Symmetries and Conservation Laws: Consequences of Noether's Theorem

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We derive conservation laws from symmetry operations using the principle of least action. These derivations, which are examples of Noether's theorem, require only elementary calculus. They help provide physical understanding of the relation between symmetry and fundamental mechanics for introductory university physics students. We extend these arguments to the transformation of coordinates due to uniform motion to show that a symmetry argument applies more elegantly to the Lorentz transformation than to the Galilean transformation.

I. INTRODUCTION

It is increasingly clear that the symmetry group of nature is the deepest thing that we understand about nature today.

Steven Weinberg[1]

Many of us have heard statements like: *For each symmetry operation there is a corresponding conservation law*. *The conservation of momentum is related to the homogeneity of space*. *Invariance under translation in time means that law of conservation of energy is valid*. Such statements come from one of the most amazing and useful theorems in physics, called **Noether's theorem**.

When the German mathematician Emmy Noether proved her theorem in 1918[2,3], she uncovered the most fundamental justification for conservation laws. Her theorem tells us that conservation laws follow from the symmetry properties of nature. Symmetries (called *principles of simplicity* by Weinberg[1]) can be regarded as a way of stating the deepest properties of nature. Modern physics uses symmetries to limit the possible forms of new physical laws. However, that deep connection between symmetry and conservation laws requires the existence of a minimum principle in nature - the principle of least action. In classical theoretical mechanics, symmetry arguments are developed using high-level mathematics. On the other hand, corresponding physical ideas are often much easier than mathematical ones. In this paper we give an elementary introduction to and explanation of the relation between symmetry arguments and conservation laws, as mediated by the principle of least action. We shall use only elementary calculus, so that considerations described in the paper can be used in introductory university physics classes.

What does symmetry mean? Feynman offers a simple description of that concept in his lectures on physics: [4]

> Professor Hermann Weyl has given this definition of symmetry: a thing is symmetrical if one can subject it to a certain operation and it appears exactly the same after the operation. For instance, if we look at a vase that is left-and-right symmetrical, then turn it 180° around the vertical axis, it looks the same.

Like Feynman, we will concentrate on symmetry in physical laws. The question is: *What can be done to a physical law so that this law remains the same?* Noether's theorem derives conservation laws from symmetries under the assumption that the principle of least action is the basic law that governs the motion of a particle in classical mechanics. This principle can be phrased as follows: "*The action is a minimum for the path (worldline) taken by the particle*."[5] This leads to the reformulation of our basic question about symmetry: *What changes can we make in the worldline that do not lead to changes in either the magnitude or the form of the action?*

In this paper we explore and apply symmetry operations to the action along an infinitesimally small path segment. Since action is additive, conclusions reached about a path segment apply to the entire path. The simplest examples of symmetry show the *independence* of the action on the *difference* in some quantity such as position, time, or angle[6]. When such a symmetry exists, then Noether's theorem tells us that a physical quantity corresponding to this symmetry is a *constant of the motion*[7] which does not change along the entire path of the particle. This implies a conservation law, which we then need to identify.

The following Section II briefly describes computer software that helps students study action and its connection to conservation laws. Section III analyzes four cases of the symmetry operations: translation in space and time, rotation through a fixed angle and symmetry under uniform linear motion, namely the Galilean transformation. The first three symmetries lead to three central conservation laws – momentum, energy, and angular momentum. The final Section IV extends the analysis to symmetry in relativity, showing that these conservation laws exist in that realm. Moreover, in the case of uniform linear motion the symmetry argument applies more elegantly to the Lorentz transformation than to the Galilean transformation.

In the following we often talk about variations of the action. Consistent with standard practice, these variations are taken to first order only. To keep the arguments simple we also assume that the particle's invariant mass m ("rest mass") does not change during the motion to be studied.

II. COMPUTER SOFTWARE

Start with a well-known definition of action for a particle of mass m that moves from some initial position at time t_1 to some final position at time t_2:

$$S = \int_{t_1}^{t_2} (\mathrm{KE} - \mathrm{PE})\,\mathrm{d}t \qquad (1a)$$

or equivalently

$$S = (\mathrm{KE}_{\mathrm{av}} - \mathrm{PE}_{\mathrm{av}})\,(t_2 - t_1) \qquad (1b)$$

Here $\mathrm{KE}_{\mathrm{av}}$ denotes the time average kinetic energy, $\mathrm{PE}_{\mathrm{av}}$ the time average potential energy between t_1 and t_2. We use double letters KE and PE as symbols for kinetic and potential energies respectively, since they are more mnemonic than the traditional symbols T and V.

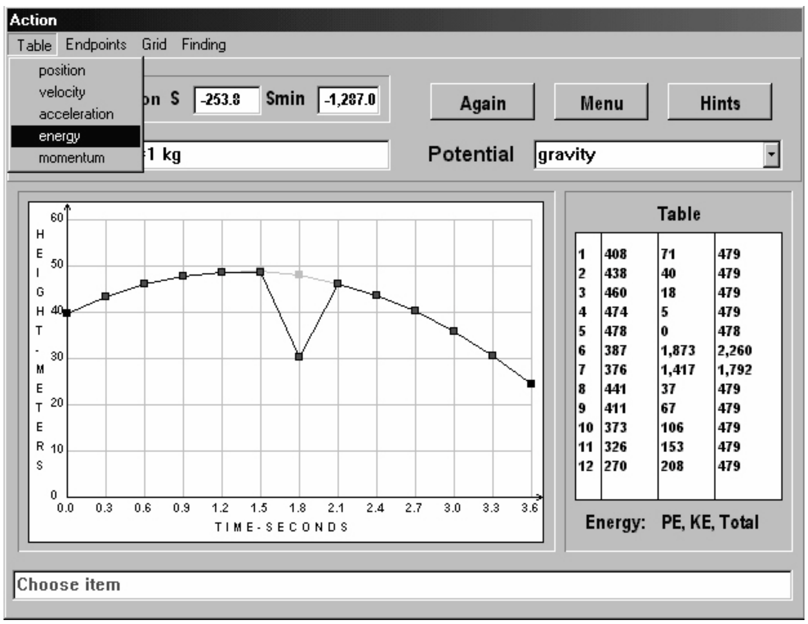

Figure 1. Software helps students to study the action along a worldline for a particle moving vertically in a gravitational field (as shown) or in other conservative potentials. The user clicks on events to create a worldline and then drags the event-dots to minimize the action S, which the computer continuously calculates and displays. The computer also displays a table of energy, momentum or other quantities that demonstrate conservation of these quantities. Students discover that for the worldline of minimum action, momentum is conserved for the motion of a free particle and that in a gravitational field total energy is conserved.

Action is not a familiar quantity[8] for many students, so we employ an interactive computer program[9] to help them develop an intuition about action and the principle of least action. Using an interactive computer display, the student can not only explore the operation of the principle of least action works but also demonstrate in specific cases the relation between this principle and conservation laws (Figure 1). In carrying out this manipulation, the student naturally and immediately works with the central concepts *worldline* (a graph of time dependence of particle's position) and *event* (a point on a worldline). Unlike the path (trajectory in space), the worldline specifies completely the motion of a particle. As background in the symmetry properties of nature, students read a selection from Richard Feynman's *The Character of Physical Law*.[10]

III. SYMMETRY AND CONSERVATION LAWS IN NEWTONIAN MECHANICS

A. Translation in space

First examine symmetry related to translation in space. We carry out an experiment at some location and then repeat the same experiment with identical equipment at another location. We expect the results of the two experiments to be the same (assuming we are not making astronomical observations). So the physical laws should be symmetrical with respect to space translation.

As a simple example, consider the action of a free particle (in zero potential or uniform potential) moving along x-axis between two events 1 [t_1, x_1] and 2 [t_2, x_2] infinitesimally close to one another along its worldline. Since the worldline section is considered as straight, the particle moves at constant velocity $v = (x_2 - x_1)/(t_2 - t_1)$ and therefore with a constant kinetic energy $(1/2)mv^2$. According to (1b) the expression for action along this straight segment in zero potential is:

$$S_{\text{for segment}} = \frac{1}{2} m \frac{(x_2 - x_1)^2}{(t_2 - t_1)} \qquad (2)$$

If we change the positions of both observed events through a fixed displacement a, the action remains unchanged (invariant), because the value of the action depends only on the *difference* between positions: $x_2 + a - (x_1 + a) = x_2 - x_1$. The principle of least action, the law governing the motion of this particle, is symmetrical with respect to a fixed displacement of position. Noether's theorem implies that this symmetry is connected with some conservation law. In the following subsections we demonstrate that the conservation law related to symmetry under space translation is the conservation of momentum.

1. PRINCIPLE OF LEAST ACTION AND MOMENTUM

Think of the motion of a free particle along x-axis. To explore the connection between the principle of least action and the conservation of momentum, we take advantage of the additive property of action to require that the action along an arbitrary infinitesimal section of the true worldline have a minimal value.[11] Thus we consider three successive infinitesimally close events, 1, 2, and 3 on the particle's worldline and approximate a real worldline by two connected straight segments, *A* and *B* (Fig. 2).

Because we are considering translation in space, we fix the first and last events 1 and 3 and change the space coordinate x_2 of the middle event 2 so as to minimize the value of the total action S. This minimum condition corresponds to a zero value of the derivative of S with respect to x_2:

$$\frac{\mathrm{d}S}{\mathrm{d}x_2} = 0 \qquad (3)$$

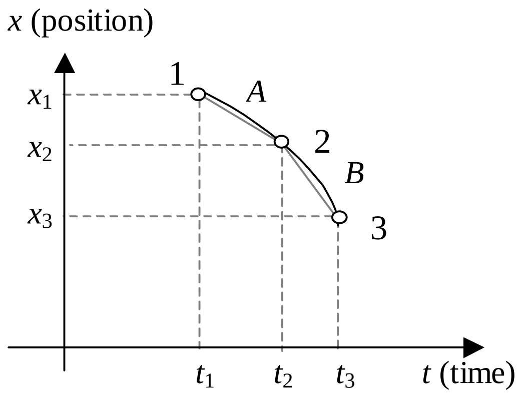

Figure 2. Segment of the worldline of a particle that passes through three infinitesimally close events, for which every smooth curve can be approximated by two connected straight segments.

Since action is an additive quantity, we can write that total action equals sum of actions for segment *A* and *B*, so $S = S(A) + S(B)$. Using (2), we write for the total action *S*:

$$S = \frac{1}{2} m \frac{(x_2 - x_1)^2}{(t_2 - t_1)} + \frac{1}{2} m \frac{(x_3 - x_2)^2}{(t_3 - t_2)} \qquad (4)$$

Carrying out on (4) the derivative indicated in (3), we derive the condition:

$$m \frac{(x_2 - x_1)}{(t_2 - t_1)} = m \frac{(x_3 - x_2)}{(t_3 - t_2)} \qquad (5)$$

The expression on the left side of Eq. (5) is the momentum p_A for segment *A* while the expression on the right side is the momentum p_B for segment *B*, so $p_A = p_B$. Now we could continue and add other segments *C*, *D*, *E*.... to cover the whole worldline that describes particle motion. For all these segments the momentum will have the same value, which yields the conservation law of momentum. The action for this free particle depends only on the *change* of the coordinate *x* and the result of this dependence is the conservation of the particle's momentum.

However, the derivation of this section uses only the displacement of one event on the worldline. Therefore we have not yet demonstrated the relation between the conservation of momentum and the symmetry of translation in space in which all three events are displaced.

2. SYMMETRY AND THE CONSERVATION OF MOMENTUM

Now we show the straightforward relationship between the symmetry of translation in space and conservation of momentum, one consequence of Noether's theorem mediated by the principle of least action.

Again consider three infinitesimally close events on the actual worldline $x(t)$ of a free particle shown in Figure 3. (Extension to the entire worldline will be discussed later.)

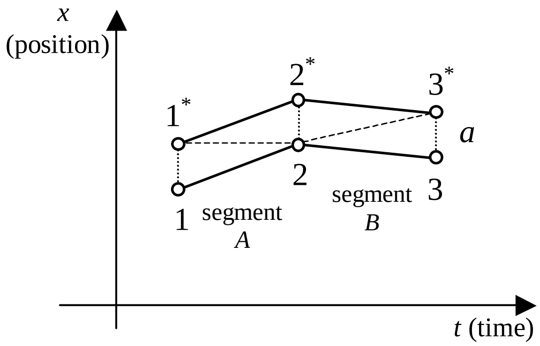

Figure 3. Three infinitesimally close events 1, 2, 3 on the actual worldline. We shift this worldline through a fixed infinitesimal displacement *a*. An arbitrary displacement can be composed from a sequence of such infinitesimal displacements.

Under translation in space we shift the worldline $x(t)$ so that every event changes position by a fixed infinitesimal x-displacement a. These new events create a shifted worldline which we indicate with an asterisk: $x^*(t)$. As pointed out earlier, the form of the action for $x^*(t)$ remains unchanged and does not depend on the parameter a. Thus the change in action with respect to displacement a is always zero:

$$\Delta_a S \equiv S(1^*2^*3^*) - S(123) \equiv 0 \qquad (6)$$

Notice that the worldline $x^*(t)$ is just as valid as the original one. Therefore the worldline $x^*(t)$ also obeys the principle of least action. In translating from $x(t)$ to $x^*(t)$ we do not need to shift all events simultaneously. The same effect is obtained if we first change the position of event 1 (see Fig. 3; only coordinate x_1 changes, which creates the worldline 1*23), then event 3 (only x_3 changes, which creates 1*23*) and finally event 2 (only x_2 changes, which creates 1*2*3*). The total change in action for displacement a can be written as:

$$\Delta_a S = \Delta S_{1\to 1^*} + \Delta S_{3\to 3^*} + \Delta S_{2\to 2^*} \qquad (7)$$

where $\Delta S_{1\to 1^*}, \Delta S_{3\to 3^*}, \Delta S_{2\to 2^*}$ denote the given changes in action after shifts in the corresponding events.

But equation (6) tells us that $\Delta_a S$ is always zero. The final change $\Delta S_{2\to 2^*}$ must also be zero, from the principle of least action applied to the new worldline. Hence Eqs. (6) and (7) give:

$$-\Delta S_{1\to 1^*} = \Delta S_{3\to 3^*} \qquad (8)$$

If we now calculate the changes in action in (8), we obtain the conservation law of momentum. Because the displacement a is infinitesimal, one can write:

$$\Delta S_{1\to 1^*} \equiv S(1^*23) - S(123) = \frac{\mathrm{d}S}{\mathrm{d}x_1}a \qquad \text{(9a)}$$

$$\Delta S_{3\to 3^*} \equiv S(1^*23^*) - S(1^*23) = \frac{\mathrm{d}S}{\mathrm{d}x_3}a \qquad \text{(9b)}$$

After substituting (9a), (9b) to (8) and using the fact that the fixed infinitesimal displacement a is arbitrary, we have[12]:

$$-\frac{\mathrm{d}S}{\mathrm{d}x_1} = \frac{\mathrm{d}S}{\mathrm{d}x_3} \qquad \text{(10)}$$

Applying the derivatives in (10) to the expression for action in equation (4) yields the identical result for a free particle as (5), but this time as a result of space translation of the entire incremental worldline segment. Then the left side of (10) can also be interpreted as the momentum at event 1 and the right side as momentum at event 3.

The preceding consideration can be applied to the entire worldline $x(t)$. We did not specify the location of the segments A and B. Therefore an arbitrary number of additional segments can be added between them. Then we shift segments as before (Figure 4). By the same analysis we conclude that the momentum for segment A (effectively the momentum at event 1) has the same value as for segment B (effectively at event 3). This means that the value of the momentum remains constant at every event on the worldline. Result: In classical mechanics, the symmetry of translation in space means that for a free particle *momentum is conserved.*

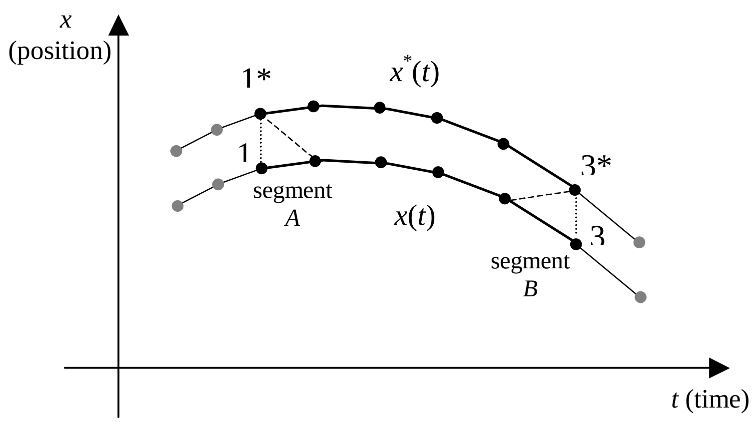

Figure 4. Following the same analysis as before, we conclude that the momentum at event 1 is the same as at event 3. The events 1 and 3 can be chosen arbitrarily. This implies the same value of momentum along the whole worldline of the moving object.

Summary:
The invariance of the action with respect to translation in space is called the *homogeneity of space*, which means that all points in space are equivalent as the origin of our reference frame. In other words, it does not matter where an experiment is carried out.
The momentum conservation law results from space homogeneity.

B. Translation in time

It is easy to envision the symmetry related to translation in time. We believe that exactly repeating an experiment on identical initial systems yields the same result when the two experiments are separated by a lapse of time. Our conclusion is that physical laws should not change with translation in time.

Again we show mathematically the relation of translation-in-time symmetry to a relevant conservation law, our second consequence of Noether's theorem. Start with an expression for the action of a particle moving in *x*-direction along an infinitesimally small worldline segment in the conservative potential field described by PE(x). Similarly as above (Sec. IIIA) the action for this segment can be written (according to Eq.(1b)) as

$$S_{\text{for segment}} = \frac{1}{2} m \frac{(x_2 - x_1)^2}{(t_2 - t_1)} - \mathrm{PE}\left(\frac{x_1 + x_2}{2}\right)(t_2 - t_1) \tag{11}$$

where the potential energy is evaluated at the average position along the segment. Now translate time t through a value τ. It is easy to see that the action will not change, since only the *difference* of time appears in the equation for action: $t_2 + \tau - (t_1 + \tau) = t_2 - t_1$. So the action of the particle is symmetrical with respect to a fixed displacement of time t. What conservation law is related to this time symmetry? We will show that it is the conservation of energy.

Follow the same line of reasoning as in the case of translation in space, but in this case fix all position and time coordinates with the exception of t_2. Think of a particle that moves along x-axis in a potential field with potential energy PE(x). To simplify the algebra, denote space and time differences by:

$$\begin{gathered} \Delta x_A = x_2 - x_1 \;\; ; \;\; \Delta x_B = x_3 - x_2 \\ \Delta t_A = t_2 - t_1 \;\; ; \;\; \Delta t_B = t_3 - t_2 \end{gathered} \tag{12}$$

According to (11) and (12), the values of actions $S(A)$ and $S(B)$ for segments A and B are equal

$$S(A) = \frac{1}{2} m \frac{{\Delta x_A}^2}{\Delta t_A} - \mathrm{PE}\left(\frac{x_2 + x_1}{2}\right)\Delta t_A \tag{13a}$$

$$S(B) = \frac{1}{2} m \frac{{\Delta x_B}^2}{\Delta t_B} - \mathrm{PE}\left(\frac{x_3 + x_2}{2}\right)\Delta t_B \tag{13b}$$

The principle of least action leads to the following condition for total action S:

$$\frac{\mathrm{d}S}{\mathrm{d}t_2} = \frac{\mathrm{d}\left[S(A) + S(B)\right]}{\mathrm{d}t_2} = 0 \tag{14}$$

After substituting Eqs. (13) to Eq. (14), differentiating, and rearranging, we obtain:

$$\frac{1}{2}m\frac{\Delta x_A{}^2}{\Delta t_A{}^2}+\mathrm{PE}\left(\frac{x_2+x_1}{2}\right)=\frac{1}{2}m\frac{\Delta x_B{}^2}{\Delta t_B{}^2}+\mathrm{PE}\left(\frac{x_3+x_2}{2}\right) \qquad (15)$$

The expressions on both sides of (15) are sums of average kinetic and potential energies. In the case of infinitesimally close events, this equation (15) gives equality for instantaneous values $(1/2)mv_A^2+\mathrm{PE}_A=(1/2)mv_B^2+\mathrm{PE}_B$. Equation (15) expresses the conservation of mechanical energy.

Next we carry out an argument that translates all three times t_1, t_2, and t_3 by the same increment τ, similar to the way we translated positions for the momentum case. Equations (7) and (8) apply to the present case as well, and also equations (9) when the derivatives are taken with respect to time rather than position. This leads to an equation similar to (10):

$$-\frac{\mathrm{d}S}{\mathrm{d}t_1}=\frac{\mathrm{d}S}{\mathrm{d}t_3} \qquad (16)$$

which yields to the equation (15) multiplied by (–1). We again obtained the conservation of energy, but this time as a result of the symmetry under time translation. For infinitesimally close events, the left side of (16) can also be interpreted as minus the total energy at event 1 and the right side as minus the energy at event 3. Energy is a constant of the motion for the entire worldline $x(t)$. The result: the symmetry of translation in time means *conservation of energy*.

Summary:
The invariance of action with displacement in time is called the *homogeneity of time*, which means that all times can equally be taken as origins for measuring time. Experimentally it does not matter when an experiment was carried out. Similar to the case of space, we say that the *conservation law of energy of a system results from the homogeneity of time.*

C. Rotation through a fixed angle

We now trace the consequences of another symmetry, symmetry under rotation in space. If we rotate an experimental setup through a fixed angle, the experiment will yield the same result. If this symmetry were not true, a laboratory in New York would not be able to verify what is measured in another laboratory in Los Angeles. Indeed, repeating the experiment in New York must lead to the same results as the earth rotates (again assuming that one is not making astronomical observations). So physical laws should remain invariant with respect to rotation.

Deploy polar coordinates to determine what conservation law corresponds to this symmetry and consider planar motion of a particle in a spherically symmetric potential field of energy PE(r). As before, take the expression for the action along the infinitesimal segment.

Definition (1b) shows that the action is equal to:

$$S_{\text{for segment}} = \frac{1}{2} m \frac{\Delta s^2}{\Delta t} - \text{PE}(r_{\text{av}})\Delta t \qquad (17)$$

The increment Δs is the length of a path segment traveled by the particle during the time interval Δt and r_{av} is an average position of the particle on that segment.

Consider three infinitesimally close points on the real path of a particle and approximate the real path by a once-broken line consisting of two infinitesimally small segments A and B (Figure 5). (In this case we do not display a worldline because this requires curves in three-dimensional spacetime.)

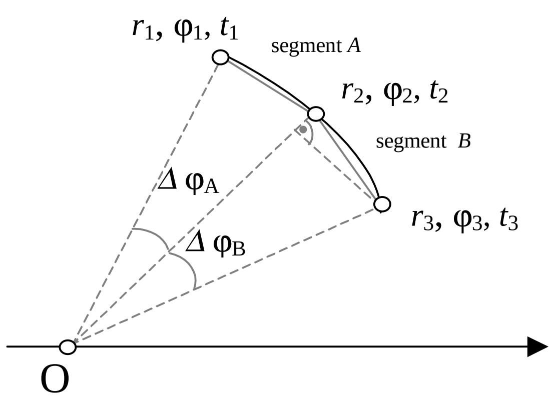

Fig.5 Path segment of planar motion with three infinitesimally close points whose positions are described by polar coordinates. The radius r_A is the average position coordinate of particle on segment *A*, and r_B on segment *B*. All coordinates of points 1, 2, 3 are fixed with the exception of the angle coordinate φ_2, which we vary to satisfy the principle of least action.

To find the required expression for the action in polar coordinates, use the Pythagorean theorem. Infinitesimal lengths Δs_A and Δs_B of segments *A* and *B* are

$$\begin{aligned} \Delta s_A{}^2 &= \Delta r_A{}^2 + (r_A \Delta\varphi_A)^2 \\ \Delta s_B{}^2 &= \Delta r_B{}^2 + (r_B \Delta\varphi_B)^2 \end{aligned} \qquad (18)$$

where

$$\Delta r_A = r_2 - r_1 \quad \text{and} \quad \Delta\varphi_A = \varphi_2 - \varphi_1$$
$$\Delta r_B = r_3 - r_2 \quad \text{and} \quad \Delta\varphi_B = \varphi_3 - \varphi_2 .$$

Substitute (18) into (17) to find values of the action for segments *A, B*:

$$S(A) = \frac{1}{2} m \frac{\Delta r_A{}^2 + r_A^2 \Delta\varphi_A{}^2}{\Delta t_A} - \text{PE}(r_A)\Delta t_A \qquad (19a)$$

$$S(B) = \frac{1}{2} m \frac{\Delta r_B{}^2 + r_B^2 \Delta\varphi_B{}^2}{\Delta t_B} - \text{PE}(r_B)\Delta t_B \qquad (19b)$$

Once again, note that the action for these two segments depends only on the *difference* in φ-coordinate, and not on the φ-coordinate itself. As before, we conclude that neither *S*(*A*) nor *S*(*B*) will change as we increase all φ-coordinates by a fixed angle Φ, since $\varphi_2 + \Phi - (\varphi_1 + \Phi) = \varphi_2 - \varphi_1$. As a result, the law of the motion of the particle is symmetrical with respect to a fixed change in angle φ. The conservation of angular momentum results from this symmetry, derived as follows.

The condition of least action *S* is expressed as:

$$\frac{\mathrm{d}S}{\mathrm{d}\varphi_2} = \frac{\mathrm{d}\left[S(A) + S(B)\right]}{\mathrm{d}\varphi_2} = 0 \qquad (20)$$

After substituting (19a) and (19b) into (20), differentiating and some rearranging we obtain:

$$m\frac{r_A^2 \Delta\varphi_A}{\Delta t_A} = m\frac{r_B^2 \Delta\varphi_B}{\Delta t_B} \qquad (21)$$

Equation (21) represents the conservation law of a quantity that is called angular momentum *L*, so $L_A = L_B$. We know the change in angle per second as angular velocity ω. Thus the equation (21) can be expressed as $mr_A^2\omega_A = mr_B^2\omega_B$.

By carrying through a derivation similar to those of previous cases of translations in space and time, we obtain the equation

$$-\frac{\mathrm{d}S}{\mathrm{d}\varphi_1} = \frac{\mathrm{d}S}{\mathrm{d}\varphi_3} \qquad (22)$$

which immediately provides the conservation of angular momentum (21). Moreover the left side of (22) can be interpreted as the angular momentum at point 1 and the right side as the angular momentum at point 3. Angular momentum is also conserved for the entire path. The result: symmetry under rotation through a fixed angle implies *conservation of angular momentum.*

Summary:
Angular momentum is a constant of the motion. The condition that physical laws remain invariant with respect to rotation through a fixed angle is called *the isotropy of space*. In other words space has the same properties in every direction. *Conservation of angular momentum results from the isotropy of space.*

D. Galilean transformation

Finally, we present a simple example that deals with an interesting and very important symmetry — the symmetry under uniform linear motion. In classical mechanics it is known as

Galileo's principle of relativity, and all classical laws are invariant under this symmetry. For example the uniform motion of a spacecraft does not affect results of any experiment carried out in this spaceship. Express Galilean relativity formally as follows: *The laws of classical mechanics are invariant under a Galilean transformation.* We will be surprised to learn that the classical action is *not* invariant under a Galilean transformation.

Consider again a free particle moving along *x*-axis between closely adjacent events 1 and 2 as observed in a laboratory frame. In this frame the action for the particle takes the form (2):

$$S_{\text{for segment}} = \frac{1}{2} m \frac{(x_2 - x_1)^2}{t_2 - t_1} \qquad \text{(23a)}$$

The (slowly-moving) rocket observer calculates the rocket action given by the same equation

$$S'_{\text{for segment}} = \frac{1}{2} m \frac{(x'_2 - x'_1)^2}{t'_2 - t'_1} \qquad \text{(23b)}$$

(Here we use primes for rocket coordinates, not for the derivative.) Now suppose that the principle of least action *S* in the laboratory frame gives the correct laws of motion. The question is, will the principle of least action *S´* for the rocket also give the same correct laws of motion for the laboratory frame?

Assume the validity of the classical Newtonian mechanics and use the Galilean transformation for coordinates

$$\begin{aligned} x' &= x - v_{rel} t \\ t' &= t \end{aligned} \quad , \qquad \text{(24)}$$

where v_{rel} is the rocket velocity with respect to the laboratory. Apply the transformation (24) to Eq. (23b) to obtain the action *S* for the laboratory frame:

$$S'_{\text{for segment}} = \frac{1}{2} m \frac{(x_2 - x_1)^2}{t_2 - t_1} - v_{rel} m (x_2 - x_1) + \frac{1}{2} m v_{rel}^2 (t_2 - t_1) \qquad \text{(25)}$$

Which is *not* the same as Eq. (23a). The action is *not* invariant under the Galilean transformation with uniform velocity in a straight line. It might look like there are two different laws of motion for one object, but according to Galileo's principle of relativity this conclusion is nonsense. Which action, *S* or *S´*, governs the motion of the particle? Or is the Galilean transformation incorrect? According to Appendix A everything is all right. The two actions *S* and *S´* differ by function that depends only on the coordinates of a given event $F(x,t) = -v_{rel} m x + \frac{1}{2} m v_{rel}^2 t$, so mechanical laws are the same as determined by using *S* as they are by using *S´*.

Using a slightly more general and complicated consideration[13] but very similar to that employed multiple times above, we can demonstrate that a corresponding conservation law to Galilean transformation is related to uniform motion of the center of mass.

IV. SYMMETRY AND CONSERVATION LAWS IN RELATIVITY

A. Action in Relativity

We have shown that the classical action is, paradoxically, not symmetrical with respect to uniform linear motion, but all laws of motion remain unchanged under a Galilean transformation. We believe that this asymmetry for the principle of least action is not accidental, but rather results from the fact that the Galilean transformation and Newton's laws are only approximate laws of motion. Symmetry under uniform linear motion is a basic assumption of Einstein's special relativity.

Consider the same free particle as before, but now use the special theory of relativity. The action for linear segment between 1 and 2 has the form:[14]

$$S_{\text{for segment}} = -mc^2\left(1-\frac{v^2}{c^2}\right)^{1/2}\Delta t \qquad (26)$$

where $\Delta t = t_1 - t_2$ and $v = \dfrac{x_2 - x_1}{t_2 - t_1}$ is the particle's speed.

It can be seen from the expression (26) for the action that Newtonian mechanics is the special case of relativistic mechanics in the low-velocity limit ($v << c$):

$$S \approx -mc^2\left(1-\frac{1}{2}\frac{v^2}{c^2}\right)\Delta t = \frac{1}{2}mv^2\Delta t - mc^2\Delta t \qquad (27)$$

According to appendix A, taking $F(x,t) = -mc^2 t$, the expression (27) for the action will then give the same laws of motion for a free particle as classical Newtonian action (2).

B. Lorentz transformation

Now we outline the symmetry argument connected to the relativistic Lorentz transformation:

$$\begin{aligned} x &= \gamma(x' + v_{rel}t') \\ t &= \gamma(t' + v_{rel}x') \end{aligned} \quad \text{where } \gamma = 1\big/\left(1 - v_{rel}^2\right)^{1/2} \qquad (28)$$

Express the action (26) along a segment of the worldline using the same unit for space and time (c = 1).

$$S_{\text{for segment}} = -m\left[(t_2 - t_1)^2 - (x_2 - x_1)^2\right]^{1/2} \qquad (29)$$

The expression in the square root is the particle's proper time (wristwatch time) between the two events, which is easily verified to be an invariant under the Lorentz transformation. Hence the relativistic action *is* symmetrical under transformation connected with uniform linear motion.

Noether's theorem can be used also in relativity. The same procedure as in paragraphs III.A (translation in space), III.B (translation in time) and III.C (rotation through a fixed angle) can be repeated also in special relativity, which yields the laws of conservation of relativistic energy, momentum and angular momentum:

$$\begin{aligned} \frac{m\Delta x}{\Delta\tau} &= p_{relativistic} = \text{const} \\ \frac{m\Delta t}{\Delta\tau} &= E_{relativistic} = \text{const} \\ \frac{mr^2\Delta\varphi}{\Delta\tau} &= L_{relativistic} = \text{const} \end{aligned} \qquad (30)$$

where $\Delta\tau$ is the particle's proper time. As for the Lorentz transformation, there also exists corresponding conservation law, but its derivation goes beyond the scope of this paper.[15]

We see that the theory of relativity eliminates the asymmetry of the action under translation. The invariance of the action under all the transformations we have considered makes the theory of relativity a more beautiful and elegant theory than the Newtonian theory of classical mechanics.

If one finds the right expression for the action (or proper time), the constants of motion can be derived also in the field of general relativity without complicated or advanced mathematics.[16]

V. SUMMARY

We have presented the connection between symmetries and conservation laws provided by Noether's theorem using only elementary calculus. This approach can be used in first year university physics to help familiarize students with the powerful consequences of symmetry in the physical world. In addition, students can see a *unified* and *systematic* approach to *all* conservation laws, mediated by Noether's theorem and the powerful principle of least action.

All our presented considerations can be easily generalized to three-dimensional case. We have to note that all symmetries in this paper are one-parameter transformations, which provide central conservation laws using the most common form of Noether's theorem related to invariance of Lagrangian (see appendix B). Finally, reference 19 and pedagogically adapted references 17, 18 give clear, elegant and more mathematically precise (but *more* mathematically oriented) applications of Noether's theorem to particle dynamics.

ACKNOWLEDGMENTS

This paper was written after we read Taylor and Wheeler's general relativity book[16] sent to us along with other materials by Taylor, who also made very helpful suggestions for this paper. The authors also wish to thank Nilo C. Bobillo Ares for helpful advice. Slavko Chalupka provided important discussion and encouragement and gave us the opportunity to teach an experimental course in quantum mechanics using some of these ideas[20].

APPENDIX

A. Addition of certain terms to the action has no effect on the laws of motion

Think of two expressions for the action $S(12)$ and $S^*(12)$ for a given worldline between any two events 1 and 2 in spacetime. Suppose that these two expressions are related to each other in the following way:

$$S^*(12) = S(12) + F(2) - F(1) \qquad (31)$$

where F is an arbitrary function that depends only on the space and time coordinates of a given event. For example $F(1)$ is the value of F at the event 1. Then the conservation laws or mechanical laws are the same for both forms of action. Why?

Answer this question by repeating the same procedure as for earlier symmetries, starting with three events 1, 2, 3. Applying (31) we get following equations relating action S and S^* for segment 1-2 and 2-3 :

$$S^*(12) = S(12) + F(2) - F(1) \qquad (32a)$$

$$S^*(23) = S(23) + F(3) - F(2) \qquad (32b)$$

The total action $S^*(123)$ is the sum of (32a) and (32b):

$$S^*(123) = S(123) + F(3) - F(1) \qquad (33)$$

The two total actions S^* and S in Eq. (33) differ only in the difference in F at the fixed events 3 and 1. If we change space or time coordinate (generally u_2) of the middle event 2, this difference remains constant. So the minima of S and S^* yield the same position of event 2, or in other words the first derivatives of S and S^* with respect to u_2 are the same (all other variables being fixed):

$$\frac{\mathrm{d}S^*}{\mathrm{d}u_2} = \frac{\mathrm{d}S}{\mathrm{d}u_2} \qquad (34)$$

According to Eq. (34) the principle of least action for S and S^* gives the same particle's path. The laws of motion are unchanged if an additive constant (the *difference* in an arbitrary function between final position and initial position of a particle) is added to its action.[21]

B. Noether's theorem and the Lagrangian

Noether's theorem determines the connection between constants of the motion and conditions of invariance of the action under different kinds of symmetry. The function $KE - PE$ in Newtonian mechanics is called the *Lagrange function* or in short the *Lagrangian* and given by the symbol L (sometimes a script L). So we can write (1a or 1b) $S_{\text{for segment}} \equiv \Delta S = L \,.\, \Delta t$ (do not confuse the symbol L for the action with the symbol L for angular momentum used in Sec. III.C). If we discuss symmetry transformations such that time is transformed identically: $t^* = t$ or transformations involving a uniform time translation: $t^* = t + \tau$, where $\Delta t = \Delta t^*$, then the invariance of the Lagrangian implies the invariance of the action. Therefore most physical textbooks use the following form of Noether's theorem: *For each symmetry of the Langrangian, there is a conserved quantity.*

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[1]R. P. Feynman and S. Weinberg, *Elementary particles and the laws of physics*, (Cambridge University Press, 1999, ISBN 0 521 65862 4), p.73

[2]In reality, there are two Noether's theorems. Our paper dealing with one-parametrical symmetry transformations is connected with the first one. See E. Noether, "Invariante Variationsprobleme," Nachr. v. d. Ges. d. Wiss. zu Göttingen 1918, pp 235-257; English translation by M. A. Tavel, "Invariant variation problem", Transport Theory and Statistical Mechanics" 1(3), 183-207 (1971); all available at
<http://www.physics.ucla.edu/~cwp/Phase2/Noether,_Amalie_Emmy@861234567.html>

[3]N. Byers, *E. Noether's discovery of the deep connection Between Symmetries and Conservation laws*, Conference proceedings vol. 12, 1999 presented at the symposium – The Heritage of Emmy Noether in Algebra, Geometry and Physics.

[4]R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1963), Vol. I., Chapter 11, p. 11-1 or Chapter 52, p.52-1

[5]More accurately the principle says that a particle moves along that path for which the action has a *stationary* value. So it should be called The Principle of Stationary Action; see e.g. D. J. Morin:"Chapter 5 – The Lagrangian Method" at
<http://www.courses.fas.harvard.edu/~phys16/handouts/textbook/ch5.pdf >

[6]Generally such quantity is a coordinate called *cyclic* or *ignorable*; H. Goldstein, *Classical Mechanics*, (Addison-Wesley, 1970), p. 48 or Reference 5

[7]Every quantity that depends on position coordinates and velocities and whose value does not change along actual trajectories is called *constant of motion*.

[8]We recommend to follow and use a more detailed described procedure for introducing action in our paper J. Hanc, S. Tuleja and M. Hancova, "Simple derivation of Newtonian mechanics from the principle of least action" (accepted in AJP, October 2002)

[9]The idea of using computers comes from E. F. Taylor; see E. F. Taylor, S. Vokos, J. M. O´Meara, and N. S. Thornber, *"Teaching Feynman's Sum Over Paths Quantum Theory*, Comp. Phys. 12 (2), 190-199, 1998 or E. F. Taylor: *Demystifying Quantum Mechanics*, workbook for his course Boundaries of Nature; see his web site http://www.eftaylor.com. Our software is based on Taylor's software.

[10]R. P. Feynman, *The Character of Physical Law*, (Random House, Inc. New York 1994, ISBN 0-679-60127-9), Chapter 4

[11]Reference 4, Vol.II, Chapter 19, p. 19-8 or more detailed in Ref. 8

[12]Strictly speaking, in these and the following cases we should use the more traditional notation of partial instead of total derivative. But in all cases it is clear which coordinates are variable and which are fixed.

[13]This more complicated consideration is connected to a more general form of Noether's theorem dealing with *invariance of action up to an additive constant* which gives *conservation of motion of the center of mass* in the case of Galilean transformation (see Ref. 19 or 17). It can be also found at our website (see Ref. 20).

[14]Relativistic formula of the action – Reference 4, Vol. II, Chap. 19; We use the concept of invariance of mass that is used by E. F. Taylor and J. A. Wheeler in their book *Spacetime Physics: Introduction to Special Relativity*, W H Freeman & Co.; ISBN: 0716723271, second edition, 1992

[15]The conservation law corresponding to the Lorentz transformation is derived in chapter 2, paragraph 14: *Angular momentum* in L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields, Course of Theoretical Physics*, (Pergamon press, 1975, ISBN 0080181767), Vol. 2, pp. 41 – 42. We present a derivation less mathematically oriented than Landau's one and similar to those used in the paper at our website (See Ref. 20).

[16]E. F. Taylor and J. A. Wheeler, *Exploring Black Holes: An Introduction to General Relativity*, (Addison Wesley Longman, 2000, ISBN 0-201-38423-X), Chapter 1 and 4; see http://www.eftaylor.com; The authors use a very similar, easy and effective variational method.

[17]N. C. Bobillo-Ares, "Noether's theorem in discrete classical mechanics", Am. J. Phys. 56 (2), pp. 174-177 (1988).

[18]C. M. Giordano and A. R.Plastino, "Noether's theorem, rotating potentials, Jacobi's integral of motion", Am.J. Phys., 66(11), pp. 989-995 (1998).

[19]P. Havas, J. Stachel, "Invariances of approximately relativistic Lagrangians and the center of mass Theorem. I.", Phys. Rev., 185 (5), pp. 1636-1647 (1969)

[20]The substance of this article was used by the authors as student projects for a special topic "Principle of least action" in a semester quantum mechanics course developed after reference 9 for students – future teachers of physics – at the Faculty of Science, P.J. Safarik University, Kosice, Slovakia. To obtain our materials and corresponding software, see a web site <http://www.LeastAction.host.sk> or mirror site <http://leastaction.topcities.com>

[21]L. D. Landau and E. M. Lifshitz, *Mechanics* (Butterworth-Heinemann, Oxford, 1976), Sec. 1.2.