many of our rules for action.

A more typical case is the quartic oscillator (Fig. 10) which moves with potential energy proportional to the fourth power of its displacement: $U(x) = Cx^4$. In this case alternative worldlines starting from initial event P can cross anywhere along the original worldline (some crossing events indicated by little squares in Fig. 10). When the alternative worldline coalesces with the original worldline, the crossing point has reached the kinetic focus Q.

Another typical case is the piecewise-linear oscillator, Fig. 9. This oscillator has a V-shaped potential energy $U(x) = C|x|$. For the piecewise-linear oscillator, as for the quartic oscillator, alternative worldlines starting from P can cross at various events along the original worldline. Note that for the piecewise-linear oscillator an alternative worldline that crosses the original worldline before its kinetic focus lies *below* the original worldline instead of above it (as for the quartic oscillator). This makes no difference in the definition of the kinetic focus as the event nearest to P at which two worldlines from P coalesce. In all cases we can equally well use an alternative worldline which crosses from below or one which crosses from above to define the coalescing worldline and kinetic focus.

Notice the gray line labeled *caustic* in Figs. 9 and 10, and also in Fig. 11 which shows worldlines for a repulsive potential. The caustic is the line along which the kinetic foci lie for a particular family of worldlines (such as the family of worldlines that start from P with positive initial velocity in Fig. 9 and Fig. 10). A caustic is also an *envelope* to which all worldlines of a given family are tangent. The caustics in Figs. 9 to 11 are space-time caustics, envelopes for space-time trajectories (worldlines). Figure 12 shows a purely spatial caustic/envelope for a family of parabolic paths (orbits) in a linear gravitational potential. The word *caustic* is derived from optics[36] (along with the word *focus*). When your cup of coffee is illuminated at an angle, a bright curved line with a cusp appears on the surface of the coffee (Fig. 3). Each point on this spatial *optical caustic* or *ray envelope* is the focus of light rays reflected from a small portion of the circular inner surface of the cup.

In Figs. 9 through 12 the caustic for a family of worldlines (or paths) represents a limit for those worldlines (or paths). *No* worldline of that family exists for final events outside the caustic. *At least one* worldline can pass through any event *inside* the caustic. *Exactly one* worldline can pass through an event *on* the caustic, and that event is the worldline's kinetic focus. This observation is consistent with the definition of the kinetic focus as an event at which two separate worldlines become one (coalesce).

At the kinetic focus the worldline is tangent to the caustic. When two curves touch but do not cross and have equal slope at the point where they touch, the curves are said to *osculate* or *kiss*, which leads to a summary preview of the results of this paper:

> *When a worldline terminates before it kisses the caustic, the action is minimum; when the worldline terminates after it kisses the caustic, the action is a saddle point.*

This means that when you use a computer to plot a family of worldlines (by whatever means), you can eyeball the envelope/caustic and locate the kinetic focus of each worldline visually.

This summary covers every case (but one), because when no kinetic focus exists, there is no caustic so a worldline of any length has minimum action. The one case not covered by this rule is the harmonic oscillator; for the harmonic oscillator (and also for the sphere geodesics of Fig. 1), the caustic collapses to a single point at the kinetic focus. In this case there is no caustic *curve*; only one caustic *point* (a focal point) exists. A corresponding optical case is a concave reflecting parabolic surface of revolution illuminated with incoming light rays parallel to its axis; the optical caustic collapses into a single point at the focus (focal point) of the parabolic mirror. When the optical caustic reduces to a point for a lens or mirror system, the resulting images have minimum distortion ("minimum aberration").

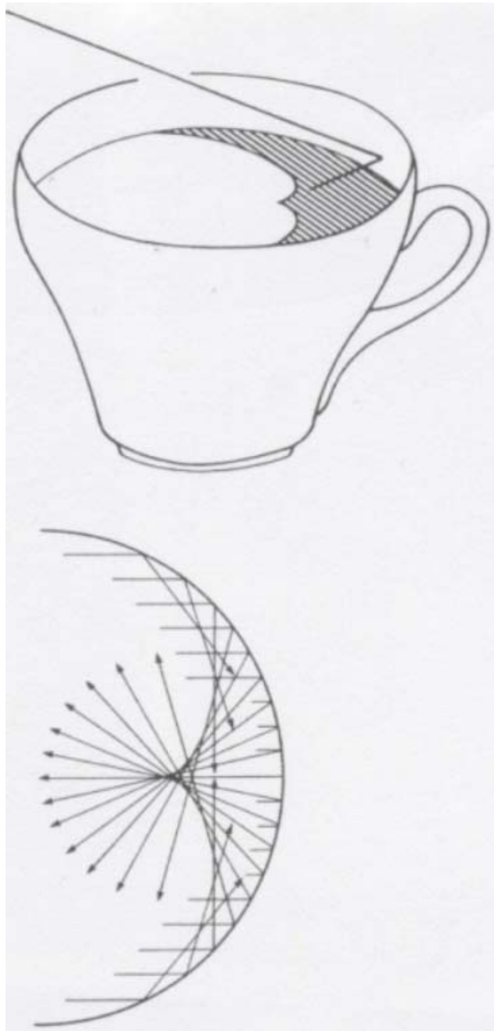

Figure 3. The coffee-cup optical caustic. The caustic shape (a nephroid) was derived by Johan Bernoulli in 1698. (from ref. 42)

For the quartic oscillator and the piecewise-linear oscillator (and the harmonic oscillator) subsequent crossing points exist at which two worldlines can coalesce. We have defined *the* kinetic focus as the *first* of these, the one nearest initial event P. The procedure for locating the subsequent kinetic foci is identical to that for locating the first one, and is discussed briefly in the examples of Secs. VIII and IX. For 1D potentials $U(x)$, subsequent kinetic foci[43] exist for the bound worldlines but not for the scattering worldlines, for example those in Fig. 11. We shall not be much concerned with subsequent kinetic foci; when we refer to *the* kinetic focus we mean the first one, as we have defined it. We shall show in what follows that for a few potentials $U(x)$ arising in practice (e.g., $U(x) = C$, $U(x) = Cx$) kinetic foci do not exist, because true worldlines beginning at a common initial event P do not cross again.

The definition of kinetic focus in terms of coalescing worldlines provides a "practical" way to find the kinetic focus. In Fig. 2 look at the slopes of nearby worldlines 1 and 0 at the initial event P. The initial slope of curve 1 is only slightly different from that of worldline 0; as that difference approaches zero the crossing event approaches the kinetic focus Q. The slope of a worldline at any point measures the velocity of the particle at that point. This leads to a method for finding the kinetic focus: Launch an identical second particle from event P (therefore simultaneously with the original launch) but with a slightly different initial velocity (that is with a slightly different slope of the worldline). Worldline number 1 is also a true worldline. Then in the limiting case of vanishing difference in initial velocities at event P (vanishing angle between the initial slopes) the two worldlines will cross again, and the two particles collide, at the kinetic focus event Q.

Convert this "practical" (actually heuristic) idea into an analytical method, often easily applied when we have an analytic expression for the worldline. Let the original worldline be described by the function $x(t, v_0)$, where v_0 is the initial velocity. Then the second worldline is the same function with incrementally increased initial velocity $x(t, v_0 + \Delta v_0)$. Form the expansion in Δv_o

$$x(t, v_0 + \Delta v_0) = x(t, v_0) + \frac{\partial x}{\partial v_0}\Delta v_0 + O(\Delta v_0{}^2)\ , \tag{II-1}$$

where $O(\Delta v_o^2)$ means "terms of order Δv_o^2." At an intersection point R we have $x(t_R, v_0 + \Delta v_0) = x(t_R, v_0)$. For intersection point R near Q we therefore have

$$\frac{\partial x}{\partial v_0}\Delta v_0 + O(\Delta v_0{}^2) = 0 \quad , \tag{II-2}$$

which implies that for R $\rightarrow$ Q when $\Delta v_0 \rightarrow 0$ we have

$$\frac{\partial x}{\partial v_0} = 0 \quad . \tag{II-3}$$

Equation (II-3) is an analytic condition for the incrementally different worldline that crosses the original worldline at the kinetic focus, so it locates the kinetic focus Q. Sections VIII and IX contain applications of this method.

III. WHY WORLDLINES CROSS

Section II defines the kinetic focus in terms of recrossing worldlines. The burden of this paper is that when a kinetic focus exists, the action along a worldline is a minimum if it terminates before the kinetic focus Q of the initial event P, whereas the action is a saddle point when the worldline terminates beyond the kinetic focus. In the present section we consider only actual worldlines and describe qualitatively why two worldlines originating at the same initial event cross again at a later event. We also examine the special initial conditions at P under which the coalescing worldlines determine the position of the kinetic focus Q. The key parameter turns out to be the second spatial derivative $\partial^2 U/\partial x^2 \equiv U''$ of the potential energy function U. Sufficiently long worldlines can cross again only if they traverse a space in which U'' is positive. For simplicity we restrict the discussion here to time-independent potentials $U(x)$, but continue to use partial derivatives of U with respect to x to remind ourselves of this restriction. New features which can arise for time-dependent potentials $U(x,t)$ are discussed in Sec. XI.

Think of two identical particles that leave initial event P with different velocities and hence different slopes of their space-time worldlines, so that their worldlines diverge. The following description is valid whether the difference in initial slopes is small or large. (Figures 8 through 10 illustrate the following narrative.) At every event on its worldline each particle experiences the force $F = -U' = -\partial U/\partial x$ evaluated at that location. For a short time after the two particles leave P they are at essentially the same displacement x, so they feel nearly the same force $-U'$ at that common displacement. Hence the space-time curvature of the two worldlines (the acceleration, proportional to the force) is nearly the same. Therefore the two worldlines will initially curve in concert while their initial *relative* velocity carries them apart; at the beginning their worldlines steadily diverge from one another. As time goes by, this divergence carries one particle, call it II, into a region in which the *second* spatial derivative $U'' = \partial U'/\partial x = \partial^2 U/\partial x^2$ is (let us say) *positive*. Then diverging particle II feels more force than (but still in the same x-direction as the force on) particle I. As a result the worldline of II will head back toward particle I, leading to converging worldlines. As the two particles draw close again they are once more in a region of almost equal U' and therefore experience nearly equal acceleration, so their relative velocity of convergence remains nearly constant until the worldlines intersect, at which event the two particles collide.

Notice the crucial role played by the *positive* value of the second derivative U'' in the relative space-time curvatures of worldlines I and II necessary for them to recross. Suppose instead that the second spatial derivative U'' is *negative*. Then as II moves away from I it enters a region of smaller slope U' and hence smaller force than that on particle I. Hence the two worldlines will diverge even more than they did originally, and the more they separate the greater will be their rate of divergence. As long as both particles move in a region where U'' is negative the two worldlines will never recross. (If U'' is zero, the two worldlines continue indefinitely to diverge at the initial rate.)

As a special case let the *relative* velocity of the two particles at launch be only incrementally different from one another (Fig. 2) for motion in potentials with positive values of U'', and let this difference of initial velocity approach zero. In this limit, by definition (Sec. II), the particles will collide at the kinetic focus Q of the initial event P. It may seem strange that an incremental relative velocity at P results in a recrossing at Q at a significant distance along the worldline from P. One might think that as this difference in slope increases from zero the recrossing event would start at P and move smoothly away from it along worldline I, not "snap" all the way to Q. The source of the "snap" lies in the first and second spatial derivatives of U. When both particles start from the initial event, the first derivative at essentially the same displacement leads to nearly the same force $-U'$ on particles I and II, so that any difference in the initial velocity, no matter how small, continues, increasing the separation. It is only with greater relative displacement over time that the difference in these forces, quantified by the positive second derivative U'', deflects the two worldlines back toward one another, leading to eventual recrossing. No alternative true worldline starting at P and with negligibly different initial velocity crosses the original worldline earlier than its kinetic focus (though widely divergent worldlines may cross sooner, as shown in Figs. 8 and 9). One consequence of this result will be that a worldline terminating before its kinetic focus has minimum action, as shown analytically in Sec. VI.

In other words, potentials with $U''(x) > 0$ (positive spatial curvature) are stabilizing, that is, bring back together neighboring trajectories that initially slightly diverge. Potentials with $U''(x) < 0$ (negative spatial curvature) are destabilizing, i.e. push further apart neighboring trajectories that initially slightly diverge. It is thus not surprising that the question of trajectory stability (stable *vs.* unstable) is closely related to the stationary character of the trajectory action (saddle point *vs.* minimum)[22,28-30].

Planetary orbits also exhibit crossing points distant from the location of a disturbance; an incremental change in velocity at one point in the orbit leads to initial and continued divergence of the two orbits which, for certain potential functions, reverses to bring them together again at a distant point. This later crossing point is defined as the kinetic focus for the Maupertuis action W applied to spatial orbits (Appendix B). This reconvergence has important consequences for the stability of orbits and the continuing survival of life on earth as our planet experiences small nudges from solar wind, meteor impacts, and shifting gravitational pulls from other planets.

IV. VARIATION OF ACTION FOR AN ADJACENT CURVE

...another feature in classical mechanics that seemed to be taboo in the discussion of the variational principle of classical mechanics by physicists: the second variation....

— Martin Gutzwiller

The action principle says that the worldline that a particle follows between two given fixed events P and R has stationary action with respect to *every possible alternative adjacent curve* between those two events (Fig. 4). Thus the action principle employs not only actual worldlines but also freely constructed curves adjacent to the original worldline, curves that are not necessarily worldlines themselves. In this paper the word *worldline* (or for emphasis *true worldline*) refers to a space-time trajectory that a particle might follow in a given potential. The word *curve* means an arbitrarily constructed trajectory that may or may not turn out to be a worldline. To study action we need *curves* as well as *worldlines*. (In the literature the terms *actual, true,* and *real trajectory* are used synonymously with our term *worldline*; the terms *virtual* and *trial trajectory* are used for our term *curve*.)

In the present section we set up formalism to investigate the variational characteristics of the action S of a worldline in order to determine whether S is a local minimum or a saddle point with respect to arbitrary nearby curves between the same fixed events. In Fig. 4 a true worldline labeled 0 described by the function $x_0(t)$ starts at initial event P. We construct a closely adjacent arbitrary curve, labeled 1 and described by the function $x(t)$, which starts at the same initial event P and terminates at a later event R on the original worldline. To compare the action along P0R on the worldline $x_0(t)$ with action along P1R on the arbitrary adjacent curve $x(t)$, let

$$x(t) = x_0(t) + \alpha\phi(t) \quad , \qquad \text{(IV-1a)}$$

and take the time derivative, indicated by a dot over the symbol:

$$\dot{x}(t) = \dot{x}_0(t) + \alpha\dot{\phi}(t) \quad . \qquad \text{(IV-1b)}$$

In these equations α is a real numerical constant of small absolute value and $\phi(t)$ is an arbitrary real function of time that goes to zero at both fixed end-events P and R. The action principle says that action (I-1) along $x_0(t)$ is *stationary* with respect to action along $x(t)$ for small values of the parameter α . To simplify the analysis we restrict $\phi(t)$ to be a continuous function with at most a finite number of discontinuities of the first derivative; that is, all curves $x(t)$ are assumed to be at least *piecewise smooth*[44]. Within this limitation $x(t)$ represents *all possible curves*

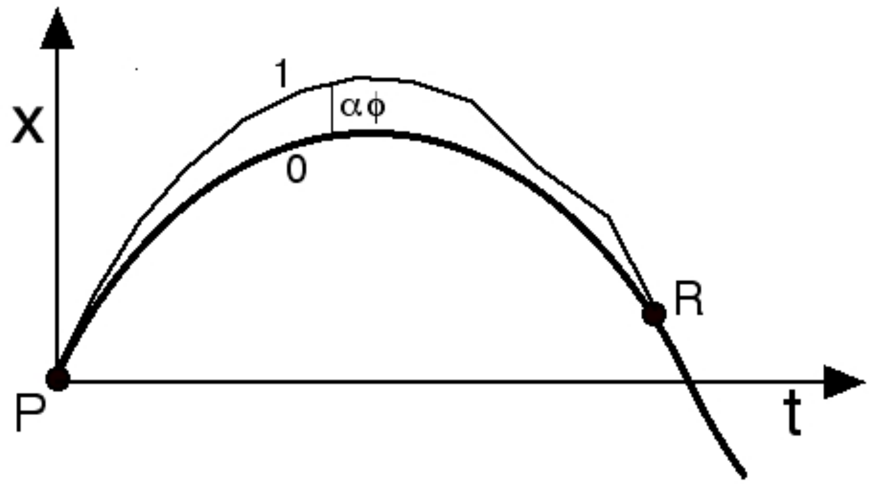

Figure 4. An original true worldline, labeled 0, starts at initial event P. We draw an arbitrary adjacent curve, labeled 1, anchored at two ends on P and a later event R on the original worldline. The variational function αϕ is chosen to vanish at the two ends P and R.

adjacent to $x_0(t)$, not only any actual nearby worldlines. From (IV-1) the Lagrangian $L(x,\dot{x},t)$ can be regarded as a function of α, and hence expanded in powers of α for small α, i.e.,

$$L = L_0 + \alpha\frac{dL}{d\alpha} + \frac{\alpha^2}{2}\frac{d^2L}{d\alpha^2} + \frac{\alpha^3}{6}\frac{d^3L}{d\alpha^3}\cdots \qquad , \tag{IV-2}$$

where $L_0 \equiv L(x_0, \dot{x}_0, t)$ and the derivatives are evaluated at $\alpha = 0$, *i.e.* along the original worldline $x_0(t)$. Apply (IV-1 a, b) to the first derivative in (IV-2):

$$\frac{dL}{d\alpha} = \frac{\partial L}{\partial x}\frac{dx}{d\alpha} + \frac{\partial L}{\partial \dot{x}}\frac{d\dot{x}}{d\alpha} = \phi\frac{\partial L}{\partial x} + \dot{\phi}\frac{\partial L}{\partial \dot{x}} \qquad , \tag{IV-3}$$

so that we can write in operator form

$$\frac{d}{d\alpha} = \phi\frac{\partial}{\partial x} + \dot{\phi}\frac{\partial}{\partial \dot{x}} \tag{IV-4}$$

and apply it twice in succession to yield a second derivative:

$$\frac{d^2L}{d\alpha^2} = \left(\phi\frac{\partial}{\partial x} + \dot{\phi}\frac{\partial}{\partial \dot{x}}\right)\left(\phi\frac{\partial L}{\partial x} + \dot{\phi}\frac{\partial L}{\partial \dot{x}}\right) = \phi^2\frac{\partial^2 L}{\partial x^2} + 2\phi\dot{\phi}\frac{\partial^2 L}{\partial x\partial \dot{x}} + \dot{\phi}^2\frac{\partial^2 L}{\partial \dot{x}^2}\,. \tag{IV-5}$$

In this paper we consider the most common case, in which the Lagrangian L is equal to the difference between kinetic and potential energy:

$$L = K - U = \frac{1}{2}m\dot{x}^2 - U(x,t), \tag{IV-6}$$

where U may be time-dependent. Then L has the partial derivatives

$$\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x}, \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x}\ ;$$

$$\frac{\partial^2 L}{\partial x^2} = -\frac{\partial^2 U}{\partial x^2}, \quad \frac{\partial^2 L}{\partial x \partial \dot{x}} = 0, \quad \frac{\partial^2 L}{\partial \dot{x}^2} = m\ ; \tag{IV-7}$$

$$\frac{\partial^3 L}{\partial x^3} = -\frac{\partial^3 U}{\partial x^3}, \quad \frac{\partial^3 L}{\partial \dot{x}^3} = 0 \quad .$$

Hence the second α derivative of L reduces to

$$\frac{d^2 L}{d\alpha^2} = -\phi^2 \frac{\partial^2 U}{\partial x^2} + m\dot{\phi}^2 \quad . \tag{IV-8}$$

Apply (IV-4) to (IV-8) and use (IV-7) to obtain the third α derivative of L for this case:

$$\frac{d^3 L}{d\alpha^3} = -\phi^3 \frac{\partial^3 U}{\partial x^3} \quad . \tag{IV-9}$$

Expansion (IV-2) for L defined by (IV-6) now becomes

$$L = L_0 + \alpha\left(-\phi\frac{\partial U}{\partial x} + m\dot{x}_0\dot{\phi}\right) + \frac{\alpha^2}{2}\left(-\phi^2\frac{\partial^2 U}{\partial x^2} + m\dot{\phi}^2\right) + \frac{\alpha^3}{6}\left(-\phi^3\frac{\partial^3 U}{\partial x^3}\right) + \ldots \quad , \tag{IV-10}$$

where the U derivatives are evaluated along the original worldline $x_0(t)$. Inserting (IV-10) into the action integral (I-1) results in an expansion of the action S in powers of α with terms for which we define the following symbols:

$$S = S_0 + \delta S_0 + \delta^2 S_0 + \delta^3 S_0 + \ldots \quad . \tag{IV-11}$$

The standard result of the action principle[45] is that along a true worldline the action is *stationary*. By this we mean that the term δS_0 in (IV-11), called the *first order variation* (linear in α), has value zero for all variations around an actual worldline $x_0(t)$. (This is the necessary and sufficient condition for the validity of Lagrange's equation of motion for $x_0(t)$.) We will need the higher-order variations $\delta^2 S_0$ and $\delta^3 S_0$ from an actual worldline, called the second and third order variations respectively. From (IV-10) and (IV-11) they take the forms

$$\delta^2 S = \frac{\alpha^2}{2}\int_P^R \left(-\phi^2\frac{\partial^2 U}{\partial x^2} + m\dot{\phi}^2\right) dt \quad , \tag{IV-12}$$

and

$$\delta^3 S = -\frac{\alpha^3}{6}\int_P^R \phi^3 \frac{\partial^3 U}{\partial x^3} dt \qquad , \qquad \text{(IV-13)}$$

where the derivatives of U are evaluated along the actual worldline $x_0(t)$. Higher-order variations (such as $\delta^4 S$, $\delta^5 S$, . . .) differ from (IV-13) in higher powers of ϕ, higher partial derivatives of U (such as $\partial^4 U/\partial x^4$, $\partial^5 U/\partial x^5$, . . .) and higher powers of α (such as α^4, α^5 . . .). In (IV-12) and (IV-13) and for most of what follows we use the briefer standard notations $\delta^2 S \equiv \delta^2 S_0$ and $\delta^3 S \equiv \delta^3 S_0$ (as well as $\delta S \equiv \delta S_0$).

In the remainder of this article we use (IV-12) and (IV-13) to determine when the action is greater or less for a particular adjacent curve than for the original worldline, paying primary attention to the second order variation $\delta^2 S$. When the action is greater for all adjacent curves than for the worldline, then $\delta^2 S > 0$ and the action along the worldline is a true minimum. The phrase "for all adjacent curves" means that the value of $\delta^2 S$ in (IV-12) is positive for *all possible variations* $\alpha\phi(t)$. Equation (IV-12) shows immediately that when $\partial^2 U/\partial x^2$ is zero or negative along the entire worldline then the integrand is everywhere positive, leading to $\delta^2 S > 0$ so that when $\partial^2 U/\partial x^2 \le 0$ a worldline of any length has minimum action. This result was previewed in the qualitative argument of Sec. III.

The outcome is more complicated when $\partial^2 U/\partial x^2$ is neither zero nor negative everywhere along the worldline. We show in Sec. V that even in this case we have $\delta^2 S > 0$ for sufficiently short worldlines, so that action is a still a minimum. Later sections show that "sufficiently short" means a worldline terminated before the kinetic focus; for a worldline terminated beyond the kinetic focus the action is greater $(\delta^2 S > 0)$ for *some* adjacent curves and smaller $(\delta^2 S < 0)$ for *other* adjacent curves. This is called a *saddle point* in the action: the value of $\delta^2 S$ in (IV-12) is positive for some variations $\alpha\phi(t)$ and negative for other variations $\alpha\phi(t)$.

When $\delta^2 S = 0$ for one or more adjacent curves, as happens at a kinetic focus[46], we need to examine the higher-order variations to see whether $S - S_0$ is positive, negative, or zero for these particular adjacent curves.

There is no worldline whose action is a true maximum, that is for which $\delta^2 S < 0$ or more generally for which $S - S_0 < 0$ for *every* adjacent curve. The following intuitive proof by contradiction was given briefly by Jacobi[21] in his seminal paper, and in more detail by Morin[47] for a Lagrangian L that is the difference between kinetic and potential energies, $L = K - U$ with K positive as in (IV-6). Consider an actual worldline for which it is claimed that S in (I-1) is a true maximum. Now modify this worldline by adding wiggles somewhere in the middle. These wiggles are to be of very high frequency and very small amplitude so that they increase the kinetic energy K compared to that along the original worldline with only a small change in the corresponding potential energy U. The Lagrangian $L = K - U$ for the region of wiggles is larger for the new curve and so is the overall time integral S. The new worldline has greater action than the original worldline, which we claimed to have maximum action. Therefore S cannot be a true maximum for any actual worldline.

V. WHEN ACTION IS A MINIMUM

We now employ the formalism of Sec. IV to analyze action along a worldline that begins at initial event P and terminates at various final events R that lie along the worldline farther and farther from P. In the present section we show that the action is a minimum for a "sufficiently short" worldline PR in all potentials, and we give a rough estimate of what "sufficiently short" means. (We showed in Sec. IV that the action is a minimum for *all* worldlines in some potentials.) In Sec. VI we show that "sufficiently short" means precisely "before the terminal event reaches the event R at which $\delta^2 S$ first vanishes" for a particular, unique variation. We also show there that this R is Q, the kinetic focus of the worldline, defined in Sec. II. In Sec. VII we show that conversely $\delta^2 S$ *must* vanish at the kinetic focus, and that when final event R is beyond Q the action along PR is a saddle point.

In considering different locations of the terminal event R along the worldline it is important to recognize that the set of incremental functions ϕ that go to zero at P and at R will be different for each terminal position R. Particular functions may have similar form for all R; for example, assuming $t_P = 0$ for simplicity we might have $\phi = A(t/t_R)(1 - t/t_R)$ or $\phi = A\sin(\pi t/t_R)$. However ϕ need not be so restricted; the only restrictions are that ϕ go to zero at both P and R and be piecewise smooth. Statements about the value of $\delta^2 S$ for each different terminal event R are taken to be true for *all possible ϕ for that particular R* that satisfy these conditions.

For a sufficiently short worldline the action is always a minimum compared with that of adjacent curves, as mentioned in Sec. III. The formalism developed in Sec IV confirms this result as follows (here we follow and elaborate Whittaker[35], apart from a qualification stated below). Rewrite (IV-12) using $U''(x) \equiv \partial^2 U/\partial x^2$:

$$\delta^2 S = -\frac{\alpha^2}{2}\int_P^R \phi^2 U'' dt + \frac{\alpha^2}{2}\int_P^R m\dot{\phi}^2 dt \quad . \tag{V-1}$$

Because $\phi = 0$ at P, we can write

$$\phi(t) = \int_P^t \dot{\phi}(t')\,dt' \le (t - t_P)\dot{\phi}_{\max} < T\,\dot{\phi}_{\max} \ , \tag{V-2}$$

where t' is a dummy variable of integration, $T = t_R - t_P$, and $\dot{\phi}_{\max}$ is the maximum value between P and R. With this substitution the magnitude of the first integral in (V-1) for $\delta^2 S$ can be bounded:

$$\left|\int_P^R \left(-\phi^2 U''\right)dt\right| < T^3\dot{\phi}^2_{\max}\left|U''_{\max}\right| \quad . \tag{V-3}$$

The second integral in (V-1) can be rewritten as

$$\int_P^R m\dot{\phi}^2 dt = mT\left\langle \dot{\phi}^2 \right\rangle , \tag{V-4}$$

where $\langle\dot{\phi}^2\rangle$ is the mean square of $\dot{\phi}$ over the time interval T. Compare (V-3) and (V-4) and note that $\dot{\phi}^2_{\max}$ and $\langle\dot{\phi}^2\rangle$ have the same order of magnitude for all values of *T*; the reader can check the special case $\phi(t) = A\sin(n\pi(t-t_P)/T)$, where n is any nonzero integer. Here we assume for simplicity that *φ(t)* is nonzero for all times *t* in the range $T = (t_R - t_P)$ except possibly at discrete points; a similar argument can be given if this condition is violated. Also note that $|U''_{\max}|$ will not increase as R takes positions closer to P. Thus if the range is sufficiently small the most important term in $\delta^2 S$ is the one that contains $\dot{\phi}$. In this limit (V-1) reduces to

$$\delta^2 S \to \frac{1}{2}\alpha^2 m\int_P^R \dot{\phi}^2 dt > 0\ , \text{ for sufficiently short worldlines .} \tag{V-5}$$

This quantity has a positive value since m, α^2, and $\dot{\phi}^2$ are all positive. (See footnote 48 for a more careful statement of this result.) Therefore the second order variation $\delta^2 S$ adds to the action, which demonstrates that the action is always a true minimum along a sufficiently short worldline. We shall use this result repeatedly in the remainder of this paper.

We can give a rough estimate[52] of the largest possible value of T such that $\delta^2 S > 0$ (the exact value is given in Sec. VI). Using (V-3) and (V-4) in (V-1) we see that

$$\delta^2 S > \frac{\alpha^2}{2}\left[mT\langle\dot{\phi}^2\rangle - \dot{\phi}^2_{\max}|U''_{\max}|T^3\right]\ , \tag{V-6}$$

so that we will have $\delta^2 S > 0$ if

$$mT\langle\dot{\phi}^2\rangle > \dot{\phi}^2_{\max}\,|U''_{\max}|\,T^3\ ,$$

or

$$T < \frac{\dot{\phi}_{rms}}{\dot{\phi}_{\max}}\frac{T_0}{2\pi}\quad , \tag{V-7}$$

where $T_o/2\pi = (m/U''_{\max})^{1/2}$ and $\dot{\phi}_{rms} \equiv \langle\dot{\phi}^2\rangle^{1/2}$ is the root-mean-square value of $\dot{\phi}$. For the harmonic oscillator T_0 is exactly equal to the period, and for a general oscillator T_0 is a time of the order of the period. Assume for simplicity that φ(t) is nonvanishing over the whole range T with exceptions only at discrete points, e.g. $\phi(t) = A\sin(n\pi(t-t_P)/T)$; a similar argument can be constructed if this condition is violated. The ratio $\dot{\phi}_{rms}/\dot{\phi}_{\max}$ is then of order unity; for example, for φ(t) of the form $A\sin(n\pi(t-t_P)/T)$ we have $\dot{\phi}_{rms}/\dot{\phi}_{\max} = 1/\sqrt{2}$. Thus for times T less than about $(1/\pi)T_0/2$ we have $\delta^2 S > 0$. For the various oscillators studied quantitatively in Secs. VIII and IX we will see that $T_0/2$ is a better estimate for the time limit for which $\delta^2 S > 0$. For example, for the harmonic oscillator we show that for all times up to $T_0/2$ exactly, we have $\delta^2 S > 0$, so that action is a minimum for times less than a half-period. We show in Sec. VI that the precise time limit for which $\delta^2 S > 0$ is $(t_Q - t_P)$, the time to reach the kinetic focus.

Since the location of initial event P is arbitrary, it follows that the action is a minimum on a short *segment* anywhere along a true worldline. It is not difficult to show that a necessary and sufficient condition for a curve to be a true worldline is that all short segments have minimum action. This result, central to some computer programs for finding some true worldlines, is valid irrespective of whether the action for the total worldline is minimum or a saddle point.

As already discussed in Sec. IV, if $U''(x) \equiv \partial^2 U / \partial x^2$ has zero or negative value at every x along the worldline, then $\delta^2 S$ in (IV-12) is always positive, with the result that worldlines of every length have minimum action for particles moving in these potentials. For example the gravitational potential energy functions U_1 for vertical motion near Earth's surface and U_2 for radial motion above the Earth (radius r_E) have the standard forms

$$U_1(x) = mgx \ , \qquad 0 \le x << r_E \quad ,$$

$$U_2(x) = -\frac{GMm}{r_E + x} \quad , \qquad 0 \le x \, . \tag{V-8}$$

In both cases $U''(x)$ has zero or negative value everywhere, so that (IV-12) tells us that worldlines of any length have minimum action. (Further discussion of the nature of the stationary action for trajectories in these gravitational potentials appears in Appendix B; differences arise when we examine two-dimensional trajectories and when we compare the Hamilton action S with the Maupertuis action W.)

There is an infinite number of potential energy functions with the property $U''(x) \le 0$ everywhere (another example is the parabolic barrier $U(x) = -Cx^2$), leading to minimum action along worldlines of any length. Nevertheless the class of such functions is small compared with the class of potential energy functions for which $U''(x) > 0$ everywhere (such as the harmonic oscillator potential $U(x) = kx^2/2$) or for which $U''(x)$ is positive for some locations and zero or negative for other locations (such as the Lennard-Jones potential $U(x) = C_{12}/x^{12} - C_6/x^6$). For this larger class of potentials the particular choice of worldline (length and location) determines whether the action has a minimum or whether it falls into the class for which action is a saddle point.

In the present section we have shown with some degree of formality that (1) in all potentials, action is a minimum for sufficiently short worldlines, and (2) in some potentials the action is a minimum for worldlines of any length. The following section shows that "sufficiently short" means precisely "before the terminal event R reaches the kinetic focus Q of initial event P" defined in Sec. II.

VI. MINIMUM ACTION WHEN WORLDLINE TERMINATES BEFORE KINETIC FOCUS

The central result of the present section is that the event along a worldline nearest to initial event P at which $\delta^2 S$ goes to zero is kinetic focus Q. The key idea in this proof is that a unique true worldline connects P to Q, a worldline that coalesces with the original worldline and thus satisfies the definition of the kinetic focus in Sec. II. The primary outcome of this proof is that the action is a minimum if a worldline terminates before reaching its kinetic focus.

The discussion here is inspired by the classic work of Culverwell[19] (see Whittaker[49] for a textbook discussion). Culverwell and Whittaker focus on Maupertuis action *W*. We adapt their work to the Hamilton action *S*, extend and simplify it, and finally show that their argument is incomplete.

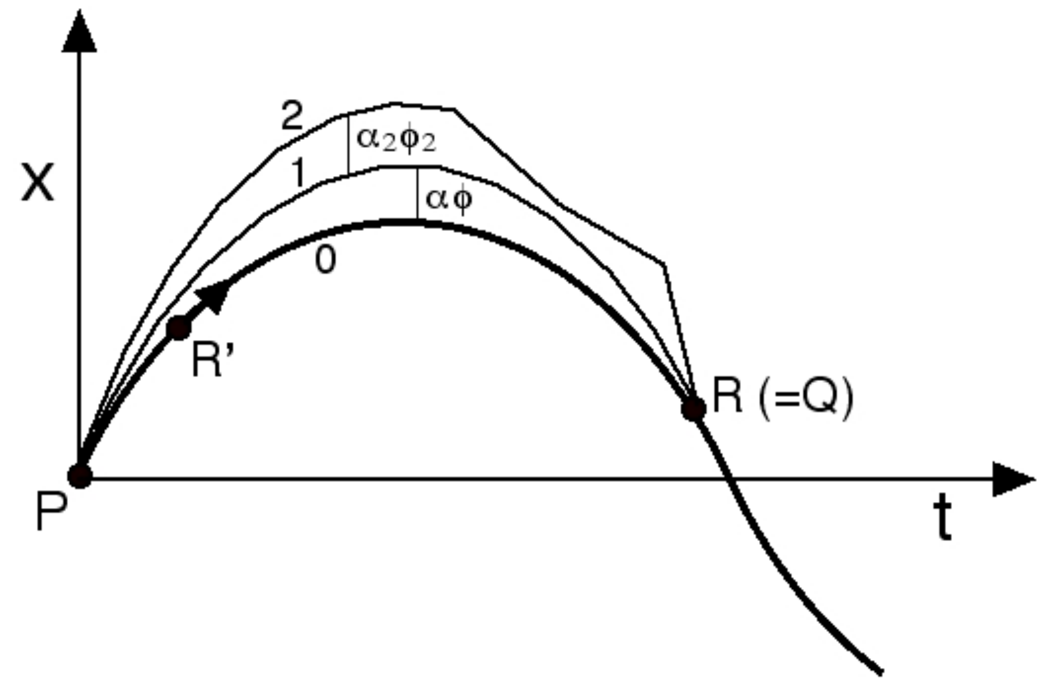

Figure 5. Let R be the *earliest* event along true worldline $x_0(t)$, labeled 0, such that $\delta^2 S = 0$ for worldline PR along $x_0(t)$. The unique variational function which accomplishes this is labeled 1. We show that for this location of R, curve 1 is a true worldline and R is the kinetic focus Q. Arbitrary curve 2 is used to verify that curve 1 is a true worldline.

Consider a true worldline $x_0(t)$, labeled 0 in Fig. 5. As we have seen in Sec. V, the action S_0 along a sufficiently short segment PR′ of worldline 0 is a minimum, which leads to $\delta^2 S > 0$ for all variations. We imagine terminal event R′ located at later and later positions along the worldline until it reaches R, the event at which, by hypothesis, $\delta^2 S \to 0$ for the first time for some variation; i.e. the integral in (IV-12) defining $\delta^2 S$ vanishes for some choice of ϕ. We shall find that the *earliest* event at which $\delta^2 S$ vanishes is connected to the initial event P by a unique type of variation, namely a true worldline which coalesces with the original worldline in the limit at which their initial velocities at P coincide. Hence the earliest event at which $\delta^2 S$ vanishes satisfies the definition of the kinetic focus Q in Sec. II.

As stated above, we assume that $\delta^2 S > 0$ for all R′ up to R′ = R, the first event for which $\delta^2 S = 0$ for a particular variation $\alpha\phi$. To prove R is the kinetic focus Q, we need to consider *small* variations ($\alpha \to 0$) since Q involves a coalescing second worldline. In the typical case, the integral in (IV-13) defining $\delta^3 S$ does not vanish, but letting $\alpha \to 0$ will ensure $\delta^3 S$ (proportional to α^3) does not exceed $\delta^2 S$ (proportional to α^2) for R′ approaching R. This keeps $S - S_0 > 0$ (not just $\delta^2 S > 0$) for R′ up to R. (The untypical case, in which the integral in (IV-13) vanishes, is discussed below.) In the limit R′ → R, we let $\alpha \to 0$ so that the varied curve $x_1(t) = x_0(t) + \alpha\phi(t)$ coalesces with true worldline $x_0(t)$, and $S - S_0 = 0$ at R′ = R.

To satisfy the definition of the kinetic focus, we need to show that $x_1(t)$ is a true worldline just short of the limit $\alpha \to 0$ (i.e. just short of the limit $R' \to R$), not just *at* the limit $\alpha = 0$. Prove this by contradiction: assume curve $x_1(t)$ (curve 1 in Fig. 5) is *not* a true worldline. Consider the arbitrary comparison curve 2 in Fig. 5, which differs from 1 by the arbitrary variation $\alpha_2\phi_2$, with α_2 small. Assume (wrongly) that curve 1 is not a true worldline, so that the first-order variation δS in S between curves 1 and 2 is nonzero for arbitrary $\alpha_2\phi_2$, and the sign of α_2 can be chosen to make $S_2 < S_1$. But since $S_1 = S_0$ to second-order, we must have $S_2 < S_0$, which is a contradiction; R is the earliest zero of $S - S_0$ for any small variation, so that small variations giving $S - S_0 < 0$ are impossible. To avoid the contradiction, curve 1 must be a true worldline. Thus we have proved that the unique variation $\alpha\phi$ that connects P and R = Q when $\delta^2 S$ goes to zero for the first time corresponds to a true worldline.

The above argument covers the typical case, where the integral in (IV-13) defining $\delta^3 S$ does not vanish. It turns out that the only untypical case[53] is the harmonic oscillator, to be discussed in Sec. VIII. The harmonic oscillator has the potential $U = kx^2/2$, for which $\partial^3 U/\partial x^3$ in (IV-13) vanishes, so that $\delta^3 S = 0$ identically. Similarly variation $\delta^4 S$ and higher variations all vanish since $\partial^4 U/\partial x^4$ and higher potential derivatives vanish. Thus for the harmonic oscillator $\delta^2 S = S - S_0$ and $S - S_0$ remains positive up to $R' = R$ for arbitrary α (not just small α). The above argument with $\alpha \to 0$ is valid also for the harmonic oscillator, so that the coalescing true worldline at R again shows that R is the kinetic focus Q. However, it is not necessary here to take the limit $\alpha \to 0$. Figure 7 for the harmonic oscillator shows that *all true* worldlines beginning at P intersect again where $\delta^2 S$ first vanishes which is the kinetic focus Q. By varying the amplitude of the alternative true worldlines for the harmonic oscillator, we can find one that coalesces with the original worldline and thus satisfies the definition of the kinetic focus (Sec. II).

In summary we have shown that as terminal event R takes up positions along the worldline farther away from initial point P, in all cases the special varied curve which leads to the earliest vanishing of $\delta^2 S$ is a unique[54] true worldline that can coalesce with the original worldline. This R satisfies the definition of the kinetic focus Q (Sec. II). Since the varied worldline for which $\delta^2 S = 0$ for the first time is unique, it follows that all other curves PQ adjacent to the original worldline have $\delta^2 S > 0$.

For bound motion in a time-independent potential, worldlines that can coalesce will typically cross more than once. In the literature, *all* of these sequential limiting crossings are called kinetic foci. The above argument is valid *only* for the *first* such crossing, which in this paper we refer to as *the* kinetic focus.

The present section shows that a *sufficient* condition for the kinetic focus Q is that it is the earliest terminal event R for which $\delta^2 S = 0$. In the following section we show the converse *necessary* condition: Given the definition (Sec. II) of kinetic focus Q as the first event at which a second true worldline can coalesce with PQ, the necessary consequence is that $\delta^2 S = 0$ for worldline PQ for the variation leading to coalescence. Taken together, the arguments in these two sections prove the following theorem, which is the fundamental analytical result of our paper:

> *A necessary and sufficient condition for Q to be a kinetic focus of worldline PQ is that Q is the earliest event on the worldline for which* $\delta^2 S = 0$.

This earliest vanishing of $\delta^2 S$ occurs for one special type of variation $\alpha\phi$ (which turns out to correspond to a true worldline), with ϕ unique (up to a factor) and typically $\alpha \to 0$; for all other variations $\delta^2 S$ remains positive at the kinetic focus. The Culverwell-Whittaker argument (more complicated than that above) is incomplete in that (a) it addresses only the sufficiency part of the theorem (the necessary part given in Sec. VII is new), and (b) it overlooks the typical case above, which is the usual one applicable to all nonlinear systems.

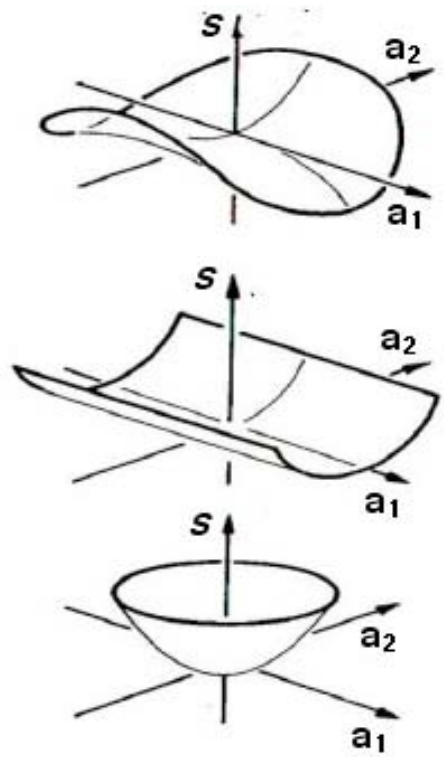

Figure 6. Schematic illustration of topological evolution of minimum $\rightarrow$ trough $\rightarrow$ saddle of action S for two "directions" in function space. Read from bottom to top. (Adapted from ref. 55)

A simple topological picture of the action landscape in function space is emerging (Fig. 6). For a short worldline PR, action S is a minimum: action increases in all directions away from the stationary point in function space (bottom panel in Fig. 6). For longer PR, we may reach a kinetic focus R = Q, for which S is trough-shaped, i.e., flat in one special direction and increasing in all other directions away from the stationary point (middle panel in Fig. 6). (The trough is completely flat for the harmonic oscillator, and flat to second-order for all other systems.) As we shall see in Sec. VII, as R moves beyond Q the trough bends downward, placing the action at a saddle point; that is, *S* decreases in one direction in function space and increases in all other directions away from the stationary point (top panel in Fig. 6). Although not discussed in this paper, the pattern may continue as R moves still further beyond the kinetic focus Q. If R reaches a second event at which $\delta^2 S = 0$ (called the second kinetic focus in the literature), label it Q_2, a trough again develops for one special variational function ϕ (different from the first special ϕ); at Q_2 the action is flat in one direction in function space, decreases in one direction, and increases in all other directions. Beyond Q_2, the trough becomes a maximum, and we have a saddle point which is a maximum in two directions and a minimum in all others. Similar topological changes occur if we reach still later kinetic foci events Q_3, Q_4, and so forth. This is in agreement with Morse's theorem[33], which states that the number of directions *n* in function space for which action is a maximum at a saddle point for worldline PR is equal to the number *n* of kinetic foci between the end events P and R of the worldline.

VII. SADDLE POINT IN ACTION WHEN WORLDLINE TERMINATES BEYOND KINETIC FOCUS

In Sec. VI we showed that the earliest event at which $\delta^2 S = 0$ is connected to the initial event P by a unique true coalescing worldline is a *sufficient* condition for that earliest event to be the kinetic focus Q. All other alternative curves PQ lead to $\delta^2 S > 0$. In the present section we demonstrate the corresponding *necessary* condition, namely: (1) Given an alternative true worldline between P and R that coalesces with the original worldline as $R \to Q$ and therefore defines Q as the kinetic focus, then we have $\delta^2 S = 0$ for that worldline. Using an extension of this analysis, we also show in the present section that (2) when R lies beyond the kinetic focus Q the action of worldline PQR is a saddle point.

The essence of the proof of (1) is outlined in the following heuristic argument of Routh[28]. Consider two intersecting true worldlines P→R connecting P to R. Assume R is close to kinetic focus Q so that the two worldlines differ infinitesimally, as required in the definition of Q (Sec. II). Let action along the two worldlines be S and $S + \delta S$ respectively. Because both are true worldlines, the first order variation of each is equal to zero: we have $\delta S = 0$ and $\delta(S + \delta S) = 0$. Subtracting these two relations gives $\delta^2 S = 0$ for R near Q and hence $\delta^2 S = 0$ for R = Q. In the following we show that Routh is correct in the sense that $\delta^2 S$ vanishes not only at R = Q but also vanishes to the usual second order $O(\alpha^2)$ for R near Q, differing from zero by third order $O(\alpha^3)$ for R near Q.

To make the Routh argument rigorous, think of two alternative true worldlines $x_0(t)$ and $x_1(t)$ in Fig. 6 that connect initial event P to terminal event R, where R is close to the kinetic focus Q of $x_0(t)$. When R reaches Q the two worldlines coalesce according to the definition of kinetic focus Q. For definiteness, take $x_1(t)$ to be the top worldline in Fig. 7 which is closely adjacent to true worldline $x_0(t)$, so it is a member of the set of adjacent curves used for variation in Sec. IV and therefore we can employ the formalism of that section. Conversely we can regard $x_0(t)$ as a varied curve of $x_1(t)$, as it is closely adjacent to the other true worldline $x_1(t)$. Equations similar to (IV-1) in Sec. IV are

$$x_1(t) = x_0(t) + \alpha\phi(t), \qquad x_0(t) = x_1(t) - \alpha\phi(t) \tag{VII-1a}$$

and

$$\dot{x}_1(t) = \dot{x}_0(t) + \alpha\dot{\phi}(t), \qquad \dot{x}_0(t) = \dot{x}_1(t) - \alpha\dot{\phi}(t) \quad . \tag{VII-1b}$$

We have

$$S_1 = S_0 + \delta S_0 + \delta^2 S_0 + \delta^3 S_0 + \ldots \tag{VII-2a}$$

In this case both $x_0(t)$ and $x_1(t)$ are true worldlines, so that we can also write the inverse expression

$$S_0 = S_1 + \delta S_1 + \delta^2 S_1 + \delta^3 S_1 + \ldots \quad . \tag{VII-2b}$$

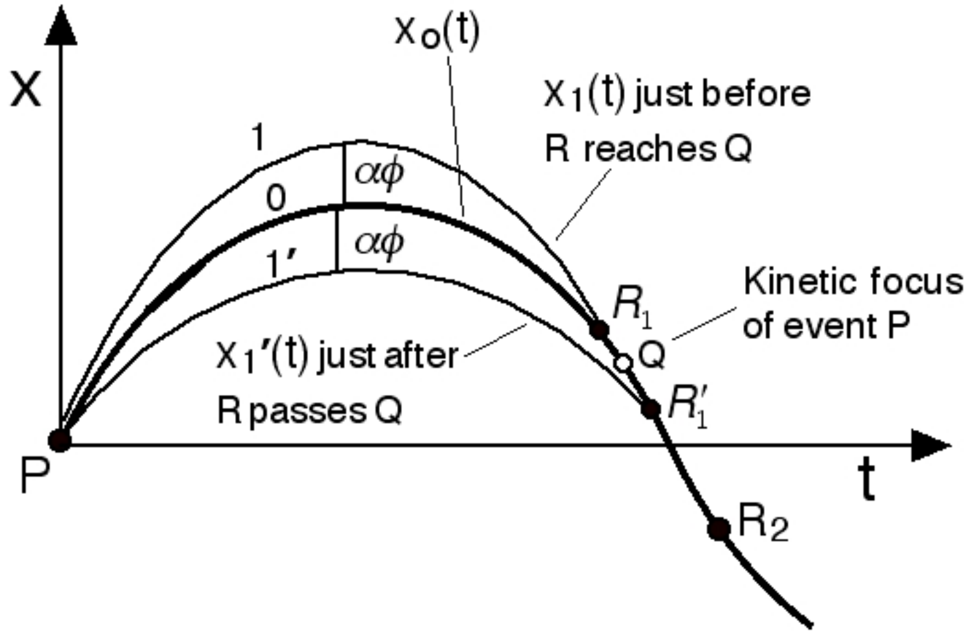

Figure 7. By definition the kinetic focus Q of initial event P is the event at which two adjacent true worldlines $x_0(t)$ and $x_1(t)$ coalesce. We show that $\delta^2 S = 0$ at the kinetic focus Q for this variational function ϕ in the limit $R_1 \to Q$, and that the action is a saddle point when the terminal event R_2 lies *anywhere* on the worldline beyond the kinetic focus Q. (The lower $\alpha\phi$ has opposite sign from the upper $\alpha\phi$, and the upper and lower functions ϕ are slightly different due to the end-events R_1 and R_1' being slightly different.)

Subtract (VII-2b) from (VII-2a) and use $\delta S_0 = 0$ and $\delta S_1 = 0$, since both $x_0(t)$ and $x_1(t)$ are true worldlines. This gives

$$2(S_1 - S_0) = \left(\delta^2 S_0 - \delta^2 S_1\right) + \left(\delta^3 S_0 - \delta^3 S_1\right) + \ldots \qquad \text{(VII-3)}$$

Find expressions for $\delta^2 S$ from (IV-12) using the simplified notation for partial derivatives $U'' = \partial^2 U / \partial x^2$:

$$\delta^2 S_0 = \frac{\alpha^2}{2}\int_P^R \left(-\phi^2 U''(x_0) + m\dot{\phi}^2\right)dt, \qquad \delta^2 S_1 = \frac{\alpha^2}{2}\int_P^R \left(-\phi^2 U''(x_1) + m\dot{\phi}^2\right)dt \quad . \qquad \text{(VII-4)}$$

Then the first parenthesis on the right side of (VII-3) has the form

$$\left(\delta^2 S_0 - \delta^2 S_1\right) = \frac{\alpha^2}{2}\int_P^R dt\left[U''(x_1) - U''(x_0)\right]\alpha\phi \quad , \qquad \text{(VII-5)}$$

where terms in $\dot{\phi}$ have cancelled. Expand $U''(x_1)$ to linear power in α:

$$U''(x_1) \approx U''(x_0) + U'''(x_0)(x_1 - x_0) = U''(x_0) + U'''(x_0)\alpha\phi \quad . \qquad \text{(VII-6)}$$

When (VII-6) is substituted into (VII-5), the resulting integral contributes a further factor of α, yielding a result of $O(\alpha^3)$:

$$\left(\delta^2 S_0 - \delta^2 S_1\right) \approx \frac{\alpha^3}{2}\int_P^R dt\, U'''\left(x_0\right)\phi^3 \qquad . \tag{VII-7}$$

The fact that (VII-7) is proportional to α^3 means that in (VII-3) we cannot neglect terms in $\delta^3 S$, which are also proportional to α^3. (Later terms are proportional to α^4 or higher.) From (IV-13), taking account of the signs of α in (VII-1a) and using our simplified notation U''' for $\partial^3 U/\partial x^3$, we have

$$\delta^3 S_0 = -\frac{\alpha^3}{6}\int_P^R \phi^3 U'''\left(x_0\right)dt \quad , \qquad \delta^3 S_1 = +\frac{\alpha^3}{6}\int_P^R \phi^3 U'''\left(x_1\right)dt \quad , \tag{VII-8}$$

and hence the second term on the right side of (VII-3) becomes

$$\left(\delta^3 S_0 - \delta^3 S_1\right) = -\frac{\alpha^3}{6}\int_P^R \left[U'''(x_1) + U'''(x_0)\right]\phi^3 dt \approx -\frac{\alpha^3}{3}\int_P^R U'''(x_0)\phi^3 dt \ , \tag{VII-9}$$

where we have set $U'''(x_1) \approx U'''(x_0)$ which is correct to $O(\alpha^3)$ in (VII-9).

Substituting (VII-7) and (VII-9) in (VII-3) then gives

$$S_1 - S_0 \approx \frac{\alpha^3}{12}\int_P^R U'''\left(x_0\right)\phi^3\, dt \quad , \qquad \text{(for R near Q, to O}(\alpha^3)\text{)}. \tag{VII-10}$$

Equation (VII-10) makes precise the earlier Routh heuristic argument: We see that $S_1 - S_0$ is of $O(\alpha^3)$ for R near Q and therefore vanishes for R→Q (i.e. $\alpha \to 0$). Comparing (VII-10) and (VII-2a) (with $\delta S_0 = 0$) and noting that the coefficient $\alpha^3/12$ in (VII-10) differs from the coefficient $-\alpha^3/6$ in (VII-8) for $\delta^3 S_0$, we see that not only is $\delta^2 S_0 - \delta^2 S_1 = O(\alpha^3)$, but $\delta^2 S_0$ itself is also $O(\alpha^3)$. In footnote 56 we again show this important relation $\delta^2 S_0 = O(\alpha^3)$, more directly but less elegantly (using equations of motion rather than purely variational arguments). This yields our desired *necessary* condition: Given the definition of Q (Sec II) involving two true worldlines coalescing as R → Q, we have $\delta^2 S = 0$ for worldline PQ for the special variation leading to the coalescence which defines kinetic focus Q (Sec. II).

Next we extend our results to show that when R lies immediately beyond the kinetic focus Q the action of worldline PQR is a saddle point. For this to occur the sign of $S_1 - S_0$ in (VII-10) must change from positive to negative as R passes through the kinetic focus Q. To interpret the sign in (VII-10) consider the varied worldline $x_1(t)$, the uppermost worldline in Fig. 7, which crosses original worldline $x_0(t)$ at R_1 slightly earlier than the kinetic focus Q of worldline $x_0(t)$. We have seen that $\delta^2 S_0 > 0$ for all variations for short PR along $x_0(t)$ and that $\delta^2 S_0$ does not vanish until R reaches Q. Thus $\delta^2 S_0$ and $S_1 - S_0$ are positive for R_1 slightly earlier than Q. Figure 7 shows, and (VII-10) makes quantitative, the fact that as R takes positions from

R_1 to Q and then positions at Q and beyond Q to R_1', the particular variational function $\alpha\phi$ vanishes, and then changes sign to become negative. From (VII-10) we see that when the variation of $x_0(t)$ is the adjacent true worldline $x_{1'}(t)$, we have $S_{1'} - S_0 < 0$ for R_1' slightly later than Q, so that $S(P1'R_1') < S(P0R_1')$. However we know of other variations in $x_0(t)$ which generate $S_{1'} - S_0 > 0$, such as displacing or adding wiggles to a short segment (recall discussion at the end of Sec. IV). Thus when R_1' is just beyond Q we can increase or decrease the action compared to S_0, depending on which variation we choose. This means that for worldline $x_0(t)$, or $P0R_1'$, the action S_0 is a saddle point for R_1' just beyond Q.

We now demonstrate that worldline $x_0(t)$ has a saddle point in action not only for R_1' just beyond Q but also for *all* terminal events on the worldline beyond Q, no matter how far beyond Q they lie. To show this, imagine some point R_2 further along $x_0(t)$ from R_1' by an arbitrary amount, so that the true worldline in Fig. 6 is now $P0R_1'R_2$. Use the bottom worldline $P1'R_1'$ in Fig. 7 to construct the comparison curve $P1'R_1'R_2$ (which is not a true worldline due to the kink at R_1'). Because $P0R_1'R_2$ and $P1'R_1'R_2$ have the segment $R_1'R_2$ in common, and because $S(P1'R_1') < S(P0R_1')$ as shown above, we therefore have $S(P1'R_1'R_2) < S(P0R_1'R_2)$. Hence we have found a variation $P1'R_1'R_2$ with a smaller action than the original worldline $P0R_1'R_2$. But we know it is easy to find other variations giving a larger action (just add wiggles somewhere). Thus $P0R_1'R_2$ has a saddle point[58] in action for all R_2 later than the kinetic focus Q.

The above demonstrations have been for the typical (nonlinear potential) case. The untypical case is the harmonic oscillator potential. The proofs of all the above points are given for the harmonic oscillator separately in Sec. VIII. All systems are therefore included in the theorem.

We have established the two central results of our paper: (1) Worldline PR has minimum action if the terminal event R is earlier than the kinetic focus event Q of initial event P. (2) Worldline PQR has a saddle point in action when terminal event R lies beyond the kinetic focus event Q of initial event P.[55] We stress that these results are correct wherever on the worldline one freely chooses to place the fixed initial event P with respect to which the later kinetic focus event Q of P is established. In the literature[24] these results are often expressed as follows: Jacobi's necessary and sufficient condition for a weak[44] local minimum of the stationary action is that a kinetic focus does not occur between the end events P and R.

VIII. HARMONIC OSCILLATOR

We apply our general results to the bound worldlines of three potentials and the scattering worldlines of one potential. From analytic equations for these worldlines, we show explicitly that they satisfy the general results derived above. The first bound system is the harmonic oscillator. For the harmonic oscillator it is particularly easy to use Fourier series to compare the action along a worldline with the action along *every* (at least every piecewise-smooth) curve alternative to the worldline[61]. The harmonic oscillator Lagrangian has the form

$$L = K - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 , \qquad \text{(VIII-1)}$$

so that (IV-12) becomes

$$\delta^2 S = \frac{\alpha^2}{2}\int_P^R \left(-k\phi^2 + m\dot{\phi}^2\right)dt = \frac{\alpha^2 m}{2}\int_P^R \left(\dot{\phi}^2 - \omega_0^2\phi^2\right)dt, \qquad \text{(VIII-2)}$$

where

$$\omega_0 \equiv \left(\frac{k}{m}\right)^{1/2} = \frac{2\pi}{T_0} \text{ and } T_0 \text{ is the natural period.} \qquad \text{(VIII-3)}$$

All third and higher partial derivatives of L with respect to x and $\dot{x}$ are zero, so there are only second-order variations in S due to ϕ (see (IV-13)). Therefore we have

$$S - S_0 = \delta^2 S, \quad \text{for the harmonic oscillator}. \qquad \text{(VIII-4)}$$

Set P = $(x_P, 0)$ and R = (x_R, t_R). Express the variational function $\phi(t)$ using a Fourier sine series:

$$\phi(t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\omega t}{2}\right) \quad , \text{ where } \omega = \frac{2\pi}{t_R}. \qquad \text{(VIII-5)}$$

This function $\phi(t)$ automatically goes to zero at initial and final events P and R respectively. The constants a_n can be chosen arbitrarily, corresponding to our free choice of $\phi(t)$. Because of the completeness of the Fourier series, the a_n taken together can represent *every possible* piecewise-smooth trial curve alternative to any true worldline. Substitute (VIII-5) into (VIII-2). The squares of $\dot{\phi}$ and ϕ lead to double summations:

$$\delta^2 S = \frac{\alpha^2 m}{2}\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty} a_n a_{n'}\int_0^{t_R}\left[\frac{nn'\omega^2}{4}\cos\left(\frac{n\omega t}{2}\right)\cos\left(\frac{n'\omega t}{2}\right) - \omega_0^2 \sin\left(\frac{n\omega t}{2}\right)\sin\left(\frac{n'\omega t}{2}\right)\right]dt. \qquad \text{(VIII-6)}$$

As t goes from zero to t_R, the arguments of the harmonic functions go from zero to $n\pi$ or $n'\pi$, both of which represent an integer number of half-cycles. Because the different harmonics are orthogonal, terms with $n' \neq n$ integrate to zero for any number of completed half-cycles. This simplifies (VIII-6) to the form

$$\delta^2 S = \frac{\alpha^2 m}{2}\sum_{n=1}^{\infty} a_n^2 \int_0^{t_R}\left[\left(\frac{n\omega}{2}\right)^2 \cos^2\left(\frac{n\omega t}{2}\right) - \omega_0^2 \sin^2\left(\frac{n\omega t}{2}\right)\right]dt. \qquad \text{(VIII-7)}$$

The integrals from $t = 0$ to $t = t_R$ in (VIII-7) are over n half-cycles. For any integer number of half-cycles the average cosine squared and the average sine squared are each equal to one-half. Therefore we have, finally:

$$\delta^2 S = \frac{\alpha^2 m t_R}{4}\sum_{n=1}^{\infty} a_n^2 \left[\left(\frac{n\omega}{2}\right)^2 - \omega_0^2\right] \quad , \qquad \text{(VIII-8)}$$

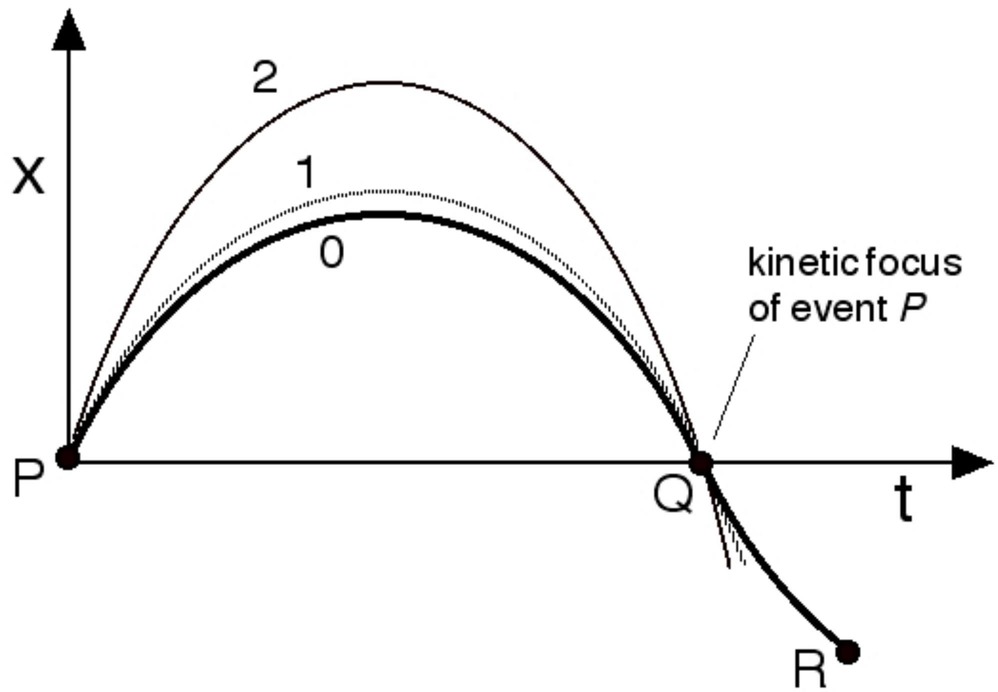

Figure 8. Several true harmonic oscillation worldlines with initial event P = (0,0) and initial velocity $v_0 > 0$. Starting at initial fixed event P at the origin, all worldlines pass through the same event Q. That is, Q is the *kinetic focus* for all worldlines of the family starting at initial event P = (0,0). Worldlines 1 and 0 differ infinitesimally; worldlines 2 and 0 differ by a finite amount.

where ω and ω_0 are defined in (VIII-3) and (VIII-5), and recall, Eq. (VIII-4), that all higher variations are zero. The harmonic oscillator is atypical in that the period does not depend on amplitude, so that all worldlines that start at the same initial event recross at the same kinetic focus, as shown in Fig. 8.

We now use (VIII-8) to give examples and verify, for the atypical case of the harmonic oscillator, the results derived for the typical case in earlier sections of this paper.

Case I: $t_R < T_0/2$. Final time less than one half-period $(\omega/2 = \pi/t_R > \omega_0 = 2\pi/T_0)$

In this case $\delta^2 S$ is positive for all choices of a_n and hence for *every* adjacent curve. For final times less than one half-period (that is, for worldlines that terminate before they reach the kinetic focus of the initial event) the action along the worldline is a minimum with respect to every adjacent curve.

Case II: $t_R = T_0/2$. Final time equal to one half-period $(\omega/2 = \pi/t_R = \omega_0 = 2\pi/T_0)$

Here the final event is the kinetic focus of the initial event, where (Eq. (VIII-4)) $S - S_0 = \delta^2 S$ goes to zero for the first time for a special type of variation. Choose $a_1 = A$ and all other $a_n = 0$, giving $\delta^2 S = 0$ in (VIII-8). This corresponds to the trial curves 1 and 2 pictured in Fig. 8, which are alternative curves, which here are also true worldlines; if we take the limit $\alpha \to 0$ (or the zero-displacement limit $A \to 0$) we have true worldline 1 coalescing with true worldline 0. (All half-period harmonic oscillator worldlines starting from P pass through the same kinetic focus, Fig. 8.) For all other choices of the a_n (*e.g.* $a_2 = A$, other $a_n = 0$) we have $S - S_0 = \delta^2 S > 0$ in (VIII-8), a result established for the typical case in Sec. VI.

Think of each choice of all the coefficients a_1, a_2, a_3, . . . as a point in function space. Then we see that at the kinetic focus the action is a minimum for all "directions" in function space except for the direction with $a_1 \neq 0$, all other $a_n = 0$, for which $S - S_0 = \delta^2 S = 0$. Thinking of each a_n as plotted along a different direction in function space, we can picture this exceptional stationary S case at the kinetic focus as an action "trough" in function space, i.e. flat in one special direction, increasing in all others. (In the typical case[46] the trough is flat only to second-order.) In contrast, for Case I where $t_R < T_0/2$, then $\delta^2 S$ is positive along every direction in function space.

Case III: $t_R > T_0/2$. Final time greater than one half-period $(\omega/2 = \pi/t_R < \omega_0 = 2\pi/T_0)$
In this case we can choose our a_n in order to find adjacent curves with action either greater or less than the action along the original worldline. For $a_1 = A$ and all other $a_n = 0$ the value of $\delta^2 S$ is negative, so the action for the worldline is greater than that for the adjacent curve. In contrast, if we choose $a_n = A$ for any term $n = N$ for which $(N\omega/2)^2 > \omega_0^2$ and all other $a_n = 0$, then $\delta^2 S$ is positive and the action for the worldline is smaller than for the curve. In brief, for a final time greater than half a period of the harmonic oscillator the action for the worldline is neither a true maximum nor a true minimum; it is a saddle point. The corresponding result was shown for the typical case in Sec. VII. Figure 6 shows schematically the evolution of S from Case I to Case II to Case III as t_R increases.

All three cases above apply to the second variations $\delta^2 S$ for all harmonic oscillator true worldlines $x_0(t)$, for example those that do not start from $(x_P, t_P) = (0,0)$, such as $x_0(t) = A_0 \sin(\omega_0 t + \theta_0)$, and includes the no-excursion or equilibrium worldline $x_0(t)=0$. In all cases, for actual worldlines S is a minimum (or a trough as in the special case II) or a saddle point, never a true maximum, in agreement with the general theory.

For the harmonic oscillator all worldlines starting at initial event P = (0, 0) as in Fig. 8, for example, converge next on event Q = (0, $T_0/2$), which is therefore the kinetic focus of P. We verify this result analytically using the general method of Sec. II. For the harmonic oscillator with P = (0, 0), express the amplitude of displacement in terms of the initial velocity v_0:

$$x = \frac{v_0}{\omega_0} \sin \omega_0 t, \tag{VIII-9}$$

where ω_0 is independent of v_0 for the harmonic oscillator. According to (II-3), time t_Q of the kinetic focus is found by taking the partial derivative of (VIII-9) with respect to v_0 and setting the result equal to zero:

$$\frac{\partial x}{\partial v_0} = \frac{1}{\omega_0} \sin \omega_0 t_Q = 0 \quad . \tag{VIII-10}$$

Therefore the kinetic focus of the initial event P = (0, 0) occurs at the time when $\omega_0 t_Q = \pi$ or $t_Q = T_0/2$, as expected. (What the general literature calls "later kinetic foci" occur for $\omega_0 t = 2\pi, 3\pi$, … but we limit the term *kinetic focus* to the first of these.) A similar calculation with

P $\neq$ (0, 0) gives the same result, i.e. $t_Q - t_P = T_0/2$. The fact that $t_Q - t_P$ is independent of initial event P is exceptional for the harmonic oscillator and does not carry over to nonlinear oscillators, considered next. In Appendix B we discuss the location of the kinetic focus for the trajectories of two-dimensional harmonic oscillators.

In summary, four characteristics of the harmonic oscillator worldlines are exceptional; these characteristics are *not* true for worldlines in most potential energy functions. For an arbitrary initial event P: (1) all worldlines from P pass through the same point (the kinetic focus), (2) the time of the kinetic focus Q of P is half a period $T_0/2$ later, (3) the time interval is $T_0/2$ between all successive kinetic foci, and (4) when the final event R is not a kinetic focus, only one true worldline connects it to P. Underlying these four exceptional characteristics is the basic exceptional property of the harmonic oscillator: frequency is independent of amplitude, which reflects the linearity of the system.

IX. NONLINEAR OSCILLATORS

We could analyze the action S for the worldlines of an arbitrary oscillator with potential $U(x)$ by methods similar to those used for the harmonic oscillator in Sec. VIII. The second order variation $\delta^2 S$ from the value S_0 for a worldline $x_0(t)$ is given by (IV-12):

$$\delta^2 S = \frac{\alpha^2}{2}\int_P^R \left[m\dot{\phi}(t)^2 - U''\left(x_0(t)\right)\phi(t)^2\right]dt, \tag{IX-1}$$

where $U''(x) = d^2U(x)/dx^2$ and $\alpha\phi(t)$ is an arbitrary variation from $x_0(t)$ which vanishes at the end-events P and R. The analysis is complicated[62] for arbitrary $U(x)$; it is more instructive to consider instead two examples: (a) the piecewise-linear oscillator with V-shaped potential $U(x) = C|x|$ and (b) the quartic oscillator with U-shaped potential $U(x) = Cx^4$. Figure 9 illustrates the fact that the piecewise linear oscillator is representative of the class whose period *increases* with increasing amplitude of oscillation. Figure 10 illustrates the fact that the quartic oscillator is representative of the class whose period *decreases* with increasing amplitude of oscillation. (The period of the harmonic oscillator is independent of oscillation amplitude.)

(a) Piecewise-linear Oscillator

The piecewise-linear oscillator we consider has a symmetric potential $U(x) = C|x|$, with $C > 0$. As an example, a star oscillates back and forth through the plane of the galaxy and perpendicular to it (Misner et al[63]). We approximate the galaxy as a (freely penetrable) sheet of zero thickness and uniform mass density and express the gravitational potential energy of this configuration as $U(x) = mg|x|$, with the value of $g = C/m$ calculated from the mass density per unit area of the galaxy surface. On earth a piecewise-linear potential of the form[64,65] $U(x) = C|x|$ (with C proportional to the conventional value of g) models the horizontal component of the oscillations of a particle sliding without friction between two equal-angle inclined planes that meet at the origin. The same form of potential roughly models that between two quarks where x is their separation. The classical, semiclassical, and quantum motions of three quarks on a line interacting with mutual piecewise-linear potentials have been studied by variational methods (reference 69 and references therein).

For concreteness we use the example of the star oscillating back and forth perpendicular to the galaxy. We know the solution to the star's oscillation on either side of the galaxy from elementary analysis of the vertical motion near the surface of earth. With the initial event chosen as $P(x_P, t_P)=(0, 0)$ and $v_0 > 0$ as in Fig. 9, the first half-cycle follows the parabolic worldline

$$x = v_0 t - \frac{1}{2} g t^2 , \qquad t \le T_0/2 , \tag{IX-2}$$

where the numerical value of g for galactic oscillation derives from the surface mass density of the galaxy sheet. The time $T_0/2$ of the first half cycle is the time to return to $x = 0$:

$$\frac{T_0}{2} = \frac{2v_0}{g} . \tag{IX-3}$$

After crossing into negative values of x, the worldline equation has a form similar to (IX-2):

$$x = -v_0\left(t - \frac{2v_0}{g}\right) + \frac{1}{2} g \left(t - \frac{2v_0}{g}\right)^2 , \qquad T_0/2 \le t \le T_0 . \tag{IX-4}$$

If we were dealing with motion in the region of positive x (or negative x) alone, there would be no kinetic focus because the second derivative U'' is zero in either region, leading to a positive second order variation in the action derived from (IX-1) as discussed in Sec. V. It is the infinite second derivative U'' at the origin of the potential $U(x) = C|x|$ that creates the kinetic focus for the piecewise-linear oscillator. The second derivative $U''(x)$ of this potential is

$$U''(x) = 2C\,\delta(x) , \tag{IX-5}$$

where $\delta(x)$ is the Dirac delta function.

Now consider the second order variation $\delta^2 S$ for a worldline with $(x_P, t_P)= (0,0)$ and the time t_R of the terminal point R in the range $T_0/2 \le t_R \le T_0$. As our variational function we choose the sine function

$$\phi(t) = a_n \sin\left(\frac{n\omega t}{2}\right) , \tag{IX-6}$$

where a_n is arbitrary and $\omega = 2\pi / t_R$. The variational function $\phi(t)$ vanishes at the end-points P and R, as it should, and is a slowly oscillating variation for $n = 1$ and a rapidly oscillating variation for n large. We substitute (IX-6) and (IX-5) into (IX-1) and carry out the integrations. The integration over $\delta(x)$ is most easily carried out by changing integration variable from t to x using $dt = dx/\dot{x}$. The other integration follows the same pattern as in Sec. VIII. We find

$$\delta^2 S = \frac{1}{4}\alpha^2 m a_n^2 t_R \left[\left(\frac{n\omega}{2}\right)^2 - \frac{4}{\pi^2}\frac{T_0}{t_R}\,\omega_0^2 \sin^2\left(\frac{n\omega T_0}{4}\right)\right] , \tag{IX-7}$$

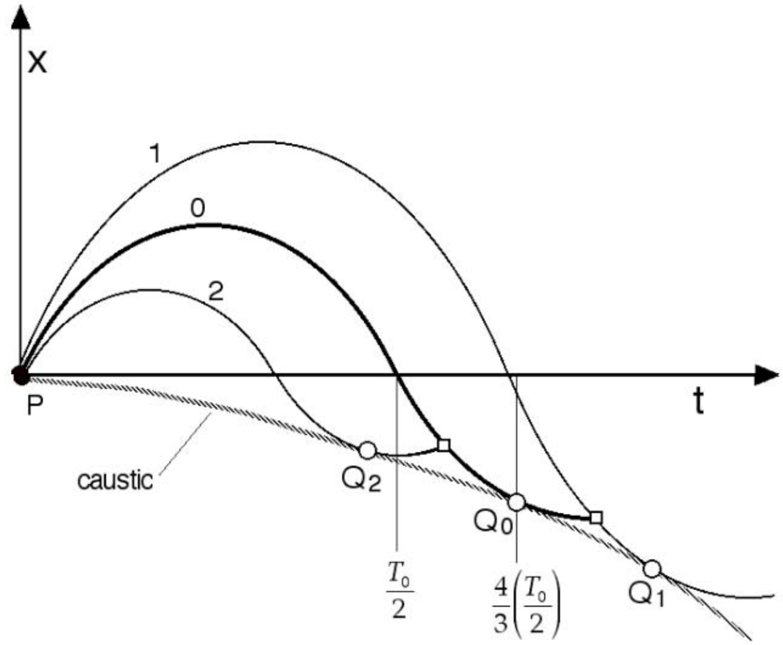

Figure 9. Schematic space-time diagram of a family of true worldlines for a piecewise-linear oscillator, with initial event P = (0,0) and initial velocity $v_0 > 0$. Kinetic focus Q_0 of worldline 0 occurs at 4/3 of its half-period $T_0/2$. Similarly circles Q_1 and Q_2 are the kinetic foci of worldlines 1 and 2 respectively. The heavy gray curve is the *caustic*, the locus of all kinetic foci of different worldlines of this family (originating at the origin with positive initial velocity). Squares indicate events at which the other worldlines recross worldline 0.

where $\omega_0 = 2\pi / T_0$ and T_0 is given by (IX-3). For sufficiently short t_R (*i.e.* sufficiently large $\omega = 2\pi / t_R$) the positive $(n\omega / 2)^2$ term in (IX-7) will dominate for any n, so that we always have $\delta^2 S > 0$. Action S is therefore a minimum for a worldline with sufficiently short t_R. For large t_R (*i.e.* ω small), the $(n\omega / 2)^2$ term will again dominate for a variation with sufficiently large n. In this case we again have $\delta^2 S > 0$. But for the n=1 variation the negative term in (IX-7) dominates for small enough ω (*i.e.* t_R sufficiently large). In this case we have $\delta^2 S < 0$. Thus for sufficiently large t_R the action is a saddle point. These results are consistent with the general theorems derived in earlier sections.

The dividing line between small and large t_R in the preceding paragraph is the time of the kinetic focus. To find the time t_Q at the kinetic focus Q of initial event P (see Fig. 9) we use the analytic method developed in Sec. II. Because the oscillator frequency decreases with increasing amplitude (see Fig. 9) we know that the kinetic focus time t_Q will exceed $T_0/2$. We apply condition (II-3) to (IX-4), giving

$$\frac{\partial x}{\partial v_0} = 0 = -\left(t_Q - \frac{2v_0}{g}\right) + \frac{2v_0}{g} + g\left(t_Q - \frac{2v_0}{g}\right)\left(-\frac{2}{g}\right) \quad , \tag{IX-8}$$

which yields

$$t_Q = \frac{8v_0}{3g} = \frac{2T_0}{3} = \frac{4}{3}\left(\frac{T_0}{2}\right) \quad . \tag{IX-9}$$

Thus for the piecewise-linear oscillator the time of the kinetic focus is *later* than the half-period by a factor of 4/3, as displayed in Fig. 9.

The spatial location x_Q of the kinetic focus Q of a particular worldline is found from t_Q using (IX-4):

$$x_Q = -v_0\left(t_Q - \frac{2v_0}{g}\right) + \frac{1}{2}g\left(t_Q - \frac{2v_0}{g}\right)^2 \quad , \tag{IX-10}$$

where v_0 is the initial velocity of the particular worldline. The locus of the various kinetic foci Q of the family of worldlines in Fig. 9, is the *caustic* or *envelope* and can be found by relating v_0 to t_Q. From (IX-9) we have

$$\frac{2v_0}{g} = \frac{3}{4}t_Q \quad . \tag{IX-11}$$

Substituting (IX-11) into (IX-10) gives the equation for the caustic of the family of piecewise-linear oscillator worldlines with P = (0,0) and $v_0 > 0$:

$$x_Q = -\frac{1}{16}g t_Q^2 \quad . \tag{IX-12}$$

This caustic is a parabola, shown as the heavy gray line in Fig. 9. It divides space-time. Each final event (x_R, t_R) above the caustic can be reached by one or more worldlines of this family of worldlines; each final event on the caustic can be reached by just one worldline of the family; and each final event below the caustic can be reached by no worldline of the family. For the harmonic oscillator all kinetic foci for a given initial event P fall at the same point (a focal point[38]), the limiting case of a caustic. Caustics for other systems are discussed in Secs. IX (b) and X and Appendix B.

Unlike the harmonic oscillator, for the piecewise-linear oscillator the time t_Q-t_P to reach a kinetic focus depends on coordinates (x_P,t_P) of the initial event P. For example we have already shown that t_Q-t_P has the value (4/3) $T_0/2$ when x_P=0. On the other hand if x_P is nonzero but still small we find that t_Q-t_P is smaller than (4/3) $T_0/2$ by the approximate amount $(8x_P/gT_0^2)T_0/2$.

(b) Quartic Oscillator

The quartic oscillator has the potential $U(x) = Cx^4$, with C>0. Pure quartic potentials are rare in nature,[70] but a mechanical model is easily constructed.[75] A particle is linked by harmonic springs on both sides lying along the y-axis. The equilibrium position is $y = 0$ and both springs are assumed to be relaxed in this position. Oscillations along the y-axis are harmonic, but for small transverse oscillations in the x direction the potential has the form $U(x) = Cx^4 + O(x^6)$.

Figure 10 shows a family of worldlines for the quartic oscillator. The second order variation of the action for a worldline $x_0(t)$ is given from (IX-1) for the potential $U(x) = Cx^4$ as

$$\delta^2 S = \frac{\alpha^2}{2}\int_0^{t_R}\left[m\dot{\phi}(t)^2 - 12C\,x_0(t)^2\;\phi(t)^2\right]dt\;, \qquad \text{(IX-13)}$$

where we assume $t_P = 0$ and $\phi(t)$ is an arbitrary variational function. An exact analysis for a general worldline $x_0(t)$ is complicated. We therefore analyze an approximate worldline which brings out the salient points.

Consider a periodic worldline $x_0(t)$ which starts from P=(0,0) with $v_0>0$, as in Fig. 9. For a given energy or amplitude of motion the worldline can be approximated by[12,69]

$$x_0(t) \approx A_0 \sin\left(\omega_0 t\right) \quad . \qquad \text{(IX-14)}$$

Unlike the harmonic oscillator, the frequency ω_0 depends on the amplitude A_0. Action principles can be used in the direct (Rayleigh-Ritz) mode[12,69] to estimate ω_0, giving

$$\omega_0 = \frac{2\pi}{T_0} \approx \left(\frac{3C}{4m}\right)^{1/2} A_0 \quad . \qquad \text{(IX-15)}$$

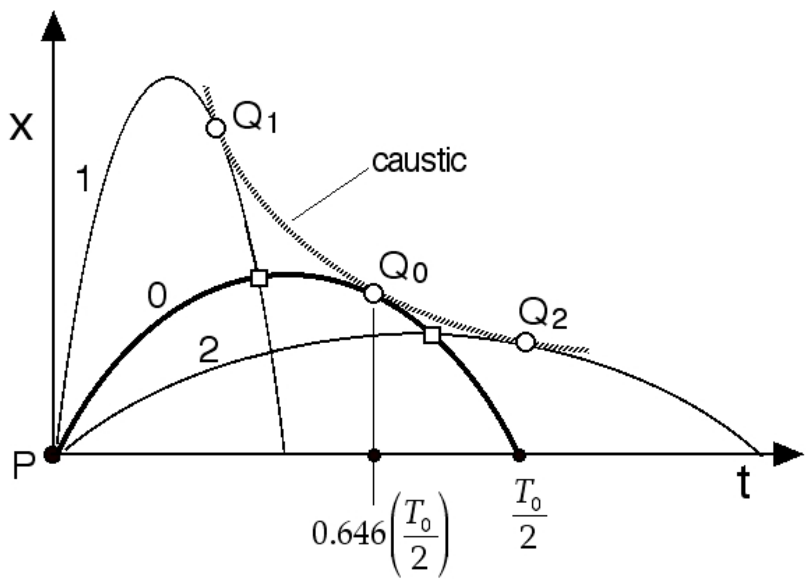

Figure 10. Schematic space-time diagram of a family of true worldlines for the quartic oscillator starting at P = (0,0) and with $v_0>0$. The kinetic focus occurs at a fraction 0.646 of the half-period $T_0/2$, illustrated here for worldline 0. The kinetic foci of all worldlines of this family lie along the heavy gray line, the *caustic*. Squares indicate recrossing events of worldline 0 with the other two worldlines.

As discussed in reference 12, the variational result (IX-15) is accurate to better than 1%, and (IX-14) and (IX-15) can both be improved systematically with the direct variational method if required. (A direct variational method finds true trajectories directly from a variational principle (here an action principle) without any use of equations of motion.)

We can analyze $\delta^2 S$ for the quartic oscillator in the same manner as for piecewise-linear oscillator: substitute (IX-6) and (IX-14) into (IX-13) and carry out the integrations. The results are similar to the preceding section so we omit the details.

As discussed earlier, the cut-off time for minimum action trajectories is the kinetic focus time t_Q; beyond this time the action is a saddle point. We recall (see (II-3) and argument there) that for a family of worldlines $x(t,v_0)$ all starting at event P and with differing initial velocity v_0, the kinetic focus of the worldline with initial velocity v_0 occurs when $\partial x(t,v_0)/\partial v_0 = 0$. To apply this condition to the worldline in (IX-14), first express A_0 and ω_0 in (IX-14) in terms of v_0. For brevity we write (IX-15) as $\omega_0 = \beta A_0$, where $\beta \equiv (3C/4m)^{1/2}$. We also have $v_0 = \omega_0 A_0$ from differentiation with respect to time of (IX-14). From these two relations we get $A_0 = \beta^{-1/2} v_0^{1/2}$ and $\omega_0 = \beta^{1/2} v_0^{1/2}$. Our condition for the kinetic focus is then

$$\frac{\partial}{\partial v_0}\left[\beta^{-1/2} v_0^{1/2} \sin\left(\beta^{1/2} v_0^{1/2} t\right)\right] = 0 \quad , \tag{IX-16}$$

or

$$\frac{1}{2}\beta^{-1/2} v_0^{-1/2} \sin\left(\beta^{1/2} v_0^{1/2} t\right) + \beta^{-1/2} v_0^{1/2} \cos\left(\beta^{1/2} v_0^{1/2} t\right)\left(\frac{1}{2}\right) \beta^{1/2} v_0^{-1/2} t = 0 \quad .$$

Resubstituting $\beta^{1/2} v_0^{1/2} = \omega_0$ we obtain

$$\tan(\omega_0 t) = -\omega_0 t \quad . \tag{IX-17}$$

Equation (IX-17) is satisfied for $t = t_Q$ (and for times of later kinetic foci). MAPLE yields the smallest positive root θ_Q of $\tan\theta = -\theta$ as $\theta_Q \approx 0.646\pi$. The kinetic focus time is then given by $\omega_0 t_Q \approx 0.646\pi$, or $t_Q \approx 0.646(T_0/2)$ for worldline 0 in Fig. 10 and the same fraction of the half-period for the other worldlines shown there. Since our worldline (IX-14) is approximate, this location of the kinetic focus is also approximate. Note that for the quartic oscillator t_Q is *earlier* than the half-period $T_0/2$.

The spatial location x_Q of the kinetic focus $Q(x_Q,t_Q)$ of a particular worldline is found from t_Q using (IX-14):

$$x_Q = A_0 \sin(\omega_0 t_Q) \equiv A_0 \sin\theta_Q , \tag{IX-18}$$

where $\theta_Q \equiv 0.646\pi$ and A_0 is the amplitude of the particular worldline. The locus of the various kinetic foci $Q(x_Q,t_Q)$ of the family of worldlines in Fig. 10, or *caustic*, can be found by relating A_0 to t_Q. From (IX-15) we have $A_0 = \omega_0/\beta$, where $\beta = (3C/4m)^{1/2}$, and we also have $\omega_0 t_Q \equiv \theta_Q$. Thus we find

$$x_Q = \frac{B}{t_Q} \quad , \tag{IX-19}$$

where $B = \theta_Q \sin\theta_Q/\beta = 1.82(4m/3C)^{1/2}$. This caustic in Fig. 10 is a simple hyperbola and it too divides space-time (see discussion of piecewise-linear oscillator caustic). Other families of quartic oscillator worldlines can have different caustics.

The quartic oscillator is a typical nonlinear oscillator, having properties different from the linear harmonic oscillator. For example, the time interval t_Q-t_P to reach the first kinetic focus $Q \equiv Q_1$ is not universal, but depends on the location of initial event P. Also, the time interval between the first and second kinetic foci, for example for the worldlines of Fig. 10 where $t_{Q_2} - t_{Q_1} = 0.914\ (T_0/2)$, differs from the time interval $t_{Q_1} - t_P = 0.646\ (T_0/2)$.

We have now seen examples (the quartic oscillator, the harmonic oscillator, the piece-wise linear oscillator) for which the kinetic focus time is earlier than, equal to, and later than the half-period, respectively. These correspond, respectively, to oscillators whose frequency increases with amplitude, is independent of amplitude, and decreases with amplitude.

X. REPULSIVE INVERSE SQUARE POTENTIAL

The previous examples were systems with exclusively bound motions (oscillators). We now demonstrate corresponding results for a system whose unbound worldlines all describe scattering from the potential

$$U(x) = \frac{C}{x^2} \quad , \tag{X-1}$$

where $C > 0$. It may seem surprising that a worldline in a scattering potential, where motion is unbound, can have a kinetic focus, since there is no kinetic focus for free particle worldlines or for worldlines in the scattering potential $U(x) = Cx$. The difference is in the curvatures of the potentials: the inverse square potential (X-1) has $U''(x) > 0$ whereas the free particle and the linear potential have $U''(x) = 0$. As discussed qualitatively in Sec. III, potentials with positive curvature ($U'' > 0$) are stabilizing/focusing, which can lead to a kinetic focus.

For a given initial position x_P and final position x_R in the potential (X-1), a worldline may be "direct" (direct motion from x_P to x_R) or "indirect" (backward motion from x_P to a turning point x_T followed by forward motion from x_T to x_R). For indirect worldlines the turning point (x_T, t_T) occurs where the kinetic energy is equal to zero, so that the total energy E is equal to the potential energy (X-1), yielding

$$x_T^2 = \frac{C}{E} \quad . \tag{X-2}$$

The actual worldlines $x(t)$ for the potential (X-1) are calculated by integrating the energy conservation relation. Assuming $(x_P, t_P) = (x_P, 0)$, we find

$$x^2 = x_T^2 + \frac{2E}{m}\left(t \pm t_T\right)^2 \quad , \tag{X-3}$$

where +/– apply to a direct/indirect worldline respectively. For an indirect worldline with the initial event P(x_P, 0), (X-3) can be solved for the turn-around time t_T:

$$t_T = \left(\frac{m}{2E}\right)^{1/2}\left(x_P^2 - x_T^2\right)^{1/2} \tag{X-4}$$

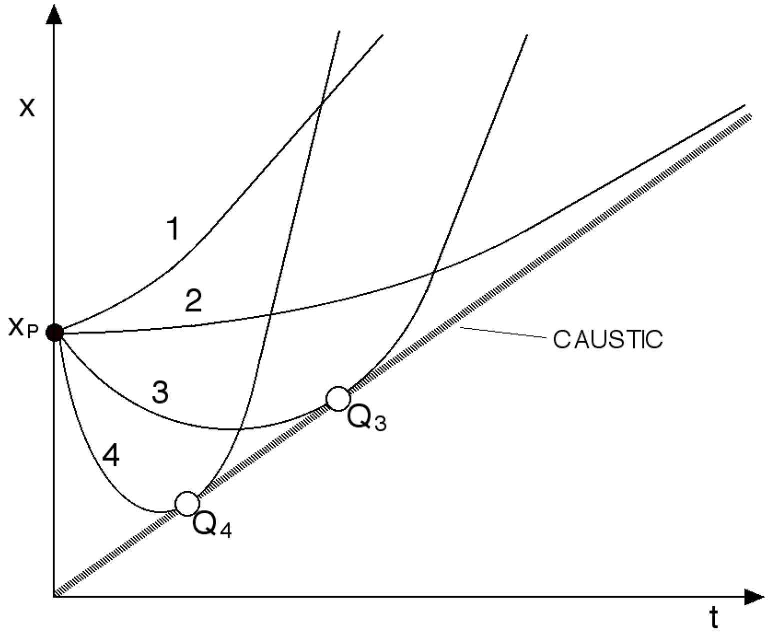

Figure 11. Schematic space-time diagram for the repulsive inverse square potential, with a family of worldlines starting at P(x_P, 0) with various initial velocities. Intersections are events where two worldlines cross. The heavy gray straight line $x_Q = (\gamma / x_P) t_Q$, where $\gamma = (2C/m)^{\frac{1}{2}}$, is the caustic, the locus of kinetic foci Q (open circles) and envelope of the indirect worldlines. Worldline 2, with zero initial velocity, is asymptotic to the caustic, with kinetic focus Q_2 at infinite space and time coordinates. The caustic divides space-time: each final event above the caustic can be reached by two worldlines of this family of worldlines, each final event on the caustic by one worldline of the family, and each final event below the caustic by no worldline of the family.

Some typical worldlines with P(x_P, 0) are shown in Fig. 11. Note the two types, direct and indirect. A direct worldline of arbitrary length has minimum action (no kinetic focus). For indirect worldlines the kinetic foci Q are not the minimum-x turning points but rather the tangent points to the straight-line caustic (heavy gray line) with the equation

$$x_Q = \left(\frac{2C}{m}\right)^{1/2} \frac{t_Q}{x_P} \quad . \tag{X-5}$$

The derivation and discussion of $\delta^2 S$ for comparison curves are similar to those given for the piecewise-linear oscillator in Sec. IX, so we omit them.

The derivation of the kinetic foci (x_Q, t_Q) and caustic equation (X-5) by our standard method is cumbersome for this system, so we use an alternative argument (cf. reference 51). We can eliminate t_T and x_T from (X-3) using (X-2) and (X-4), giving a relation involving x, t, and the (conserved) energy E. Following some routine algebra, we can solve for E:

$$\left(2t^2/m\right)E = x^2 + x_P^2 \pm 2xx_P\left[1 - \frac{2C}{m}\frac{t^2}{x^2 x_P^2}\right]^{\frac{1}{2}} \quad . \tag{X-6}$$

Here the +/– signs refer either to an indirect/direct pair of worldlines, or to two indirect worldlines; both situations are possible, as seen in Fig. 11. A kinetic focus arises here when two indirect worldlines coalesce into one; the locus of kinetic foci forms the caustic or envelop (heavy gray line) in Fig. 11. When the trajectories coalesce, their energies coincide. From (X-6) we see that the condition for coinciding energies is the vanishing of the term in square brackets. The caustic relation is thus found to be (X-5). Equation (X-5) for the caustic is seen to be plausible by the following argument. Note from Fig. 11 that worldline 2 starting from rest ($v_0 = 0$, or zero initial slope) has the caustic as its asymptote. The equation for this asymptote is easily calculated from (X-3) by setting $t_T = 0$ and $E = C/x_P^2$ and taking t large. We find (X-5) immediately.

As we have seen, the indirect worldlines each have a kinetic focus. In contrast to the oscillator systems studied earlier, subsequent kinetic foci do not exist for this system.

XI. GENERALIZATIONS

Extensions of the results of this paper to two-dimensional (2D) and three-dimensional (3D) motion and multi-particle systems is formally straightforward, primarily because both action and energy are scalars; adding dimensions or particles merely sums the corresponding scalar quantities. Let x_i denote the coordinates, for example $(x_1,x_2) \equiv (x,y)$ for 2D motion of a single particle; $(x_1,x_2,x_3) \equiv (x,y,z)$ for 3D motion of a single particle; $(x_1, \ldots, x_6)$ for two particles in 3D, where $(x_1,x_2,x_3) = (x,y,z)$ for particle one and $(x_4,x_5,x_6) = (x,y,z)$ for particle two, and so forth. Equation (IV-1a) generalizes to

$$x_i = x_i^{(0)} + \alpha\, \phi_i \quad , \tag{XI-1}$$

where $x_i^{(0)}$ and x_i are the coordinates of a point on the actual worldline and varied curve, respectively. This leads to obvious generalizations of the subsequent equations of Section IV. In particular, for the common case of one or more particles of mass m in 1D, 2D or 3D, the Lagrangian is

$$L = \sum_i \frac{1}{2} m \dot{x}_i^2 - U \quad , \tag{XI-2}$$

where for particles interacting with each other and/or an external field U can be a function of all x_i and time. Equation (IV-12) then generalizes to

$$\delta^2 S = \frac{\alpha^2}{2} \int_P^R dt \left[-\sum_{ij} \phi_i\, \phi_j\, \frac{\partial^2 U}{\partial x_i\, \partial x_j} + \sum_i m\, \dot{\phi}_i^2 \right] \quad . \tag{XI-3}$$

The kinetic energy in Lagrangian (XI-2) is quadratic in the velocities and thus positive, and this leads to S having a minimum or saddle point (never a maximum) for true worldlines (see argument at the end of Sec. IV). It would be of interest to investigate the possible extension of this result (and the corresponding result for Maupertuis action W – see Appendix A) to more general Lagrangians, including relativistic Lagrangians, Lagrangians containing terms linear in the velocities (e.g. magnetic field terms, gyroscopic terms), etc.

Appendix B describes 2D motion of a particle in two types of gravitational potential and in harmonic oscillator potentials. The criteria for minimum action and location of kinetic foci[76,55] are essentially the same for 2D as for 1D, but, unlike 1D, for the attractive 1/r potential two trajectories connecting P to R can both have minimum action. The formal analysis generalizes easily to other 2D systems, to 3D and multi-particle systems, but the calculations will be more complicated for complex worldlines, e.g. when motion is chaotic (following paragraph). Also, for many-particle systems it is unlikely that we want to specify in advance the complete final as well as initial configuration, since a major purpose of mechanics is to *find* the final configuration[77]. In such cases these powerful deterministic tools may be less useful than modern statistical mechanical methods, although it is interesting to note that historically[69] Helmholtz, Boltzmann, Planck and others attempted to base the second law of thermodynamics on action principles for the molecular motions.

In our derivations we have been careful to use partial derivatives of the potential function with respect to position, $\partial U / \partial x \equiv U'$, $\partial^2 U / \partial x^2 \equiv U''$, etc., because potential energy $U(x,t)$ can also be an explicit function of time in the defining equations of action and its variations, such as (I-1) and (IV-12). Results of this paper can be applied in principle to motion in time-dependent potentials, in which the energy of the moving particle may not be a constant of the motion. Qualitatively new features may arise if U is explicitly time-dependent; for example, we expect that $\delta^2 S$ can remain positive for some long worldlines in potentials with $U''>0$.[80] As an example, consider the quartic oscillator studied in Sec. IX but with time-dependent external forcing *F(t)* added. The potential is now

$$U(x,t) = Cx^4 - xF(t) \quad , \tag{XI-4}$$

and has $U''>0$ for all x except $x = 0$. One often chooses periodic forcing $F(t) = F_0 \cos \omega t$, but other choices are also of interest, e.g. quasiperiodic forcing $F(t) = F_1 \cos\omega_1 t + F_2 \cos\omega_2 t$, with ω_2/ω_1 irrational. Alternatively, one can introduce parametric forcing by modulating C. The unforced oscillator has only equilibrium and periodic worldlines, which are stable. Depending on the initial conditions and potential parameters such as C, F_0, and ω, the forced oscillator can have in addition unstable periodic worldlines[29], quasiperiodic worldlines[83,84], and chaotic (aperiodic, exponentially unstable) worldlines[86]. Similarly the potential $U(x) = C|x|$ of the piecewise-linear oscillator of Sec. IX can be made time-dependent[87]. Second variations and kinetic foci have been studied for some worldlines of various oscillators with time-dependent potentials[88,89,90], but as far as we are aware not specifically for chaotic worldlines.

Chaotic behaviour can arise in higher dimensions even without explicitly time-dependent potentials. As an example, worldlines for the 2D nonlinear Henon-Heiles oscillator with potential

$$U(x,y) = \frac{1}{2}k(x^2 + y^2) + \alpha x^2 y - \frac{1}{3}\beta y^3 \tag{XI-5}$$

are chaotic for certain values of the initial conditions and parameters[91]. Some studies of $\delta^2 S$ and kinetic foci have been done on periodic worldlines for this system[89], but again we are not aware of any studies for chaotic worldlines.

It will be interesting and challenging to study $\delta^2 S$ for chaotic worldlines in any system[92]. We hypothesize that kinetic foci will not exist if the worldline is sufficiently chaotic. In such cases worldlines with incremental difference in velocity at initial event P may recross pseudo-randomly in time, but the severe instability may prevent the two worldlines from smoothly coalescing, as required for the existence of a kinetic focus Q. Worldlines PR lacking kinetic foci have $\delta^2 S > 0$ for arbitrary final events R, so that action is expected to remain a minimum in such cases, even for long worldlines in potentials having U″ > 0, such as (XI-4).

XII. SUMMARY

We examined the nature of the stationary value of the Hamilton action S for the worldlines of a single particle moving in 1D with potential energy function $U(x)$. We showed that when no kinetic focus exists the action is a minimum for worldlines of arbitrary length. When a kinetic focus exists, and when a worldline terminates before reaching its kinetic focus, then the action is still a minimum. In contrast, when a worldline terminates beyond its kinetic focus, its action is a saddle point. The value of the action S is never a true maximum for a true worldline. These results were illustrated with the harmonic oscillator, two anharmonic oscillators, and a scattering system. Extensions to time-dependent 1D potentials $U(x,t)$, and to multidimensional potentials $U(x, y)$ etc., were discussed briefly. Appendices supply parallel results for spatial orbits described by Maupertuis' action W and give examples for 2D motion for both S and W. Corresponding results for some newer action principles have not as yet been derived, and open questions about these newer action principles are sketched in the final Appendix C.

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APPENDIX A: THE MAUPERTUIS ACTION PRINCIPLE

In the literature there are two major versions of action and two corresponding action principles. The Hamilton or time-dependent action S, and the corresponding Hamilton action principle were introduced in Sec. I. The Maupertuis or time-independent action W is defined along an arbitrary trial trajectory connecting P(x_P, t_P) to R(x_R, t_R) by

$$W = \int_{x_P}^{x_R} p dx = \int_{t_P}^{t_R} m\dot{x}\frac{dx}{dt}dt = \int_{t_P}^{t_R} 2K dt, \tag{A-1}$$

where the first (time-independent) form is the general definition with $p = \partial L / \partial \dot{x}$ the canonical momentum (pdx is replaced by $\boldsymbol{p} \bullet d\boldsymbol{x}$ for multidimensional systems), and the last (time-dependent) form is valid more generally for so-called normal systems[69], in which the kinetic energy K is quadratic in the velocity components. For normal systems W is positive for all trajectories in all potentials (unlike S). The Maupertuis action principle states that in conservative systems W is stationary ($\delta W = 0$) for an actual trajectory when comparing trial trajectories all of the same fixed energy E and the same fixed start and end positions x_P and x_R. Note that in Maupertuis' Principle the energy E is fixed and the duration $T \equiv (t_R - t_P)$ is not, the opposite conditions from those occurring in Hamilton's Principle. The constraint of fixed end positions x_P and x_R is common to both principles. Hamilton's Principle is valid for both conservative systems, and nonconservative systems with $U = U(x,t)$. The conventional

Maupertuis Principle just given is valid only for conservative systems; the extension to nonconservative systems is discussed in reference 69. Maupertuis' Principle can be used in its time-independent form to find spatial orbits (e.g. $(x_P, y_P) \longrightarrow (x_R, y_R)$ in 2D), and in its time-dependent form to find space-time trajectories or worldlines (e.g. $(x_P, y_P, t_P) \longrightarrow (x_R, y_R, t_R)$).

The Hamilton and Maupertuis action principles can be stated[69] succinctly in terms of constrained first variations as $(\delta S)_T = 0$ and $(\delta W)_E = 0$ respectively, where the constraints of fixed T and fixed E are denoted explicitly as subscripts, and the constraint of fixed end-positions x_P and x_R is left implicit. Along an arbitrary trial trajectory P → R, S and W are related[69] by a Legendre transformation, i.e.

$$S = W - \overline{E}T\ , \tag{A-2}$$

where $\overline{E} = \int_{t_P}^{t_R} H\ dt/T$ is the mean energy along the arbitrary trajectory, and $T \equiv (t_R - t_P)$ is the duration. Equation (A-2) follows simply from integrating over time between t_P and t_R, along the arbitrary trajectory, the corresponding Legendre transform relation $L = p\dot{x} - H$ between Lagrangian L and Hamiltonian H, and deserves to be better known. Along an actual trajectory of a conservative system, (A-2) reduces to the well-known relation[93] $S = W - ET$, where E is the constant energy of the actual trajectory.

Parallel yet distinct discussions have developed for the second variations of actions S and W because the kinetic foci, which play such an important role in determining the second variation (see Secs. II and VI) can differ for the two actions[29,94,95].

An intuitive argument why W can never be a true maximum for actual paths was given by Routh[96] for normal systems (defined above). Consider an actual path $x_P \to A \to B \to x_R$ which makes stationary the first form of W in (A-1). Here A and B are two arbitrary intermediate positions between the end-positions x_P and x_R. Consider a second, trial path $x_P \to A \to B \to A \to B \to x_R$ which has an extra "loop" inserted, with momentum p exactly reversed at every point along B→A compared to A→B. This comparison path satisfies the proper constraint of having the same energy as the actual path, but clearly has a larger action since $p\,dx$ is always positive. Thus W for the actual path cannot be a true maximum.

APPENDIX B: TWO-DIMENSIONAL TRAJECTORIES

(a) Gravitational Fields

We have shown that the Hamilton action S is a minimum for all 1D radial/vertical trajectories in the $1/r$ and linear gravitational potentials discussed in Sec. V. This minimum action property may not hold for 2D trajectories, as we shall presently discuss. A possible non-minimum in the action is more evident for the Maupertuis action W, Eq. (A-1), for which the true trajectories are defined by giving the two end-positions (x_P, y_P) and (x_R, y_R) and the energy E; we therefore discuss 2D orbits for W first. (We choose our x,y axes in the plane of the orbit.)

For the linear gravitational potential $U(x, y) = mgy$, with y the vertical direction and x horizontal, it is well-known that *two* actual spatial orbits (parabolas) with the same energy E can connect two given positions, the origin $(x_P,y_P) = (0,0)$ say, and final position (x_R, y_R), provided that position $(x_R\ , y_R)$ lies within the so-called "parabola of safety" — the envelope[97,98] of the parabolic orbits of energy E originating at (0,0). See Fig. 12. If (x_R,y_R) lies *on* the parabola of safety, which is the locus of the spatial kinetic foci (x_Q, y_Q), or caustic, there is one actual orbit between fixed end-locations. If (x_R, y_R) lies outside the parabola of safety, no orbit of energy E can connect it to the origin. These points are evident in Fig. 12. As we have seen in Sec. II, the intersection of two paths implies the existence of a (different) kinetic focus for each of the paths. W is a minimum for the path for which (x_R, y_R) precedes its kinetic focus, and is a saddle point for the path for which (x_R, y_R) lies beyond its kinetic focus.

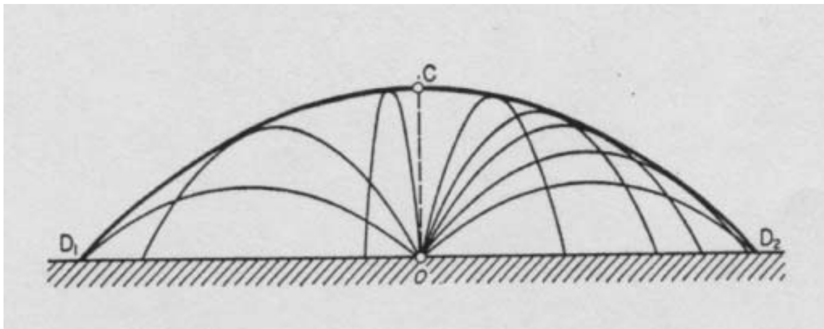

Figure 12. For the Maupertuis action, the heavy line envelope (the "parabola of safety") is the locus of spatial kinetic foci (x_Q, y_Q), or caustic, of the family of parabolic orbits of energy E originating from the origin O $(x_P, y_P) = (0,0)$ with various directions of initial velocity $\mathbf{v}_0$. The potential is U(x,y) = mgy. The horizontal and vertical axes are x and y respectively, and the caustic/envelope equation is $y = v_0^2/2g - gx^2/2v_0^2$, found by Johan Bernoulli in 1692. The caustic divides space. Each final point (x_R, y_R) inside the caustic can be reached from initial point (x_P, y_P) by two orbits of the family, each final point on the caustic by one orbit of the family, and each point outside the caustic by no orbit of the family. C is the vertex (highest reachable point $y = v_0^2/2g$) of the caustic and D_1, D_2 denote the maximum range points $(x = \pm v_0^2/g)$. (Figure from ref. 98.)

Similarly, as first shown by Jacobi[21], typically[99] two given positions (x_P, y_P) and (x_R, y_R) in the gravitational potential $(1/r)$ can be connected with *two* actual orbits (ellipses) (and therefore four paths) of the same energy E. Again this leads to a non-minimum in the action W for actual paths connecting (x_P, y_P) to (x_R, y_R) when (x_R, y_R) lies beyond the kinetic focus. An example is shown in Fig. 13. The intersection points of the orbits show two different ellipses connecting point P to other points. The outer curve is the envelope/caustic, which is also elliptical with foci at P and the force center. The kinetic foci for action W lie on this outer ellipse. The second kinetic focus occurs at P itself, following one revolution. There is no envelope for the hyperbolic scattering orbits for the attractive 1/r potential.[101]

To discuss 2D space-time worldlines for the Hamilton action S, which depends on the two given end-positions and the time interval which now specify a worldline, note that for the $1/r$ gravitational potential, typically *two*[102] actual worldlines can connect two given positions (x_P, y_P) and (x_R, y_R) in a given time interval $(t_R - t_P)$. This is illustrated in Fig. 14, showing two different elliptical trajectories connecting the initial and final points in the same time. Choose $t_P = 0$ for simplicity. The kinetic focus time t_Q for action S is here the period T_0. This is clear intuitively from Fig. 13 which shows a family of trajectories leaving point P and all converging back on P in the same time T_0. A rigorous proof that $t_Q = T_0$ can also be given.[105] Note that the

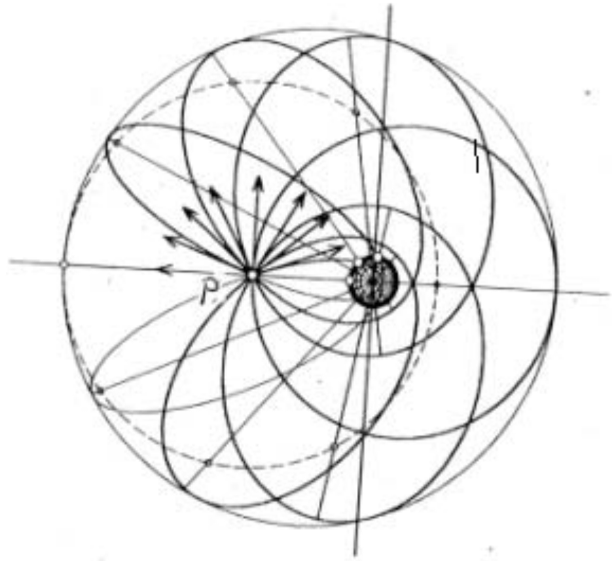

Figure 13. A family of elliptical trajectories starting at P with the same speed $|\mathbf{v}_0|$ (the directions of $\mathbf{v}_0$ differ), and hence the same energy E, the same major axis 2a, and the same period T_0, in the 1/r gravitational potential. Here the value of v_0 exceeds that necessary to generate a circular orbit. The center of force is the earth (heavy circle). The dashed circle gives the locus of the second focus of the ellipses (a circle centered at P). The outer ellipse, with foci at P and the earth, is the envelope of the family of ellipses. (Figure adapted from Butikov, ref. 97)

kinetic focus here is of the "focal point" type, as in Fig. 1 for the sphere geodesics and in Fig. 7 for the harmonic oscillator worldlines. Note also that, as seen in Fig. 14, if two trajectories connect P to R in time $(t_R - t_P) < T_0$, both have minimum action; this is in contrast to 1D, where one of the paths has a saddle point in action.

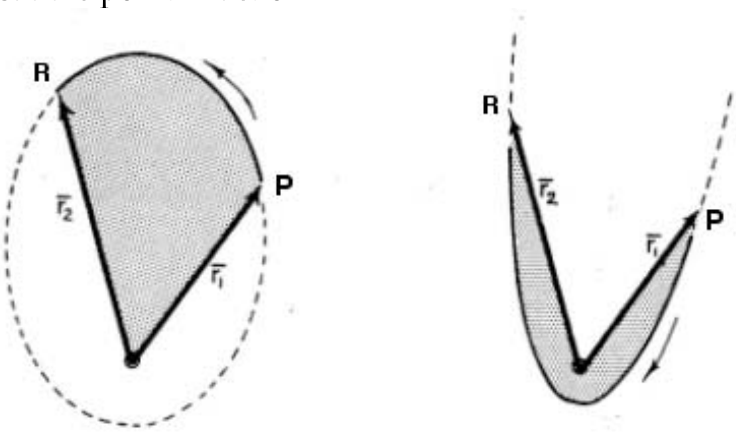

Figure 14. Two different elliptical trajectories typically can connect P = ($\mathbf{r}_1$, t_1) to R = ($\mathbf{r}_2$, t_2) in the same time ($t_2 - t_1$) for the attractive 1/r potential. (Figure from Bates et al, ref.89)

For the *linear* gravitational potential $U(x, y) = mgy$, only *one* 2D (parabolic-shaped) actual worldline can connect two given positions (x_P, y_P) and (x_R, y_R) in a given time ($t_R - t_P$). Kinetic foci for the space-time trajectories therefore cannot arise, and hence S is always a minimum for actual worldlines in this potential[107]. This is in contrast with the Maupertuis action W, for which we have seen that some pairs of positions (x_P,y_P) and (x_R, y_R) can be connected by more than one actual path of given energy E, so that kinetic foci can exist for the spatial orbits. This contrast between S and W also holds for the vertical 1D paths in this potential, $U = mgy$. For S, only one actual 1D worldline can connect a given y_P to a given y_R in a given time ($t_R - t_P$), which leads to the conclusion that all 1D actual worldlines minimize S (see discussion of Section V). For W, typically two actual 1D paths of given energy E can connect y_P to y_R, which leads to the conclusion that not all 1D actual paths minimize W (some are saddle points). For the linear gravitational potential $U(x, y) = mgy$, there is *always* one actual worldline which can connect two given spatial points in a given time, for both 2D and 1D worldlines (S is always a minimum for these worldlines). Again in contrast, there may be *no* actual path which can connect two given spatial points for a given energy, for both 2D and 1D. Fig. 12 shows examples (final points outside the caustic) of this nonexistence of actual paths.

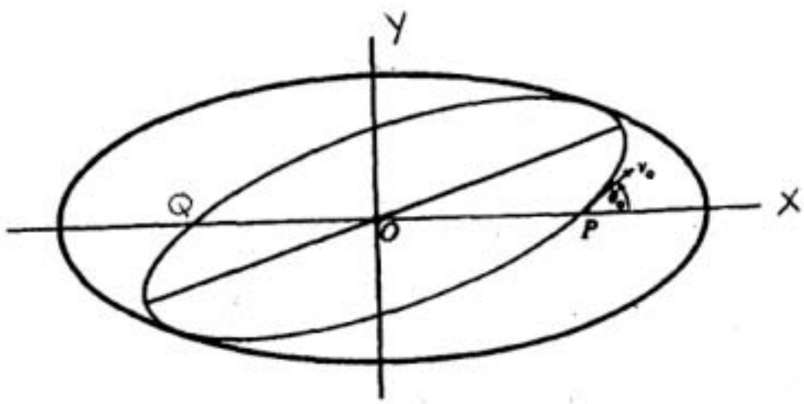

Figure 15. An elliptical orbit (tilted ellipse) in a 2D isotropic harmonic oscillator potential $U(r) = (1/2)k\,r^2$ with force center at O. A family of trajectories is launched from P with equal initial speeds $|\mathbf{v}_0|$ and various directions θ_0 of $\mathbf{v}_0$. One member of the family is shown. The envelope of the family is the outer ellipse, with foci at P and Q (coordinates $x_Q = -x_P$, $y_Q = y_P = 0$). Points on the envelope are the kinetic foci for the spatial orbits. Point Q, occurring at time $t_Q = T_0/2$, where T_0 is the period, is the kinetic focus for the space-time trajectories (wordlines). (Figure adapted from French, ref. 97)

(b) Harmonic Oscillators

The potential for a 2D *isotropic* harmonic oscillator is

$$U(x, y) = \frac{1}{2}k\left(x^2 + y^2\right) \equiv \frac{1}{2}k\,r^2 \quad . \tag{B-1}$$

The spatial orbits are ellipses with the force center (r = 0) at the center of the ellipse. A family of ellipses launched from P in Fig. 15, with equal $|\mathbf{v}_0|$'s (and hence equal energies E) and various directions of $\mathbf{v}_0$, has an envelope/caustic which is also elliptical in shape.[101] The envelope is the outer ellipse in Fig. 15. The kinetic foci for action W lie on the caustic.

To locate the space-time kinetic focus, relevant for action S, we apply the relation given in footnote 74 for 2D trajectories $\mathbf{x}(t, \mathbf{v}_0)$. The matrix $\partial x_i / \partial v_{0j}$ is here diagonal, so that the determinantal condition reduces to $(\partial x / \partial v_{0x})(\partial y / \partial v_{0y}) = 0$, and we get separate 1D conditions for the x and y motions, i.e. $\partial x/\partial v_{0x} = 0$ or $\partial y/\partial v_{0y} = 0$. Choose $t_P = 0$ for simplicity. Previously

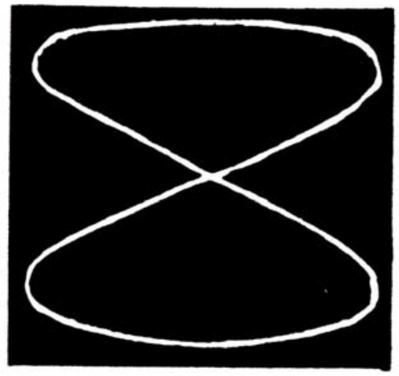

Figure 16. A periodic orbit of a 2D anisotropic harmonic oscillator with commensurate frequencies (here $\omega_1 / \omega_2 = 2$). (From ref. 108)

(Sec. VIII) we showed that the kinetic focus time t_Q for the 1D harmonic oscillator is $\mathrm{T}_0/2$, where $\mathrm{T}_0 = 2\pi/\omega$ is the period and $\omega_0 = (k/m)^{1/2}$. Thus we have $t_Q = \mathrm{T}_0/2$ for the isotropic 2D harmonic oscillator.

The space-time kinetic focus for the trajectories of the 2D *anisotropic* harmonic oscillator with potential

$$U(x,y) = \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2 \tag{B-2}$$

and $k_1 \neq k_2$ can be derived similarly. The orbits are here Lissajous figures, closed (periodic) for ω_1/ω_2 rational as in Fig. 16, and open (quasiperiodic) for ω_1/ω_2 irrational as in Fig. 17, where $\omega_i = (k_i/m)^{1/2}$. The problem is again separable into x and y motions, and the method of footnote 74 yields for the kinetic focus time t_Q the value $\mathrm{T}_0/2$, where T_0 is the smaller of T_1 and T_2 where $\mathrm{T}_i = 2\pi/\omega_i$.

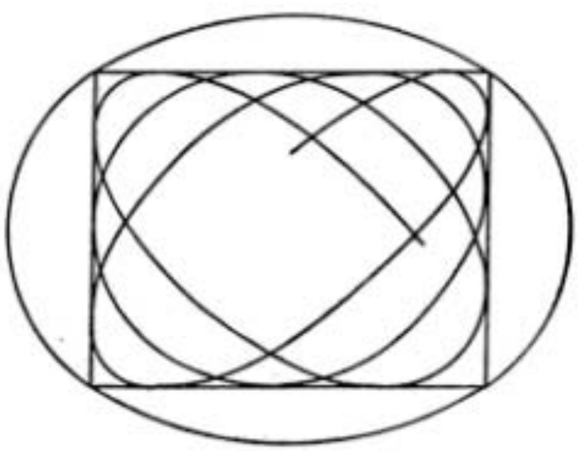

Figure 17. A quasiperiodic orbit of a 2D anisotropic harmonic oscillator with incommensurate frequencies (ω_1 / ω_2 irrational). The outer ellipse is the equipotential contour U(x,y) = E. The rectangle delimits the region of x-y space actually reached by the particular orbit. (From ref. 109)

APPENDIX C: OPEN QUESTIONS FOR SOME NEWER ACTION PRINCIPLES

Culverwell and Whittaker (Sec. VI) framed their analysis in terms of the Maupertuis action W. For sufficiently short trajectories (i.e., the final position occurs before the kinetic focus) W is always a minimum, and for longer trajectories W is a saddle point. W is never a true maximum. In this paper we amended the Culverwell-Whittaker analysis and adapted it to the Hamilton action S. For times less than a kinetic focus time t_Q, the action S is always a minimum. For longer times S is a saddle point. S is never a true maximum. We refer to these results for W and S as "no-max" theorems.

It may be possible to extend the theorems to several newer action principles[69.110]. To state the newer principles, we first recall the succinct notation for the Hamilton Principle (HP) and the Maupertuis Principle (MP) given in Appendix A:

$$(\delta S)_T = 0\,, (HP) \qquad \text{(C-1)}$$

$$(\delta W)_E = 0\,, (MP) \qquad \text{(C-2)}$$

where subscripts $T \equiv t_R - t_P$ (the duration) and E (the energy) denote the constraints. The additional constraints of fixed end-positions x_P and x_R are left implicit in (C-1) and (C-2), and are to be understood to hold here and also in all the action principles given below.

In recent years the Maupertuis Principle (C-2) has been extended to a Generalized Maupertuis Principle (GMP)[69,110],

$$(\delta W)_{\overline{E}} = 0\,, (GMP) \qquad \text{(C-3)}$$

where $\overline{E} \equiv \int_0^T H \; dt / T$ is the mean energy along the arbitrary trial trajectory, with H the Hamiltonian, and where for simplicity we choose $t_P \equiv 0$, $t_R \equiv T$. The constraint of fixed E in (C-2) has been weakened to one of fixed mean energy $\overline{E}$ in (C-3). Conservation of energy for actual trajectories is now a consequence of the principle (C-3), rather than an assumption as in the original principle (C-2).

Both the GMP (C-3) and HP (C-1) have associated *reciprocal* principles[69,110]:

$$(\delta \overline{E})_W = 0\,, (RMP) \qquad \text{(C-4)}$$

$$(\delta T)_S = 0\,, (RHP) \qquad \text{(C-5)}$$

i.e. a Reciprocal Maupertuis Principle (RMP) (C-4) and a Reciprocal Hamilton Principle (RHP) (C-5). The newer principles (C-3)-(C-5) have several advantageous features, computational and conceptual, discussed in references 12, 66 and 91. Additionally, the RMP (C-4) is the direct classical analogue[110,111] (in fact, the classical $\hbar \to 0$ limit) of the well known Schrödinger quantum variational principle involving the quantum mean energy.

It would be of interest to prove the existence or nonexistence of a no-max or no-min theorem for these newer action principles. A Routh-type intuitive argument (see Appendix A) suggests that the GMP (C-3) obeys a no-max theorem, but the examples worked out to date[12,69,110] provide no compelling evidence one way or the other for the other principles. Other newer action principles are also discussed in references 69 and 110. One can completely relax the constraints of fixed T in (C-1) and fixed $\overline{E}$ in (C-3) with the help of Lagrange multipliers. One finds an Unconstrained Hamilton Principle (UHP), $\delta S = - E\, \delta T$, and an Unconstrained Maupertuis Principle (UMP), $\delta W = T\, \delta \overline{E}$, where the Lagrange multipliers E and T are the energy and duration of the actual trajectory, respectively. The UHP and UMP can also be written in the more suggestive forms $\delta(S + \lambda T) = 0$ and $\delta(W + \lambda \overline{E}) = 0$ respectively, where $\lambda = E$ or $\lambda = -T$ is the corresponding constant Lagrange multiplier. These "unconstrained" principles still have the constraint of fixed end positions x_P and x_R; by introducing additional Lagrange multipliers these constraints can also be relaxed.[69] It would also be of interest to prove the existence or nonexistence of no-max or no-min theorems for these various principles.

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13. J.L. Lagrange, *Analytical Mechanics*, 2nd ed., translated from *Mécanique Analytique*, 1811, (Kluwer, Dordrecht, 1997), p. 183 and p. 219. The same erroneous statement occurs in work published in 1760-61, ibid, p. xxxiii.
14. Just on our shelves, we found about two dozen texts using the erroneous term "maximum". See, for example, E. Mach, *The Science of Mechanics*, English translation based on the 9th German edition of 1933 (first ed. 1883), (Open Court, La Salle, Illinois, 1960), p. 463; A. Sommerfeld, *Mechanics*, (Academic Press, New York, 1952, translated from the 1943 German edition), p. 208; P.M. Morse and H. Feshbach, *Methods of Theoretical Physics, Part 1*, (McGraw Hill, New York, 1953), p. 281; D. Park, *Classical Dynamics and its Quantum Analogues*, 2nd ed., (Springer, Berlin, 1990), p. 57; L.H. Hand and J.D. Finch, *Analytical Mechanics*, (Cambridge U.P., Cambridge, 1998), p. 53; G.R. Fowles and G.L. Cassiday, *Analytical Mechanics*, 6th ed., (Saunders, Forth Worth, 1999), p. 393;

R.K. Cooper and C. Pellegrini, *Modern Analytical Mechanics*, (Kluwer, New York, 1999), p. 34.
15. E.F. Taylor and J.A. Wheeler, *Exploring Black Holes: Introduction to General Relativity* (Addison Wesley Longman, San Francisco, 2000), pages 1-7 and 3-4.
16. The number of authors of books and papers using "extremum" and "extremal" is endless. Some examples are: R. Baierlein, *Newtonian Dynamics* (McGraw-Hill, New York, 1983), p. 125; L.N. Hand and J.D. Finch, ref. 14, p. 51; J.D. Logan, *Invariant Variational Principles* (Academic Press, New York, 1977), mentions both "extremal" and "stationary", p. 8; L. D. Landau and E. M. Lifschitz, *Mechanics*, 3rd ed. (Butterworth Heineman, Oxford, 2003), pp 2, 3, see also their *Classical Theory of Fields*, 4th ed., (Butterworth Heineman, Oxford, 1999), p. 25; S.T. Thornton and J.B. Marion, *Classical Dynamics of Particles and Systems*, 5th ed. (Thomson, Brooks/Cole, Belmont CA, 2004), p. 231; R.K. Cooper and C. Pellegrini, ref. 14.
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18. The seminal work on second variations of general functionals by Legendre, Jacobi and Weierstrass and many others is described in the historical accounts of references 20. Mayer's work (Goldstine, ref. 20) was devoted specifically to the second variation $\delta^2 S$ of the Hamilton action. In our paper, we adapt the Culverwell work (ref.19) on $\delta^2 W$ for Maupertuis action W to the Hamilton action. Culverwell's work in turn was preceeded by that of Jacobi (ref. 21) and Kelvin and Tait (ref.22).
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23. Gelfand and Fomin (ref. 24) and other modern books on calculus of variations are rigorous but rather sophisticated. A previous study (ref. 25) in this journal of the nature of the stationarity of worldline action was based on the Jacobi-Morse eigenfunction method (ref. 26), rather than on the more geometrical Jacobi-Culverwell-Whittaker approach discussed here.
24. I. M. Gelfand and S. V. Fomin, *Calculus of Variations*, tr. R.A. Silverman, (Prentice-Hall, Englewood Cliffs NJ, 1963, Russian edition 1961, reprinted by Dover 2000).

25. M.S. Hussein, J.G. Pereira, V. Stojanoff and H. Takai, "The Sufficient Condition for an Extremum in the Classical Action Integral as an Eigenvalue Problem", Am. J. Phys. **48**, 767-770 (1980). (Hussein et al. make the common error of assuming *S* can be a true maximum.) See also J.G. Papastavridis, "An Eigenvalue Criterion for the Study of the Hamiltonian Action's Extremality", Mechanics Research Commun. **10**, 171-179 (1983).
26. M. Morse, *The Calculus of Variations in the Large*, (American Mathematical Society, Providence, 1934).
27. As we shall see, the nature of the stationary value of Hamilton's action *S* (and also Maupertuis' action *W* – see appendices) depends on the sign of the second variations $\delta^2 S$ and $\delta^2 W$ (defined formally in Section IV and Appendix A), which in turn depends on the existence or absence of kinetic foci (see Sections II and VII). The same quantities (signs of the second variations, and kinetic foci) are also important in classical mechanics for the question of dynamical stability of trajectories (refs. 22 and 28-30), and in semiclassical mechanics where they determine the phase of the contribution to the semiclassical propagator due to a particular classical path (refs. 31,32). The phase depends on the so-called Morse (or Morse-Maslov) index which equals the number of kinetic foci between the end-points of the trajectory (see footnote 33). Further, in devising computational algorithms to find the stationary points of the action (either *S* or *W*) it is useful to know whether one is seeking a minimum or a saddle point, since different algorithms (ref. 34) are often used for the two cases. As we discuss in this paper, it is the sign of $\delta^2 S$ (or $\delta^2 W$) which determines which case we are dealing with. Practical applications of mechanical focal points are mentioned at the end of footnote 36.
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31. L.S. Schulman, *Techniques and Applications of Path Integration*, (Wiley, New York, 1981), p. 143.
32. M.C. Gutzwiller, *Chaos in Classical and Quantum Mechanics*, (Springer, New York, 1990), p. 184.
33. In general, saddle points can be classified (or given an "index" in the language of Morse (ref. 26)) according to the number of independent directions leading to maximum-type behavior. Thus the point of zero-gradient on an ordinary horse saddle has a Morse index of unity. The Morse index for an action saddle point is equal to the number of kinetic foci between the end-points of the trajectory (Schulman, ref. 31, p. 90). Readable introductions to Morse Theory are given by: R. Forman, "How Many Equilibria Are There? An Introduction to Morse Theory", in *Six Themes on Variation*, ed. R. Hardt, (American Mathematical Society, Providence, 2004), p. 13; B. Van Brunt, *The Calculus of Variations*, (Springer, New York, 2004), p. 254.
34. F. Jensen, *Introduction to Computational Chemistry*, (Wiley, Chichester, 1999), Ch. 14, "Optimization Methods".
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36. A closer mechanics-optics analogy is between a kinetic focus (mechanics) and a caustic point (optics) (ref. 37). The locus of limiting intersection points of pairs of mechanical spatial orbits is termed an envelope or caustic (see Appendix B for an example), just as the locus of limiting intersection points of pairs of optical rays is termed a caustic. In optics, the intersection point of a bundle of many rays is termed a focal point; a mechanical analogue occurs naturally in a few systems, e.g. the sphere geodesics of Fig. 1, and the harmonic oscillator trajectories of Fig. 8, where a bundle of trajectories recross at a mechanical focal point. In electron microscopes (refs. 38,39), mass spectrometers (ref. 40), and particle accelerators (refs. 41), electric and magnetic field configurations are designed to create mechanical focal points.
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43. Systems with subsequent kinetic foci are discussed in Secs. VIII and IX. For examples with only a single kinetic focus, see Figs. 11 and 12.
44. This type of variation, $\delta x = \alpha\phi$, $\delta\dot{x} = \alpha\dot{\phi}$, where δx and $\delta\dot{x}$ vanish together for $\alpha \to 0$, is termed a "weak" variation: see, e.g., C. Fox, *An Introduction to the Calculus of Variations*, (Oxford U.P., Oxford, 1950), reprinted by Dover, 1987, p.3.
45. H. Goldstein, C. Poole and J. Safko, *Classical Mechanics*, 3rd ed. (Addison Wesley, San Francisco, 2002), p. 44.
46. As discussed in Sec. VI, for a true worldline $x_0(t)$ or PQ where Q is the (first) kinetic focus we have $\delta^2 S = 0$ for one special variation (and $\delta^2 S > 0$ for all other variations) as well as $\delta S = 0$ for all variations. Further, we show that $\delta^3 S$ etc all vanish for the special infinitesimal variation for which $\delta^2 S$ vanishes, and that $S - S_0 = 0$ to second-order for larger such variations. In Morse Theory (refs. 26, 33) worldline PQ is referred to as a *degenerate critical* (stationary) point. Since "degenerate" has other meanings in physics we refrain from using this terminology.
47. D. Morin, *Introductory Classical Mechanics*, 2004, Ch. 5, p. V-8, http://www.courses.fas.harvard.edu/~phys16/Textbook/
48. This statement, and the corresponding one in Whittaker (ref. 49), must be qualified. It is in general not simply a matter of the time interval (t_R-t_P) being short. The spatial path of the worldline must be sufficiently short. When, as usually happens, more than one actual worldline can connect a given position x_P to a given position x_R in the given time interval (t_R-t_P), for short time intervals only the spatially shortest worldline will have

minimum action. For example, the repulsive power-law potentials $U(x) = C/x^n$ (including the limiting case of a hard-wall potential at the origin occurring for $C \to 0$) and the repulsive exponential potential $U(x) = U_0 \exp(-x/a)$ have been studied (ref. 50). No matter how short the given time interval $(t_R - t_P)$, two different worldlines can connect given position x_P to given position x_R. This leads to a kinetic focus time t_Q occurring later than t_R for the shorter of the worldlines and a (different) kinetic focus time t_Q' occurring earlier than t_R for the other worldline. For the first worldline S is a minimum and for the other worldline S is a saddle point. Another example is the quartic oscillator discussed in Sec. X, where an infinite number of actual worldlines can connect given terminal events (x_P,t_P) and (x_R,t_R), no matter how short the time interval (t_R-t_P). Only for the shortest of these worldlines is S a minimum. (The situation is different in 2D etc. – see Appendix B.)

49. E. T. Whittaker, ref. 35, pp. 250-253. Whittaker deals with the Maupertuis action W discussed in our Appendix A, whereas we adapt his analysis to the Hamilton action S. In more detail, our equation (I-1) corresponds to the last equation on p. 251 of this reference, with the changes $q_1 \to t$, $q_2 \to x$ and $q_2' \to \dot{x}$.
50. L.I. Lolle, C.G. Gray, J.D. Poll and A.G. Basile, "Improved Short-Time Propagator for Repulsive Inverse Power-Law Potentials", Chem. Phys. Lett. **177**, 64-72 (1991). In Sec. X and ref. 51 further analytical and numerical results are given for the particular case of the repulsive inverse-square potential, $U(x) = C/x^2$. For given end positions x_P and x_R, there are two actual worldlines $(x_P,t_P) \to (x_R,t_R)$ for given short times (t_R-t_P), there is one actual worldline for $t_R = t_Q$ (kinetic focus time) when the two worldlines have coalesced into one, and there is no actual worldline for longer times (remember x_P and x_R are fixed).
51. A.G. Basile and C.G. Gray, "A Relaxation Algorithm for Classical Paths as a Function of End Points: Application to the Semiclassical Propagator for Far-from-Caustic and Near-Caustic Conditions", J. Comput. Phys. **101**, 80-93 (1992).
52. Better estimates can be found using the Sturm and Sturm-Liouville theories – see Papastavridis, refs. 25 and 62.
53. There are other systems for which $\partial^3 U/\partial x^3$ etc vanish, e.g. $U(x) = C$, $U(x) = Cx$, $U(x) = -Cx^2$. However, for these systems the worldlines cannot have $\delta^2 S = 0$ (see (IV-12)), so that kinetic foci do not exist.
54. In higher dimensions more than one independent variation can occur due to symmetry. In Morse Theory the number of independent variations satisfying $\delta^2 S = 0$ is called the *multiplicity* of the kinetic focus (Van Brunt, footnote 33, p. 254; Schulman, ref. 31, p. 90).
55. J.M.T.Thompson and G.W.Hunt, *Elastic Instability Phenomena*, (Wiley, Chichester, 1984), p. 20.
56. The fact that $\delta^2 S_0 \to 0$ for $\alpha \to 0$ is not surprising since $\delta^2 S_0$ is proportional to α^2. The surprising fact is that here $\delta^2 S_0$ vanishes as α^3 as $\alpha \to 0$. This occurs because the integral involved in the definition (IV-12) of $\delta^2 S_0$ here becomes $O(\alpha)$. One can see directly that $\delta^2 S_0$ becomes $O(\alpha^3)$ for R near Q for this special variation $\alpha\phi$. Integrate the $\dot{\phi}$ term by parts in (IV-12), using $\phi = 0$ at the end-points, giving

$$\delta^2 S_0 = -\frac{\alpha^2}{2}\int_P^R \phi\left[m\ddot{\phi} + U''(x_0)\phi\right]dt \quad .$$

Since x_0 and $x_1 = x_0 + \alpha\phi$ are both true worldlines, we can apply the equation of motion $m\ddot{x} + U'(x) = 0$ to both. Subtract these two equations of motion and Taylor expand $U'(x_0 + \alpha\phi)$ as $U'(x_0) + U''(x_0)\alpha\phi + (1/2)U'''(x_0)(\alpha\phi)^2 + O(\alpha^3)$, giving

$m\ddot{\phi} + U''(x_0)\phi = -(1/2)U'''(x_0)\alpha\phi^2 + O(\alpha^2)$. (If the nonlinear terms on the right-hand side are neglected in the last equation, it becomes the well known Jacobi-Poincare' linear variation equation used in stability studies.) Using this result in the above expression for $\delta^2 S_0$ gives to lowest nonvanishing order $\delta^2 S_0 = (\alpha^3/4)\int_P^R dt U'''(x_0)\phi^3$, which is $O(\alpha^3)$ as stated above. Combining this result with (IV-13) for $\delta^3 S_0$ then gives the result (VII-10) for $S_1 - S_0$.

57. Using arguments similar to those of this section and Sec. VI, we can show $\delta^2 S$ vanishes again at the second kinetic focus Q_2, and that for R beyond Q_2 the wordline PR has a second, independent, variation leading to $\delta^2 S < 0$, in agreement with Morse's general theory (ref. 33).
58. Using $L(x,\dot{x}) = p\dot{x} - H(x,p)$, one can rewrite the Hamilton action as a phase-space integral, i.e., $S = \int_P^R \left[p\dot{x} - H(x,p) \right] dt$. Setting $\delta S = 0$ and varying *x(t)* and *p(t)* independently, one finds (ref. 59) the Hamilton equations of motion, i.e., $\dot{x} = \partial H/\partial p$, $\dot{p} = -\partial H/\partial x$. One can then show (ref. 60) that in *phase space*, the actual trajectories x(t), p(t) (which satisfy the Hamilton equations) are always saddle points of *S*, i.e., never a true maximum *or* a true minimum. In the proof, it is assumed that *H* has the normal form $H(x,p) = p^2/2m + U(x)$, i.e., quadratic in the momentum.
59. H. Goldstein et al., ref. 45, p.353.
60. M.R. Hestenes, in *Modern Mathematics for the Engineer*, ed. E.F. Beckenbach (McGraw Hill, New York, 1956), p.79
61. O. Bottema, "Beisipiele zum Hamiltonschen Prinzip", Monatshelfte für Mathematik **66**, 97-104 (1962).
62. J.G. Papastavridis, "On the Extremal Properties of Hamilton's Action Integral", J. Appl. Mech. **47**, 955-956 (1980).
63. C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*, (Freeman, San Francisco, 1973), p. 318. These authors use the Rayleigh-Ritz direct variational method (see ref. 12 for a detailed discussion of this method) with a two-term trial trajectory x(t) = $a_1 \sin(\omega t/2) + a_2 \sin(\omega t)$, where $\omega = 2\pi/t_R$ and a_1 and a_2 are variational parameters, to study the half-cycle ($t_R = T_0/2$) and one-cycle ($t_R = T_0$) trajectories, where T_0 is the oscillator period. Since the kinetic focus time t_Q is greater than $T_0/2$ for this oscillator, they find, in agreement with our results, that S is a minimum for the half-cycle trajectory (with $a_1 \neq 0$, $a_2 = 0$), and a saddle point for the one-cycle trajectory (with $a_1 = 0$, $a_2 \neq 0$). However, in the figure accompanying their calculation, which shows the stationary points in (a_1, a_2) space, they label the origin (a_1, a_2) = (0, 0) a maximum. The point (a_1, a_2) = (0, 0) represents the equilibrium trajectory *x(t)* = 0. As we have seen, a true maximum in S cannot occur, so that other "directions" a_n in function space (not considered by the authors) must give minimum-type behaviour of *S*, leading to an overall saddle point.
64. The two-incline oscillator potential has the form U(x) = C|x|, with C = mg sinα cosα where α is the angle of inclination. Here x is a horizontal direction. A detailed discussion of this oscillator is given by B.A. Sherwood, *Notes on Classical Mechanics*, (Stipes, Champaign, Ill., 1982), p. 157.
65. Other constant force or linear potential systems include the 1D Coulomb model (ref. 66) $U(x) = q_1 q_2 |x|$, the bouncing ball (ref. 67), and (for x ≥ 0) the constant force spring (ref. 68).

66. I.R. Lapidus, "One- and Two-Dimensional Hydrogen Atoms", Am. J. Phys. **49**, 807 (1981). K. Andrew and J. Supplee, "A Hydrogen Atom in d-Dimensions", Am. J. Phys. **58**, 1177-1183 (1990).
67. I.R. Gatland, "Theory of a Nonharmonic Oscillator", Am. J. Phys. **59**, 155-158 (1991), and references therein. W.M. Hartmann, "The Dynamically Shifted Oscillator", Am. J. Phys. **54**, 28-32 (1986).
68. A. Capecelatro and L. Salzarulo, *Quantitative Physics for Scientists and Engineers: Mechanics*, (Aurie Associates, Newwark, 1977), p. 162. C.-Y. Wang and L.T. Watson, "Theory of the Constant Force Spring", Trans. ASME **47**, 956-958 (1980). H. Helm, "Comment on "A Constant Force Generator for the Demonstration of Newton's Second Law" ", Am. J. Phys. **52**, 268 (1984).
69. C.G. Gray, G. Karl and V.A. Novikov, "Progress in Classical and Quantum Variational Principles", Rep. Prog. Phys. **67**, 159-208 (2004).
70. Nearly pure quartic potentials have been found in molecular physics for ring-puckering vibrational modes (ref. 71) and for the caged motion of the potassium ion K^+ in the endohedral fullerene complex $K^+ @ C_{60}$ (ref. 72), where the quadratic terms in the potential are small. Ferroelectric soft modes in solids are also sometimes approximately represented by quartic potentials (refs.73, 74).
71. R.P. Bell, "The Occurrence and Properties of Molecular Vibrations with $V(x) = ax^4$", Proc. Roy. Soc. A**183**, 328-337 (1945). J. Laane, "Origin of the Ring-Puckering Potential Energy Function for Four-Membered Rings and Spiro Compounds. A Possibility of Pseudorotation", J. Phys. Chem. **95**, 9246-9249 (1991).
72. C.G. Joslin, J. Yang, C.G. Gray, S. Goldman and J.D. Poll, "Infrared Rotation and Vibration-Rotation Bands of Endohedral Fullerene Complexes. $K^+ @ C_{60}$". Chem. Phys. Lett. **211**, 587-594 (1993).
73. A.S. Barker, in *Far-Infrared Properties of Solids*, eds. S.S. Mitra and S. Nudelman, (Plenum, New York, 1970), p.247.
74. J. Thomchick and J.P. McKelvey, "Anharmonic Vibrations of an "Ideal" Hooke's Law Oscillator", Am. J. Phys. **46**, 40-45 (1978).
75. R. Baierlein, ref. 16, p. 73.
76. For 2D, with $\mathbf{x} = (x_1, x_2)$ and $\mathbf{v}_0 = (v_{01}, v_{02})$, our analytic condition (II-3) to locate the kinetic focus of worldline $\mathbf{x}(t, \mathbf{v_0})$ becomes $\det\left(\partial x_i / \partial v_{0j}\right) = 0$, where $\det(A_{ij})$ denotes the determinant of matrix A_{ij}. The generalization to 3D etc. is obvious. This condition (in slightly different form) is due to Mayer (1868; Goldstine, ref. 20, p.269): for a clear discussion, see J.G. Papastavridis, *Analytical Mechanics*, (Oxford U.P., Oxford, 2002), p.1061. For multidimensions a caustic becomes in general a surface in space-time.
77. A dynamics problem can be formulated as an initial value problem (e.g., find x(t) from Newton's equation of motion with initial conditions $(x_P, \dot{x}_P)$), or as a boundary value problem (e.g., find x(t) from Hamilton's Principle with boundary conditions (x_P,t_P) and (x_R,t_R)). Solving a boundary value problem with initial value problem methods (e.g. the shooting method) is standard (ref. 78). Solving an initial value problem with boundary value problem methods is much less common (ref. 79).
78. See, e.g., W.H. Press, S.A. Teukolsky, W.T. Vetterling and B.P. Flannery, *Numerical Recipes in Fortran*, 2ed., (Cambridge U.P., Cambridge, 1992), p. 749.
79. H.R. Lewis and P.J. Kostelec, "The Use of Hamilton's Principle to Derive Time-Advance Algorithms for Ordinary Differential Equations", Computer Phys. Commun. **96**, 129-151 (1996); D. Greenspan, "Approximate Solution of Initial Value Problems for Ordinary Differential Equations by Boundary Value Techniques", J. Math. and Phys. Sci. **15**, 261-274 (1967).

80. The converse effect cannot occur: a time-dependent potential $U(x,t)$ with $U''<0$ at all times always has $\delta^2 S > 0$ as seen from (IV-12). If $U(x,t)$ is such that U'' alternates in sign with time, kinetic foci (and hence trajectory stability) may occur. An example is a pendulum with a rapidly vertically oscillating support point. In effect the gravitational field is oscillating. The pendulum can oscillate stably about the (normally unstable) upward vertical direction (ref. 81). Two- and three-dimensional examples of this type are Paul traps (ref. 82) and quadrupole mass filters (ref. 81) which use oscillating quadrupole electric fields to trap ions. The equilibrium trajectory $\mathbf{x}(t) = 0$ at the center of the trap is unstable for purely electrostatic fields but is stabilized by using time-dependent electric fields. Focusing by "alternating-gradients" (also known as "strong focusing") in particle accelerators and storage rings is based on the same idea (ref. 41).
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82. P.K. Gosh, *Ion Traps*, (Oxford U.P., Oxford, 1995), p. 7.
83. J.J. Stoker, *Nonlinear Vibrations in Mechanical and Electrical Systems*, (Wiley, New York, 1950), p. 112; Stoker's statements on series convergence need amendment in light of the Kolmogorov-Arnold-Moser (KAM) theory (ref. 85), see J. Moser, "Combination Tones for Duffing's Equation", Commun. Pure and Applied Math. **18**, 167-181 (1965); T. Kapitaniak, J. Awrejcewicz and W-H Steeb, "Chaotic Behaviour in an Anharmonic Oscillator with Almost Periodic Excitation", J. Phys. A**20**, L355-L358 (1987); A.H. Nayfeh, *Introduction to Perturbation Techniques*, (Wiley, New York, 1981) p. 216; A.H. Nayfeh and B. Balachandran, *Applied Nonlinear Dynamics*, (Wiley, New York, 1995), p. 234; S. Wiggins, "Chaos in the Quasiperiodically Forced Duffing Oscillator", Phys. Lett. A**124**, 138-142 (1987).
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85. G.M. Zaslavsky, R.Z. Sagdeev, D.A. Usikov and A.A. Chernikov, *Weak Chaos and Quasi-Regular Patterns*, (Cambridge U.P., Cambridge, 1991), p. 30.
86. See, e.g., M. Tabor, *Chaos and Integrability in Nonlinear Dynamics,* (Wiley, New York, 1989), p. 35; J.M.T. Thompson and H.B. Stuart, *Nonlinear Dynamics and Chaos,* 2 ed., (Wiley, Chichester, 2002), p. 3 and p. 10.
87. For example, the equilibrium position can be modulated. A similar system is a ball bouncing on a vertically oscillating table. The motion can be chaotic. See, e.g., J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, (Springer, New York, 1983), p. 102; N.B. Tufillaro, T. Abbott and J. Reilly, *An Experimental Approach to Nonlinear Dynamics and Chaos*, (Addison-Wesley, Redwood City, 1992), p. 23; A.B. Pippard, *The Physics of Vibration*, Vol. 1, (Cambridge U.P., Cambridge, 1978), p. 253 and p. 271.
88. J.G. Papastavridis, ref. 29. This author studies the forced Duffing oscillator with $U(x,t) = (1/2)kx^2 + Cx^4 - F_0 \cos\omega t$. For k=0 the Duffing oscillator reduces to the quartic oscillator.

89. R.H. G. Helleman, "Variational Solutions of Non-integrable systems", in *Topics in Nonlinear Dynamics,* ed. S. Jorna (A.I.P., New York, 1978), p. 264. This author studies the forced Duffing oscillator with $U(x,t) = (1/2)kx^2 - Cx^4 - F_0 \cos \omega t$ (note the sign change in C compared to ref. 88), and the Henon-Heiles oscillator with potential (XI-5).
90. A.G. Basile and C.G. Gray, ref. 51. These authors study a harmonic oscillator with time-dependent equilibrium position, with potential $U(x,t) = (1/2)k[x - x_e(t)]^2$. The worldlines for this system are all nonchaotic.
91. M. Henon and C. Heiles, "The Applicability of the Third Integral of the Motion, Some Numerical Experiments", Astron. J. **69**, 73-79 (1964).
92. There have been a few formal studies of action for chaotic systems, but very few concrete examples seem to be available. See, e.g., S. Bolotin, "Variational Criteria for Nonintegrability and Chaos in Hamiltonian Systems", in *Hamiltonian Mechanics*, ed. J. Seimenis, NATO ASI Series, Vol. 331, (Plenum, New York, 1994), p. 173.
93. H. Goldstein et al., ref. 45, p. 434.
94. H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. 3 (Gauthier-Villars, Paris, 1899), English transl: *New Methods of Celestial Mechanics*, Part 3, (AIP Press, New York, 1993), p. 958.
95. The situation is complicated because, as (A-1) shows, there are two forms for *W*, i.e. the time-independent (first) form, and the time-dependent (last) form. Spatial kinetic foci (discussed in Appendix B) occur for the time-independent form of *W*, whereas, as for *S*, space-time kinetic foci occur for the time-dependent form for *W*. Typically the kinetic foci for the two forms for *W* differ from each other (refs. 94 and 29), and from those for *S*.
96. E.J. Routh, *A Treatise on Dynamics of a Particle*, (Cambridge U.P., Cambridge, 1898), reprinted by Dover 1960, p. 400.
97. A.P. French, "The Envelopes of Some Families of Fixed-Energy Trajectories", Am. J. Phys. **61**, 805-811 (1993); E.I. Butikov, "Families of Keplerian Orbits", Eur. J. Phys. **24**, 175-183 (2003).
98. V.G. Boltyanskii, *Envelopes*, (MacMillan, New York, 1964).
99. We assume we are dealing with bound orbits. Similar comments apply to scattering orbits (hyperbolas). Just as for the orbits in the linear gravitational potential discussed in the preceding paragraph, here too there are restrictions and special cases (Goldstine, ref. 20, p. 164; Chetaev, ref. 100, p. 122). If the second point (x_R,y_R) lies *within* the "ellipse of safety" (the envelope (ref. 97) of the elliptical trajectories of energy E originating at (x_P,y_P)), then *two* ellipses with energy E can connect (x_P,y_P) to (x_R,y_R). If (x_R,y_R) lies *on* the ellipse of safety, then *one* ellipse of energy E can connect (x_P,y_P) to (x_R,y_R), and if (x_R,y_R) lies *outside* the ellipse of safety, then *no* ellipse of energy E can connect the two points. Usually the initial and final points (x_P,y_P) and (x_R,y_R) together with the center of force at (0,0) (one focus of the elliptical path) define the plane of the orbit. If (x_P,y_P), (x_R,y_R) and (0,0) lie on a straight line, the plane of the orbit is not uniquely defined, and there is then almost always an infinite number of paths of energy E in 3D which can connect $(x_P,y_P,z_P = 0)$ to $(x_R,y_R,z_R = 0)$.
100. N.G. Chetaev, *Theoretical Mechanics*, (Springer, Berlin, 1989).
101. A.P. French, ref. 84. For the *repulsive* 1/r potential, the hyperbolic spatial orbits *do* have a (parabolic shaped) caustic/envelope – see French.
102. The finding of the two elliptical (or hyperbolic or parabolic) shaped trajectories from observations giving the two end-positions and the time interval is a famous problem of astronomy and celestial mechanics, solved by Lambert (1761), Gauss (1801-9) and others (refs. 103).
103. R.R. Bate, D.D. Mueller and J.E. White, *Fundamentals of Astrodynamics*, (Dover, New York, 1971), p. 227; H. Pollard, *Celestial Mechanics*, (Mathematical Association of

America, Washington, 1976), p. 28; P.R. Escobal, *Methods of Orbit Determination* (Wiley, New York, 1965), p. 187. More than two trajectories typically become possible at sufficiently large time intervals; these additional trajectories correspond to more than one complete revolution along the orbit (ref. 104).
104. R.H. Gooding, "A Procedure for the Solution of Lambert's Orbital Boundary-Value Problem", Celest. Mech. and Dyn. Astron. **48**, 145-165 (1990).
105. It is clear from Fig. 13 that *a* kinetic focus occurs after time T_0 (period). To show rigorously that this is the *first* kinetic focus (unlike for W where it is the second), we can use a result of Gordon (ref. 106) that action S is a minimum for time $t = T_0$. If one revolution corresponded to the second kinetic focus, trajectory P→P would correspond to a saddle point.
The result $t_Q = T_0$ can also be obtained algebraically by applying our general relation (2.3) to the equation $r = r(t,L)$ for the radial distance, where we use angular momentum L as the parameter labeling the various members of the family in Fig. 14. We obtain t_Q from $(\partial r/\partial L)_t = 0$. This implies $(\partial t/\partial L)_r = 0$, since $(\partial r/\partial L)_t = -(\partial r/\partial t)_L(\partial t/\partial L)_r$. As is well known, at fixed energy E (or fixed major axis 2a), the period T_0 is independent of L for the attractive 1/r potential, so that the solution of $(\partial t/\partial L)_r = 0$ occurs for $t = T_0$, which is therefore the kinetic focus time t_Q.
106. W.B. Gordon, "A Minimizing Property of Keplerian Orbits", Amer. J. Math. **99**, 961-971 (1977).
107. Note that for the actual 2D trajectories in the potential $U(x, y) = mgy$, kinetic foci exist for the spatial paths of the Maupertuis action *W* but do not exist for the space-time trajectories of the Hamilton action *S*. This illustrates the general result stated in Appendix A, that the kinetic foci for W and S differ in general.
108. A.P. French, *Vibrations and Waves*, (W.W. Norton, New York, 1966), p. 36.
109. J.C. Slater and N.H. Frank, *Introduction to Theoretical Physics*, (McGraw Hill, New York, 1933), p. 85.
110. C.G. Gray, G. Karl and V.A. Novikov, "The Four Variational Principles of Mechanics", Ann. Phys. **251**, 1-25 (1996).
111. C.G. Gray, G. Karl and V.A. Novikov, "From Maupertuis to Schrödinger. Quantization of Classical Variational Principles", Am. J. Phys. **67**, 959-961 (1999).