The original Euler's calculus-of-variations method: Key to Lagrangian mechanics for beginners

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Leonhard Euler's original version of the calculus of variations (1744) used elementary mathematics and was intuitive, geometric, and easily visualized. In 1755 Euler (1707-1783) abandoned his version and adopted instead the more rigorous and formal algebraic method of Lagrange. Lagrange's elegant technique of variations not only bypassed the need for Euler's intuitive use of a limit-taking process leading to the Euler-Lagrange equation but also eliminated Euler's geometrical insight. More recently Euler's method has been resurrected, shown to be rigorous, and applied as one of the direct variational methods important in analysis and in computer solutions of physical processes. In our classrooms, however, the study of advanced mechanics is still dominated by Lagrange's analytic method, which students often apply uncritically using "variational recipes" because they have difficulty understanding it intuitively. The present paper describes an adaptation of Euler's method that restores intuition and geometric visualization. This adaptation can be used as an introductory variational treatment in almost all of undergraduate physics and is especially powerful in modern physics. Finally, we present Euler's method as a natural introduction to computer-executed numerical analysis of boundary value problems and the finite element method.

I. INTRODUCTION

In his pioneering 1744 work *The method of finding plane curves that show some property of maximum and minimum*,[1] Leonhard Euler introduced a general mathematical procedure or method for the systematic investigation of variational problems. Along the way he formulated the variational principle for mechanics, his version of the principle of least action.[2] Mathematicians consider this event to be the beginning of one of the most important branches of mathematics, the calculus of variations. Physicists regard it as the first variational treatment of mechanics, which later contributed significantly to analytic mechanics and ultimately to the fundamental underpinnings of twentieth-century physics, including general relativity and quantum mechanics.

It is not certain[3] when Euler first became seriously interested in variational problems and properties. We know that he was influenced by Newton and Leibniz, but primarily by James and Johann Bernoulli who were also attracted to the subject. The best known examples of variational calculus include Fermat's principle of least time ("between fixed endpoints, light takes the path for which the travel time is shortest"), Bernoulli's brachistochrone problem[4] ("find a plane curve between two points along which a particle descends in the shortest time under the influence of gravity"), and the so-called isoperimetric problem ("find the plane curve which encloses the greatest area for a given perimeter").

While each of Euler's contemporaries devised a special method of solution depending on the character of the particular variational problem, Euler's own approach was purely mathematical and therefore much more general. Employing geometrical considerations and his phenomenal intuition for the limit-taking process of calculus, Euler established a method that allows us to solve problems using only elementary calculus.

In 1755, the 19-year-old Joseph-Louis Lagrange wrote Euler a brief letter to which he attached a mathematical appendix with a revolutionary technique of *variations*. Euler immediately dropped his method, espoused that of Lagrange, and renamed the subject the *calculus of variations*.[5] Lagrange's elegant techniques eliminated from Euler's methods not only the need for an intuitive approach to the limit process, but also Euler's geometrical insight. It reduced the entire process to a quite general and powerful analytical manipulation which to this day characterizes the calculus of variations. Euler's method was little used by others, partly because in his time the limit-taking process was intuitive, lacking the rigorous basis provided 100 years later by Weierstrass.

At the beginning of the twentieth century, interest in the nature and existence of solutions of variational problems and partial differential equations led to developments in approximation techniques. Euler's method again attracted the attention of mathematicians, and eventually the modern analysis of variational problems and differential equations[6,7,8,9] fully vindicated Euler's intuition. Euler's method rose like a phoenix and became one of the first *direct variational methods*.[6,7,8,9] At approximately the same time other direct methods appeared: the well-known Rayleigh-Ritz method (1908) and its extension called Galerkin's method (1915). Direct methods offer a unified treatment that permits a deep understanding of the existence and nature of solutions of partial differential equations. Finally, all these methods for solving differential equations led to the formulation of the powerful Finite Element Method (FEM) (1943, 1956)[10] for carrying out accurate numerical computer predictions of physical processes.

For more than a century, students in advanced undergraduate classes in mechanics have been taught to use Lagrange's calculus of variations to derive the Lagrange equations of motion from Hamilton's principle, which is also known as the principle of least action (so renamed by Landau and Feynman). Students often meet the calculus of variations first in an advanced mechanics class, find the manipulations daunting, do not develop a deep conceptual understanding of either the new (to them) mathematics or the new physics, and end up memorizing "variational recipes."

In what follows we describe a strategy that allows us to introduce important variational treatments of mechanics and physical laws[11] as "core technology" while at the same time teaching students to apply it with understanding and insight. In recent papers[12, 13, 14] we have demonstrated that the visualization provided by Euler's method leads to elementary-calculus derivations of Lagrange's equations of motion, Newtonian mechanics, and the connection between symmetries and conservation laws. This approach is easily extended to variational treatments in all areas of physics where the calculus of variations is used.

Section II provides a description of Euler's method from his 1744 work, together with summary notes for its pedagogic use not published in our previous articles. The historical context with last developments in 20th century, which shows the significance and basic simplicity of Euler's work, is mainly available in the mathematical literature[6,7,8,9] but is not well known to many in the physics community.

Section III outlines the connection between the Euler method and computer simulations. Using interactive software students carry out their own investigations of the principle of least action and Lagrangian mechanics, a process that contemporary education research shows to be effective in developing understanding of concepts and their applications.

The final Sec. IV lists how Euler's method can effectively substitute for the Lagrange method in almost all of undergraduate physics, especially modern physics. In order to compare the methods of Euler and Lagrange, we supply references to works whose authors apply each of the two methods to the same subjects.

II. BASIC IDEAS OF EULER'S METHOD

A. Euler's original considerations

The transcription of Euler's original derivations from his 1744 work is reproduced in Goldstine's book[3]. Several other mathematics and physics books offer somewhat modified versions.[15] Here we present only the essential considerations of the Euler approach.[16]

Euler's starting point was his ingenious reduction of the variety of variational problems to a single abstract mathematical form. He recognized that solving the variational problem requires finding an extremal (or more strictly stationary) value of a definite integral.

As a first example Euler presents a solution of the simplest variational problem: A function $F = F(x, y, y')$ has three variables: the independent coordinate x, the dependent coordinate y, and its derivative y' with respect to the independent coordinate. Our problem is to determine the curve $y = y(x)$, with $0 \le x \le a$, which will make the definite integral $\int_0^a F(x, y, y')dx$ extremal. Such an integral occurs in the brachistochrone problem or in the description of motion using the principle of least action.

Euler presents three crucial procedures which allowed him to solve the problem using only elementary calculus:

(1) Divide the interval between $x = 0$ and $x = a$ into many small subintervals, each of width Δx.

(2) Replace the given integral by a sum $\sum F(x, y, y')\Delta x$. In each term of this sum evaluate the function F at the initial point x, y of the corresponding subinterval and approximate the derivative $y' = \mathrm{d}y/\mathrm{d}x$ by the slope of the straight line between initial and final points of the subinterval.

(3) Employ a visualized "geometrical" way of thinking.

Goldstine's book[3] displays (p.69) the original Euler's figure (Fig. 1) in which the curve *anz* represents the unknown extremal curve $y = y(x)$:

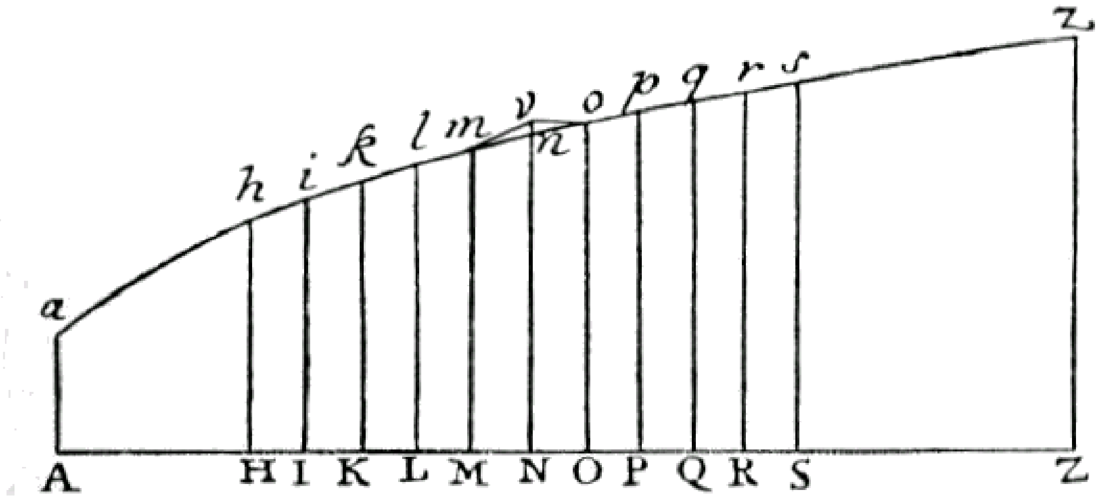

FIG. 1. Original Euler's figure used in his derivation

Figure 1 illustrates the fact that if we shift an arbitrarily chosen point on the curve, for example point *n*, up or down by an increment *nv*, then in addition to the obvious change in the ordinate y_n of the point n, there are also changes in neighboring segments mn and no and their slopes. All other points and slopes remain unchanged. These changes mean that only two terms in the integral sum $\sum F(x, y, y')\Delta x$ are affected, the first corresponding to segment mn and the second to segment no. The resulting change in the integral sum is a function of the single variable y_n . Because the curve anz is assumed to be extremal already, this function of the variable y_n must have a stationary value at n.

This new problem can be solved easily using the ordinary calculus of maxima and minima. The condition for the sum to be stationary corresponds to a zero value of its derivative with respect y_n. To calculate this derivative Euler derives the changes in both affected terms corresponding to segment mn and no caused by varying y_n and demands that these changes result in zero net change in the sum, from which Euler finally extracts the equation:

$$0 = \frac{\partial F}{\partial y_n} - \left(\frac{\partial F}{\partial y'_n} - \frac{\partial F}{\partial y'_m} \right) \Big/ \Delta x \qquad (1)$$

where y_n is again the ordinate of the point n and y'_n is its derivative at the point n and y'_m is the similar derivative corresponding to the point m.

In the limit as Δx approaches zero, the sum returns to the original integral and Eq. (1) becomes a differential equation at point m. Moreover, since the location of the triplet of points m, n, o along the curve was chosen arbitrarily, the differential equation holds for the entire interval between $x = 0$ and $x = a$:

$$0 = \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \qquad \text{for} \quad 0 \le x \le a \qquad (2)$$

This final result is usually called the *Euler-Lagrange equation* and it expresses the first necessary condition for an extremal value of the integral. Euler gives a number of specific and general examples that illustrate how to use his method, all conceptually similar to that outlined above.

To prepare for later sections of the present paper, we recall the standard physical notation used in the variational treatment of classical mechanics based on the principle of least action. Motion in one dimension is sufficient to illustrate the method. Then the description of motion includes as independent variable the time t (instead of x) along with the time-dependent generalized coordinate q (instead of y). As a generalized coordinate q one can choose not only one of the Cartesian coordinates x, y, z but also any other coordinate that describes position, such as φ or r. The role of the function F is played by the *Lagrangian function*, $L = K - U$, the difference between kinetic and potential energy. The definite integral

$$S = \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} (K - U) dt \qquad (3)$$

is called the *action integral*, or in short the *action* and is assigned the symbol S. According to this notation, the Euler-Lagrange equation (2) has the well-known form:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) = 0 . \qquad (4)$$

where a dot over the q represents differentiation with respect to time.

If we consider x as coordinate q, Euler's diagram (Fig. 1) is simply the *spacetime diagram*, and the unknown curve $x(t)$ is called the *worldline*. Finally, the variational principle of least action says that the actual path followed by particle is the one for which the action S is minimal (or more precisely *stationary*).

B. Advantages and disadvantages of the method

Two major objections can be made to the Euler procedure:

(1) There are two limit-taking processes leading to the differential equations: partial derivatives with respect to coordinates and passage to the limit $\Delta x \to 0$. From the mathematical viewpoint the use of partial derivatives is justified, but one must demonstrate the legitimacy of the limit-taking passage $\Delta x \to 0$. Euler neither established[17] the existence of this limit nor proved the fact that the limit converges to the real extremal curve. The Lagrange method of variations is free from such use of the double limit process but also does not contain geometrical insights, since geometrical examples typically cannot represent general results.

(2) Although the basic ideas of Euler's method are conceptually simple and familiar, nevertheless applying the method in detail requires manipulation[3,15] of sums containing many symbols and subscripts. As a result, this method becomes cumbersome and less transparent than the elegant Lagrange technique.[18]

In response to the first criticism, we recall that modern 20th century analysis provided an exact proof of Euler’s method, among other results. In light of these advances, we now see that Euler’s imaginative use of his procedure did not fail, and his geometrical emphasis caused no loss in generality. Moreover, although Lagrange was proud of the fact that his great work *Mécanique Analytique* (1788) contained no figures or geometrical considerations, contemporary instruction in physics and modern physics itself make use of geometrical representations, for example Feynman diagrams in quantum physics and phase diagrams in chaos theory.

We have to agree with the second objection, that many subscripts and the use of sums decreases the clarity and transparency of Euler's method for teaching purposes. Happily it is possible to remove this weakness, modifying Euler’s method to make it more accessible and simple. One of our recent papers[13] describes such modification. Here we mention only the key steps which simplify and clarify the method.

(a) Euler’s diagram (Fig.1) and the geometrical method tell us that only two terms of the integral sum are affected; all the rest remain constant and need not enter the derivation. Because the definite action integral is a summation, one can conclude “if the action integral is extremal along the entire worldline, then it is also extremal for every subsection of the wordline.”[19] Therefore it is sufficient to consider only an infinitesimal subsection of any unknown worldline: three points near to one another connected by two straight-line segments (analogous to *mno* in Fig.1). Then the action contains only the two required terms and there is no need to use subscripts and formal sums.

(b) We do not need to calculate the two terms of our Riemann integral using initial points of the subintervals. Instead we take their midpoints. The midpoint approximation leads to the same symmetrical structure for both terms of the action and is quite natural from the student’s viewpoint.

Another way to increase the power and clarity of Euler’s method is to spend some time applying it intuitively and inductively before moving on to the Lagrange formalism. Advanced textbooks[18] typically apply the deductive approach: first they derive the general Lagrange equations of motion from the principle of least action and then apply the Lagrange formalism to particular problems. If instead we pause to work out examples using the Euler method, we achieve a remarkable simplification and reinforce the later introduction of Lagrange's equations.[13]

As an example of this simplification, we analyze the motion of a particle in a uniform gravitational field. Think of a particle thrown vertically upward near Earth's surface and choose arbitrarily three infinitesimally close events 1, 2, 3 on two segments *A* and *B* of the particle's worldline (see Fig. 2). Let x_1, x_2, x_3 be the spatial coordinates of these three events and Δt be the time separation between them.

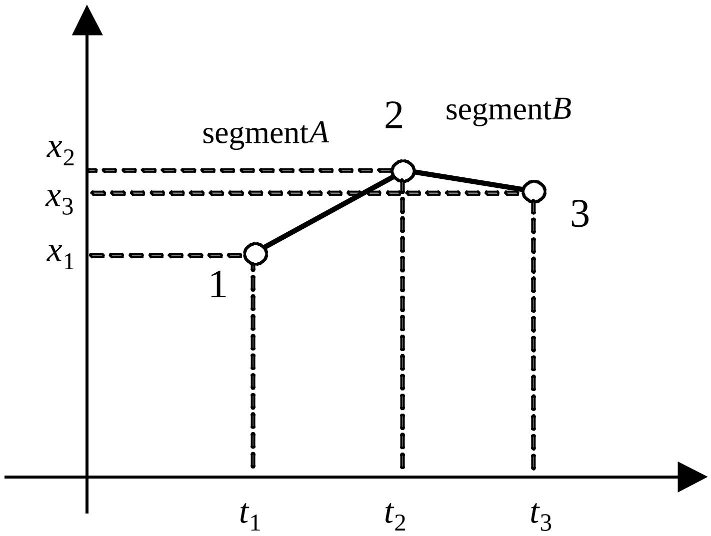

FIG. 2 Two segments of the worldline (dependence of height on time) of a particle projected vertically near Earth's surface. Points 1, 2 and 3 are three successive events on the worldline. All coordinates with the exception of x2 are fixed. We change x2 to satisfy the principle of least action.

The action *S* in Eq. (3) has two contributions:

$$S_A = \left\{ \frac{1}{2} m \left(\frac{x_2 - x_1}{\Delta t} \right)^2 - mg \frac{x_1 + x_2}{2} \right\} \Delta t \tag{5}$$

$$S_B = \left\{ \frac{1}{2} m \left(\frac{x_3 - x_2}{\Delta t} \right)^2 - mg \frac{x_2 + x_3}{2} \right\} \Delta t \tag{6}$$

where the expression in each curly bracket is the kinetic energy minus the potential energy. All space and time coordinates are fixed with the exception of the position x_2 of the middle event, which we vary to satisfy the principle of least action. Taking the ordinary derivative of the total action $S = S_A + S_B$ and setting the result equal to zero, we obtain after rearrangement:

$$-mg = m \frac{x_3 - 2x_2 + x_1}{\Delta t^2} \tag{7}$$

In the limit $\Delta t \rightarrow 0$ Eq. (7) reduces to Newton's second law of motion[12] *F=ma*.

The same result can also be obtained by direct use of Lagrange's equation, but in our experience when students start with the Lagrange equation they typically concentrate on learning the "recipe" or the "problem solving strategy" without considering the physical interpretation.

III. THE EULER METHOD AS A NUMERICAL TOOL

A. Euler's method in numerical analysis

Unlike Lagrange's purely analytic method, the Euler method provides a basis for computer modeling. Computer implementation of the method in appropriate interactive software allows students to employ basic concepts of the principle of least action and Lagrangian mechanics to visualize the variational problem, to focus on physical ideas and concepts, and to develop physical intuition. No algebraic or analytic manipulations or differential equation overlays this student activity.

We now recapitulate Euler's procedure briefly and more precisely as a numerical method.[20,21,22] To find the stationary value of a functional $S = \int_1^2 L dt$, Euler's procedure is as follows:

(1) We divide the interval into n small subintervals of equal time duration Δt, replacing the unknown curve with a piecewise-linear function. In other words, we replace the curve by a broken line of n connected segments (see Fig.3).

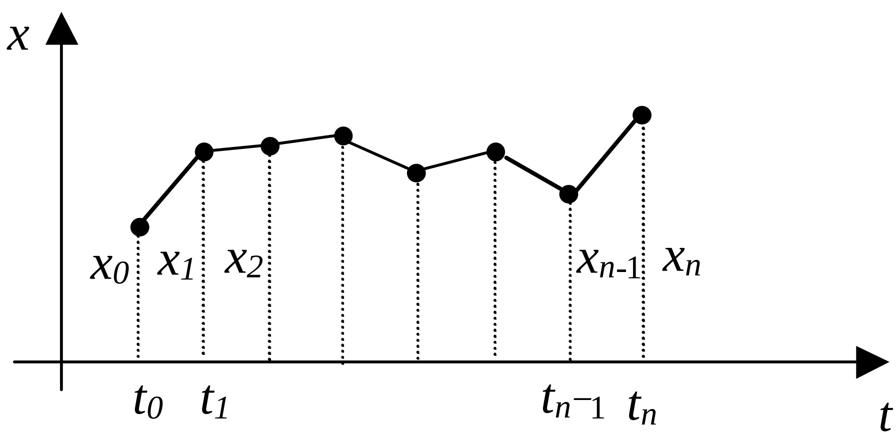

FIG. 3. Example of a piecewise-linear function approximating a smooth curve.

(2) We approximate the action integral S by the sum $S \approx \sum L\Delta t$ with terms calculated at initial points of subintervals or we can apply the more useful midpoint approximation. The derivative is given by a difference coefficient, the slope of the straight line between initial and final points of the given subinterval.

(3) Since we fix all times t_0, t_1, t_2, ..., t_{n-1}, t_n and the two endpoints x_0 and x_n, therefore S is a function $S(x_1, x_2 ..., x_{n-1})$. To find the stationary value S, we choose the values of x_1, x_2, ..., x_{n-1} to satisfy the following equations:

$$\frac{\partial S}{\partial x_1} = 0, \quad \frac{\partial S}{\partial x_2} = 0, \ \ldots, \quad \frac{\partial S}{\partial x_{n-1}} = 0 \tag{8}$$

(4) Solving the system of equations (8), with or without the help of a computer, we obtain an approximate solution of the variational problem.

Today Euler's method is one of the so-called direct variational methods for solving boundary value problems. As we will see in Sec. IV it can be regarded as a special case of the finite element method.

Technical detail: In the second step of the Euler method we could use the trapezoidal rule for the approximation of the action integral *S*, which is as efficient as the midpoint approximation.[23]

B. Visualization of Euler's method. Hunting for the least-action worldline

Our one-dimensional version of Euler's numerical method offers a great opportunity to visualize the whole process of finding the stationary curve (worldline) by minimizing the action integral. For simplicity consider a particle moving in a one dimensional potential energy $U(x)$. Then the action integral (3) has the form

$$S = \int_1^2 \left[\frac{1}{2} m v^2 - U(x) \right] dt \tag{9}$$

The computer[24] can display a trial broken worldline of the particle as shown in Fig. 3. If we calculate the action *S* with the help of the trapezoidal rule,[23] the result is:

$$S(x_1, \ldots, x_{n-1}) = \sum_{i=1}^{n} \left[\frac{1}{2} m \left(\frac{x_i - x_{i-1}}{\Delta t} \right)^2 - \frac{U(x_i) + U(x_{i-1})}{2} \right] \Delta t \tag{10}$$

The student uses the computer mouse to select an arbitrary moveable point *k* with ordinate x_k on the worldline, then drags the point up and down parallel to the *x*-axis while monitoring the displayed value of the total action in order to find the minimal (stationary) action for the time value of that point. Mathematically this procedure corresponds to finding the solution of the equation $\partial S / \partial x_k = 0$. If we take the derivative of action (10), with respect x_k, the equation is:

$$-\left.\frac{dU(x)}{dx}\right|_{x_k} = m \frac{x_{k+1} - 2x_k + x_{k-1}}{\Delta t^2} \quad . \tag{11}$$

This is the finite difference version of Newton's law of motion.

The student then drags the remaining intermediate points up and down, cycling repeatedly through all the moveable points until the time value of every point results in the least (stationary) value of the total action. This condition implies that all equations (8) or (11) are satisfied. According to the principle of least action, the resulting worldline approximates the one taken by the particle. This *method of successive displacements* or *hunting for the least-action worldline* is straightforward but tedious when many intermediate points are involved. After students have experienced this insightful but tiresome manual process, the computer can be deployed to find automatically the minimum-action location of intermediate events. This process of using the principle of least action directly to predict the motion of mechanical systems[25] encourages students to think critically and intuitively as they manipulate the mathematical machinery.

Technical details: The system of equations (8) is frequently a system of linear equations (for example for a linear potential function). In that case the method of successive displacements becomes the well-known Gauss-Seidel iterative method or Gauss-Seidel relaxation. This method is one of the basic linear iterative methods of numerical analysis and can be found in almost every introductory textbook of numerical analysis.[26,27] We do not discuss the question of convergence here, since it goes beyond the scope of the present paper.

IV. WHEN TO USE EULER'S METHOD

This section briefly describes applications of calculus of variations to many branches of physics, each with an appropriate variational principle that employs Euler's method. The section also includes references to works using Euler's method and, for comparison, references to publications that treat the same problems using Lagrange's approach.

A. Classical mechanics: Newton's laws of motion
B. Symmetries and constants of motion: Noether's theorem
C. Special Relativity: Motion and conservation laws
D. General relativity: Black holes
E. Electromagnetism: Motion of a charged particle
F. Liquids and Solids: An equilibrium state
G. Quantum mechanics: Feynman's sum over path theory
H. Ray Optics: Fermat's principle
I. Calculus of variations: Introductory elementary problems
J. Numerical mathematics: Boundary value problems & FEM

A. Classical mechanics: Newton's laws

As mentioned in Sec. II.A, Newtonian mechanics can be reformulated as a single unifying principle, the principle of least action. A recent article[28] spells out in detail the derivation of Newton's laws of motion from the principle of least action using Euler's method.

B. Symmetries and constants of the motion: Noether's theorem

The simple demonstration of the fundamental relation between symmetries of nature and conservation laws described by Noether's theorem is given in Refs. 29. The first article in that reference uses Euler's method and the principle of least action, along with elementary calculus.

C. Special Relativity: Motion and conservation laws

In special relativity the principle of least action defines action for a free particle with mass m moving from event 1 to event 2:[30]

$$S = -mc\int_1^2 ds \tag{12}$$

where ds represents the spacetime interval ("line element") between two events that are infinitesimally close on the particle's world line in flat space-time. We note that the action integral (12) does not depend on our choice of inertial reference system, because the interval ds is invariant under Lorentz transformation. Hence the principle of least action automatically satisfies the core of special relativity—the principle of relativity.

Since the interval ds also corresponds to *wristwatch* or *proper* time recorded by the clock moving with the particle ($ds = cd\tau$), the action (12) is equal to the integral

$$S = -mc^2\int_1^2 d\tau \tag{13}$$

Minimal (or stationary) action (13) along a real worldline makes the total proper time $\tau = \int_1^2 d\tau$ maximal (or stationary). Therefore, because of the minus sign in front of the integral in equation (13), the relativistic principle of least action implies the *principle of maximal proper time*, sometimes called *maximal aging*.

Simple use of the principle of least action (or maximal proper time) employing our adapted Euler method[14,31] immediately yields well-known relativistic forms of momentum and energy and verifies them to be constants of the motion.

D. General relativity: Black holes

Einstein tells us that there is no gravitational force, but only curved space-time. His general relativity theory (theory of gravitation) always allows us to create a *local* inertial frame with respect to which a particle moves along a worldline that is incrementally straight. Therefore it should not be surprising that the same[32] expressions (12) and (13) for action and the same principle of least action describes the motion of a particle in general relativity as in special relativity. In this case, however, the expression for the incremental proper time $d\tau$ of equation (13) (or ds) is provided by the metric, the solution to Einstein's field equations that describes any non-varying curved space-time such as that around spinning or nonspinning centers of gravitational attraction: planets, stars, quasars, neutron stars, and black holes. The same general[33] Euler method again allows us to describe and explore the motion of free particles, satellites, and light in the vicinity of these astronomical structures.

E. Electromagnetism: Motion of a charged particle

The Euler method and the principle of least action with the $S = \int (K - U)dt$ can also be applied to the motion of a particle with charge q in the electromagnetic field.

If we assume an electrostatic field $\boldsymbol{E}$ described by the scalar potential $\varphi(x, y, z)$, then the potential energy of the particle is $U = q\varphi$. Therefore application of the Euler method is identical with the three dimensional case in Sec. IV A and leads to the equation of motion $m\boldsymbol{a} = \boldsymbol{F}$, where $\boldsymbol{F} = -q\nabla\varphi = -q\boldsymbol{E}$.

For a charge q moving at a velocity $\boldsymbol{v}$ in a uniform magnetic field $\boldsymbol{B}$**,** which according to the classical electromagnetic theory is derivable from a vector potential $\boldsymbol{A}(x, y, z)$ by $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, we have $U = -q\boldsymbol{v} \cdot \boldsymbol{A}$. We interpret this quantity as the "interaction potential energy." Direct application of Euler's method[34] again gives $m\boldsymbol{a} = \boldsymbol{F}$ but this time with $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$.

Because action is additive, particle motion in a variable electromagnetic field leads to the potential $U = q\varphi - q\boldsymbol{v} \cdot \boldsymbol{A}$. Combining this with previous results leads to a single Lorentz force equation of motion $m\boldsymbol{a} = \boldsymbol{F}$, where $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$.

These considerations can also be extended to the relativistic motion of a charged particle merely by formally replacing the Newtonian expression for kinetic energy $(1/2)mv^2$ in the action with the relativistic expression: $-mc^2\sqrt{1 - v^2/c^2}$. But the term $-mc^2\sqrt{1 - v^2/c^2}$ in the relativistic formula for the action S is not what we have called the kinetic energy (see Feynman Ref. 19).

F. Liquids and Solids: An Equilibrium state

If we apply the principle of least action to a conservative mechanical system described by a potential $U(x)$ in an equilibrium state at rest, then the Euler method immediately provides the principle of least (stationary) potential energy. Here is a simple proof.[35] The state of equilibrium is described by constant coordinates. In our case of a three-point worldline this means $x_1 = x_2 = x_3$. Application of the Euler method to the principle of least action gives the same condition as (11) at point 2:

$$-\left.\frac{dU(x)}{dx}\right|_{x_2} = m\frac{x_3 - 2x_2 + x_1}{\Delta t^2} \tag{14}$$

Since $x_1 = x_2 = x_3$, we have:

$$\left.\frac{dU(x)}{dx}\right|_{x_2} = 0 \tag{15}$$

Equation (15) means that U is an extremum, or rather has a stationary value, at equilibrium.

The principle of least potential energy has many applications[36,37,38] such as the statics of engineering structures (theory of elasticity) and surface phenomena (liquids).

G. Quantum mechanics: Feynman's sum over paths theory

Classical action plays a fundamental role in Feynman's quantum mechanics,[39] a third formulation of the subject in addition those of Schrödinger and Heisenberg. As we can see in Ref. 24, some of the ideas of Euler's computer method can be used in student exploration of the microworld, leading to an understanding of the basic concepts of quantum physics (such as the wave function, quantum interference, evolution of quantum states, boundary states) without using complex functions or partial differential equations.

H. Ray Optics: Fermat's principle

Euler's method, particularly the interactive computer application (Sec. III B), permits a highly visual demonstration of Fermat's principle,[40] the earliest extremal principle that still survives in modern physics. Since Fermat's principle accounts for every feature of classical ray optics, hunting for the minimum-time path allows exploration and understanding of reflection, refraction (at a plane surface, in layers of glass, or in the atmosphere), mirages, and ray tracing in optical systems.

I. Calculus of variations: Introductory elementary problems

Euler's method is used in some mathematics texts to introduce the calculus of variations.[7,38] Lagrange's variational techniques can be applied to problems in parallel with interactive computer exploration of the same problems using Euler's method, leading to a deeper understanding of both and of the physical systems that they describe.

J. Numerical mathematics: Boundary value problems and Finite Element Method

Euler's method can be used to introduce the theory and basic concepts of the Finite Element Method (FEM).[41] As usual, the first step consists in dividing the interval over which the solution is defined into a number of small subintervals, called *elements* in the FEM. The second step is to replace the unknown curve with a piecewise-linear function (Fig.3) labeled φ. Starting with the trial broken worldline φ given by x_0, x_1, ..., x_n , it is not difficult to show that φ can be expressed as a linear combination of the "hat functions" φ_0, φ_1, ..., φ_n shown in Fig. 4: $\varphi = x_0\varphi_0 + x_1\varphi_1 + ... + x_n\varphi_n$.

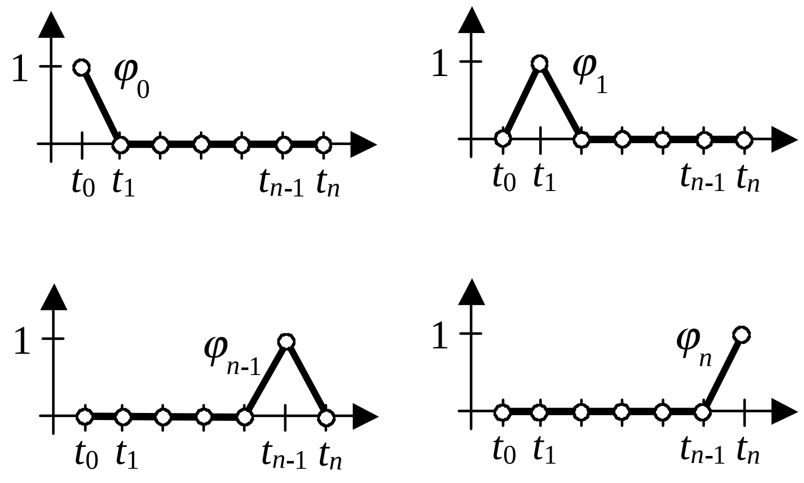

FIG.4 “Hat functions,” an example of basis functions used in the FEM.

Such a set of so-called basis functions φ_0, φ_1, ... , φ_n represents the simplest type used for one-dimensional applications of the FEM. Each function is zero except on a very small number of segments, a general feature of basis functions.

The next step of the FEM is finding positions x_1, ... , $x_{n\text{-}1}$ for the given boundary conditions $x_0 = a$, $x_n = b$ that minimize the action integral S. The integral is calculated approximately with the help of methods of numerical integration (such as the trapezoidal rule). Since the basis functions φ_0, φ_1, ... , φ_n, and endpoints x_0 and x_n are fixed, the action S due to varying φ is a function of only the coordinates x_1, ..., $x_{n\text{-}1}$. Finding the minimum value of the function S corresponds to solving a system of equations (8) with respect to unknowns x_1, ..., $x_{n\text{-}1}$. The solution of the system (8) is regarded as an approximate solution to the variational problem obtained by FEM.

The last two procedures in the FEM, approximating a function with a linear combination of basis functions and finding coefficients of a linear combination of these functions which minimizes the action integral, are known as the Rayleigh-Ritz method.[42]

The accuracy of the FEM can be increased by (1) increasing the number of elements, (2) changing the set of basis functions or (3) changing the numerical integration formula. Such improvements often employ quadratic basis functions and the Gauss-Legendre numerical integration. Such refinements permit us to solve more general problems than those in mechanics, for example heat diffusion or quantum systems described by Schrödinger’s wave equation.[43]

Euler’s method is the simplest and most fundamental example of finite element analysis . Its importance equals that of generally known Euler's one-step method for solving initial value problems in the field of ordinary differential equations. Like Euler’s one-step method, the Euler variational method can be implemented simply and transparently, and every individual step is clearly visible and easy to understand. It illustrates all essential features of finite element analysis and numerical solutions of variational problems.

V. CONCLUSIONS

The Euler variational method provides a conceptually and mathematically simple tool to introduce the principle of least action and Lagrangian mechanics. Even introductory examples offer dramatic simplification compared with the traditional Lagrange approach. Euler's method, based on elementary calculus, provides important geometric-visual insights, and its computer adaptation gives students a powerful tool to explore least-action mechanics without manipulation of equations. For beginners, Euler's method promises to be the key that unlocks the gate to the central variational treatments of physics.

The overwhelming majority of calculus-of-variations applications using Euler's method is connected to the principle of least action and demonstrates the generality of this principle and its power to illuminate almost all of undergraduate physics. Simultaneously the method allows us to present the principle of least action as a bridge between classical and contemporary physics as soon as possible without specific mathematical tools.

Applying the computer adaptation of the method permits a natural introduction to numerical analysis of boundary value problems and the powerful finite element method.

Although Euler's legacy is more than 250 years old, it remains centrally important for mathematicians and physicists and richly deserves the accolade of Pierre-Simon Laplace: "Lisez Euler, Lisez Euler, c'est notre maître à tous." ("Read Euler, read Euler, he is our master in everything.")[44]

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[1]L. Euler, *Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes sive Solutio Problematis Isoperimetrici Latissimo Sensu Accepti* (*The Method of Finding Plane Curves that Show Some Property of Maximum or Minimum...*), Laussanne and Geneva, 1744, also in L. Euler, *Opera Omnia* I, Vol. XXIV, C. Carathéodory, ed. Bern, 1952

[2]C. Lanczos, *The Variational Principles of Mechanics* (Dover, New York, 1986), Introduction

[3]H. H. Goldstine, *A History of the Calculus of Variations from the 17th Through the 19th Century* (Springer-Verlag, New York, 1980), Chap. 2.

[4]H. Erlichson, "Johann Bernoulli's brachistochrone solution using Fermat's principle of least time", Eur. J. Phys. 20 (1999), 299-304

[5]Goldstine Ref.3, Chap.3. According to this reference (p.110) in the summary of o his paper using Lagrange's variations, Euler says "*even though the author of this [Euler] had meditated a long time and revealed to friends his desire yet the glory of first discovery was reserved to the very penetrating geometer of Turin LA GRANGE, who having used analysis alone, has clearly attained the very same solution which author had deduced by geometrical considerations"* .

[6]L.E. Elsgolc, *Calculus of variations*, (Pergamon Press, London, 1961), Sec. 5.1

[7]M.A. Lavrentjev, and L.A. Ljusternik, *The Course of Variational Calculus* (in Russian, Moscow, 1959), Sec. 2.6

[8]R.Courant, D. Hilbert, *Methods of Mathematical physics*, (Interscience Publishers, New York, 1953), Vol. I, Sec. 4.2; Vol. II (original German edition, Springer, Berlin, 1937), Chap. VII.

[9]E. Giusti, *Direct methods in the calculus of variations*, (World Scientific Publishing, New York, 2003)

[10]Finite element analysis was first developed in R.Courant, "Variational method for the solution of problems equilibrium and vibrations", Bulletin of the AMS, 49,1-23,1943; M.J.

Turner et al, "Stiffness and deflection analysis of complex structures", Journal of the Aeronautical Sciences 23, 805-824 (1956), the publication widely regarded as the beginning of FEM.

[11]Variational treatments successfully describe many further cases: the elastic field, electromagnetic field, field properties of elementary particles, special and general relativity, the deep relation between symmetries and conservation laws (Noether's theorem). Almost every fundamental law can be expressed in the form of a variational principle. Numerical variational methods (e.g. FEM) represent powerful tools for understanding and simulating complex phenomena.

[12]J. Hanc, S.Tuleja, and M. Hancova, "Simple derivation of Newtonian mechanics from the principle of least action", Am. J. Phys. 71(4), 386-391 (2003)

[13]J. Hanc, E. F. Taylor, and S. Tuleja, "Deriving Lagrange's Equations Using Elementary Calculus," accepted for publication in *Am. J. Phys.* The application of the Euler method in the article provides also the constants of the motion $p = \partial L / \partial x$ and $h = \dot{x} \cdot \partial L / \partial \dot{x} - L$.

[14]J. Hanc, S. Tuleja, M. Hancova, "Symmetries and Conservations Laws: Consequences of Noether's Theorem," accepted for publication in *Am. J. Phys*.;

[15]Lanczos Ref.2, Sec.2.7; Elsgolc Ref. 6, Sec 5.2; Lavrentjev and Ljusternik Ref. 7, Sec. 1.2, Courant and Hilbert, Ref. 8, Sec. 4.2

[16]We employ a slightly modified notation, because Euler uses dashes instead of subscripts, which could lead to confusion with the standard notation of derivative. However, the logical development is not changed by this notation.

[17]In Euler's time the rigorous proof of passage to the limit $\Delta t \rightarrow 0$ was not available. The limit concepts were tinged with intuitive geometry concepts and connotations of continuous motion. (C.H. Edwards, The Historical Development of the Calculus, Springer Verlag, Berlin, 1979 p. 329) The modern theory that validated Euler's technique uses more exact, subtle and careful considerations (see e.g. Ref. 7, Chap. 2 or Ref. 8, Vol. II, Chap. VII). As an example the exact definition of the limit concept employing the so-called ε, δ technique was devised by Karl Weierstrass more than 100 years after Euler's discovery. See C.H. Edwards, Jr., *The Historical Development of the Calculus*, (Springer-Verlag, Berlin 1979), Chap. 11. The general theoretical approach to the convergence of direct methods such as that of Euler is based on works on functional analysis of the Russian mathematician S.L. Sobolev (see e.g. Ref. 9)

[18]See L.D. Landau and E. M. Lifshitz, *Mechanics* (Butterworth-Heinemann, London, 1976), 3rd ed., Chap. 1; H. Goldstein, Ch. Poole, and J. Safko, *Classical Mechanics*, 3rd ed.(Addison-Wesley, Reading, MA 2002), Chap. 2; Lanczos Ref. 2, Chap. 2

[19]The first formulation of this statement comes from Johann Bernoulli in his brachistochrone problem, see Erlichson Ref. 4. It is also mentioned in R.P.Feynman, R.B. Leighton, and M.Sands, *The Feynman Lectures on Physics* (Addison Wesley, Reading, MA, 1964), Vol. 2, Chap. 19 and Landau Ref.18

[20]See Elsgolc Ref. 6, Sec 5.2; Lavrentjev and Ljusternik Ref. 7, Chap. 1

[21]M. J. Forray, *Variational Calculus in Science and Engineering*, (McGraw-Hill Book Company, NJ, 1968).

[22]I. M. Gelfand and S. V. Fomin, *Calculus of variations* (Prentice–Hall, Englewood Cliffs, NJ, 1963), Sec. 1.6 and Sec. 8.36

[23]The midpoint approximation approximates the definite integral $\int_{u_0}^{u_1} f(u)\mathrm{d}u$ over the small interval [u_0, u_1] by $f\left(\frac{u_0+u_1}{2}\right)\cdot\Delta u$ and the trapezoidal rule uses $\int_{u_0}^{u_1} f(u)\mathrm{d}u \approx \frac{f(u_0)+f(u_1)}{2}\cdot\Delta u$.

[24]See E. F. Taylor, S. Vokos, J. M. O´Meara, and N. S. Thornber, "Teaching Feynman's sum over paths quantum theory," Comp. Phys. 12 (2), 190-199 (1998), available at <http://www.eftaylor.com>.

[25]S. Tuleja, E. F. Taylor, Software: *Principle of Least Action Interactive*, 2003, available at fo free download at the website <http://www.eftaylor.com/leastaction.html>.

[26]See J. Tobochnik, H. Gould, *An Introduction to Computer Simulation Methods: Applications to Physical Systems*, 2[d] edition, Addison-Wesley (1996), Drafts of the third edition are available at <http://physics.clarku.edu/courses/125/gtcdraft>, Chap. 10.

[27]The mathematical approach to the Gauss-Seidel method is in C.D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, 2000; available for free downloading at <http://matrixanalysis.com/>

[28]See Hanc Ref.12; the same problem is analyzed in terms of Lagrange's technique of variations in Feynman Ref. 19

[29]Hanc Ref. 14; Lagrange's approach to this subject can be found in N. C. Bobillo-Ares, "Noether's theorem in discrete classical mechanics," Am. J. Phys. 56 (2), 174-177 (1988).

[30]L. D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, (Pergamon Press, London, 1975), Sec.2.8

[31]E. F. Taylor and J. A. Wheeler, *Exploring Black Holes: Introduction to General Relativity*, Addison-Wesley Longman, New York, 2000, Sec. 1.5,1.6; Hanc et al Ref. 14 on the use of the Euler method start from the principle of least action, whereas Taylor and Wheeler begin with the corresponding principle of maximal aging.

[32]Landau Ref.30, Sec.10.87

[33]Taylor Ref. 31, Sec. 3.3 and Sec. 4.2; The Lagrangian approach to the same problem is demonstrated in M. Berry, *Principles of Cosmology and Gravitation* (Institute of Physics Publishing, Bristol, 1989), Sec 5.3

[34]In that case the difficulty of Euler's method is comparable to that of the Goldstein derivation based on Lagrange's equations. See Ref. 18, Sec. 1.5

[35]D. ter Haar, *Elements of Hamiltonian Mechanics* (Pergamon Press, Oxford, 1971), Sec. 3.1. His section contains a similar proof based on the Lagrange approach.

[36]D. S. Lemons, *Perfect Form* (Princeton University Press, Princeton, 1997), Chap. 4.

[37]L.D. Landau and E.M. Lifshitz, *The Fluid Mechanics*, (Pergamon Press, London, 1979), Sec. 12.61

[38]Ya. B. Zeldovich and A. D. Myskis, *Elements of Applied Mathematics*, tr. George Yankovsky (MIR Publishers, Moscow, 1976). Sec. 12.1. These authors use the Euler method and the principle of least potential energy in the introductory variational problem of particles (e.g. beads) on a string.

[39]R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965), Chap. 2

[40]R.P.Feynman, R.B. Leighton, and M.Sands, *The Feynman Lectures on Physics* (Addison Wesley, Reading, MA, 1964), Vol. 1, Chaps. 26, 27

[41]O.C.Zienkiewicz and K. Morgan, *Finite Elements and Approximation*, John Wiley and Sons, New York, 1983 or A. Stahel, *Calculus of variations and Finite elements*, Chap. 4, 2002, <http://www.hta-bi.bfh.ch/~sha/fem/VarFEM/VarFEM.pdf>.

[42]A simple example of the Ritz method, the calculation of the capacity of two conductors in the form of a cylinder condenser, is in Feynman Ref. 19

[43]D. J. Searles and E.I. von Nagy-Felsobuki: “Numerical experiments in quantum physics: Finite-element method“, Am. J. Phys. 56(5), 444-448 (1988). This article establishes the fact that the finite-element method can achieve the same order of accuracy more easily than the better-known Numerov–Cooley method.

[44]P. Beckmann, *A History of Pi* (Dorset Press, New York, 1989), Chap. 14.