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Deriving Lagrange's Equations Using Elementary Calculus

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We derive Lagrange's equations of motion from the principle of least action using elementary calculus rather than the calculus of variations. We also demonstrate conditions under which energy and momentum are constants of the motion.

I. Introduction

Lagrange's equations are powerful tools for obtaining the equations of motion of a mechanical system. Lagrange's equations employ scalars rather than the vectors used in Newton's second law of motion and can easily be deployed to analyze a much wider range of systems than $\boldsymbol{F} = m\boldsymbol{a}$, especially systems subject to constraints. Chapter 1 in standard advanced mechanics texts[1] uses the calculus of variations to derive Lagrange's equations from the fundamental principle of least action. In this paper we derive Lagrange's equations from the principle of least action using elementary calculus,[2] in the hope that they may be introduced earlier in the careers of physics students.

II. Differential Approximation to the Principle of Least Action

A particle moves along the *x*-axis with conservative potential energy $V(x)$. (The Appendix outlines a generalization.) For this special case, the **Lagrange function** or **Lagrangian** L has the form:[3]

$$L(x,v) = T - V = \frac{1}{2}mv^2 - V(x) \tag{1}$$

and the **action** S along a worldline is

$$S = \int_{\substack{\text{along the}\\ \text{worldline}}} L(x,v)\,dt \tag{2}$$

The **principle of least action** demands that between fixed initial event and fixed final event the particle follow a worldline such that the action S is a minimum.

Notice that the action S is an additive scalar quantity, the sum of contributions $L\Delta t$ from each segment along the entire worldline between two fixed events. Because S is additive, it follows that the principle of least action must hold for each individual infinitesimal segment of the worldline.[4] This allows us to pass from the integral equation for the principle of least action, equation (2), to Lagrange's differential equation, valid anywhere on the worldline.

We approximate a small section of the worldline by two straight-line segments connected in the middle (Figure 1).

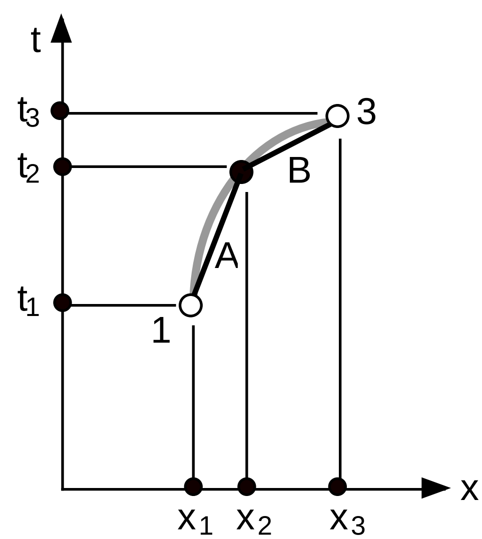

Figure 1. *An infinitesimal section of the worldline approximated by two straight line segments.*

Make the following approximations: (a) The average position coordinate in the Lagrangian along a segment is at the midpoint of that segment.[5] (b) The average velocity of the particle is equal to its displacement across the segment divided by the time span of the segment. Applied to segment A in Figure 1, these approximations yield the action S_A contributed by this segment:

$$S_A \approx L_A \Delta t = L\left(\frac{x_1 + x_2}{2}, \frac{x_2 - x_1}{\Delta t}\right) \cdot \Delta t\,, \tag{3}$$

where the dot separates the multiplier Δt from the arguments of L.

III. Derivation of Lagrange's equation

Employ the approximations of Section II to derive Lagrange's equations for the special case introduced in that section. As shown in Figure 2, we fix events numbered 1 and 3 and vary the *x* coordinate of the middle event in order to minimize the action between 1 and 3.

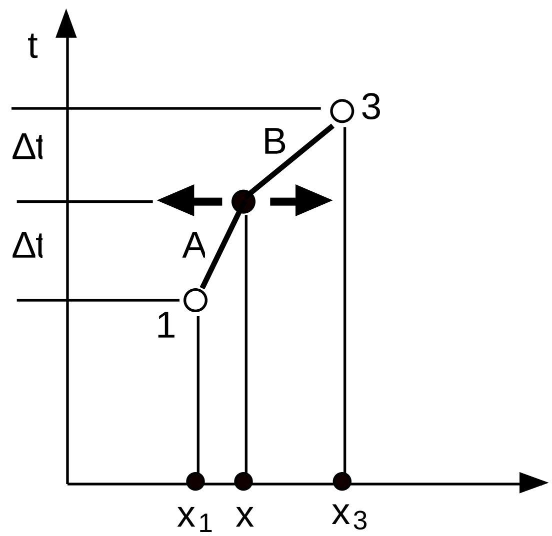

Figure 2. *Derivation of Lagrange's equations from the principle of least action. Points 1 and 3 are on the true worldline, which is approximated by two straight line segments (as in Figure 1). The arrows show that the x coordinate of the middle event is varied. All other coordinates are fixed.*

For simplicity, but without loss of generality, we choose the time increments Δt to be the same for each segment, which also equals the time lapse between the midpoints of the two segments. Calculate the average positions and velocities along segments *A* and *B*.

$$x_A = \frac{x_1 + x}{2} \text{ and } v_A = \frac{x - x_1}{\Delta t} \qquad (4a)$$

$$x_B = \frac{x + x_3}{2} \text{ and } v_B = \frac{x_3 - x}{\Delta t} \qquad (4b)$$

Expressions (4) are all functions of the single variable *x*. For later use we take the derivatives of expressions (4) with respect to *x*:

$$\frac{dx_A}{dx} = \frac{1}{2} \text{ and } \frac{dv_A}{dx} = +\frac{1}{\Delta t} \qquad (5a)$$

$$\frac{dx_B}{dx} = \frac{1}{2} \text{ and } \frac{dv_B}{dx} = -\frac{1}{\Delta t} \qquad (5b)$$

Let L_A and L_B be the values of the Lagrangian on segments *A* and *B* respectively using these approximations, and label the summed action across these two segments S_{AB}:

$$S_{AB} = L_A \Delta t + L_B \Delta t \tag{6}$$

The principle of least action requires that coordinates of the middle point *x* be chosen so to yield the smallest value of the action between fixed events 1 and 3. Setting the derivative of S_{AB} with respect to *x* equal to zero[6] and using the chain rule leads to

$$\frac{dS_{AB}}{dx} = 0 = \frac{\partial L_A}{\partial x_A}\frac{dx_A}{dx}\Delta t + \frac{\partial L_A}{\partial v_A}\frac{dv_A}{dx}\Delta t + \frac{\partial L_B}{\partial x_B}\frac{dx_B}{dx}\Delta t + \frac{\partial L_B}{\partial v_B}\frac{dv_B}{dx}\Delta t \tag{7}$$

Substitute from equations (5) into (7), divide through by Δt, and regroup to obtain

$$\frac{\left(\frac{\partial L_A}{\partial x_A} + \frac{\partial L_B}{\partial x_B}\right)}{2} - \frac{\left(\frac{\partial L_B}{\partial v_B} - \frac{\partial L_A}{\partial v_A}\right)}{\Delta t} = 0 \tag{8}$$

To first order, the initial term in (8) is the average value of the partial *x*-derivative $\partial L/\partial x$ on the two segments *A* and *B*. In the limit of small Δt this approaches the value of the partial derivative at the intermediate point *x*. In the same limit, the second term in (8) becomes the time derivative of the partial derivative of the Lagrangian with respect to velocity $d(\partial L/\partial v)/dt$. Therefore in the limit of small Δt equation (8) becomes the Lagrange equation in *x*:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) = 0 \tag{9}$$

We did not specify the location of segments *A* and *B* along the worldline. The additive property of the action (2) implies that equation (9) is valid on every adjacent pair of segments.

An essentially identical derivation applies to any particle with one degree of freedom in a conservative potential. For example, the single angle ϕ tracks the motion of a simple pendulum, so its equation of motion follows from (9) by replacing *x* with ϕ without the need to take vector components.

When the Lagrangian L is not an explicit function of *x* (for example, if the potential is zero or independent of position), then Lagrange's equation (9) tells us that $\partial L/\partial v$ does not change with time. From equation (1), we find that $\partial L/\partial v = mv$, which shows that in our case the *x*-momentum is a constant of the motion.

IV. Energy as a Constant of the Motion

Next we show that the energy is a constant of the motion for a particle moving in a conservative potential. We do this by varying the time of the middle event (Figure 3), rather than its position, while still demanding that the action be a minimum.

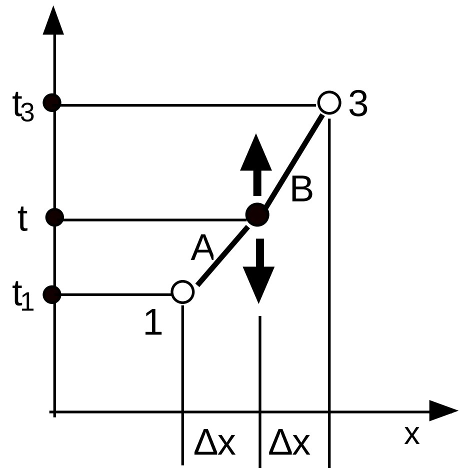

Figure 3. *Derivation to show that energy is a constant of the motion. Points 1 and 3 are on the true worldline, which is approximated by two straight line segments (as in Figures 1 and 2). The arrows show that the t coordinate of the middle event is varied. All other coordinates are fixed*.

For simplicity, but without loss of generality, we make the *x*-increments equal, with the value Δx. In this case we keep the coordinates of all three events fixed except for the time coordinate of the middle event. We have:

$$v_A = \frac{\Delta x}{t - t_1} \quad \text{and} \quad v_B = \frac{\Delta x}{t_3 - t} \tag{10}$$

These expressions are functions of the single variable *t*, with respect to which we take the derivatives

$$\frac{dv_A}{dt} = -\frac{\Delta x}{(t - t_1)^2} = -\frac{v_A}{t - t_1} \tag{11a}$$

and

$$\frac{dv_B}{dt} = +\frac{\Delta x}{(t_3 - t)^2} = +\frac{v_B}{t_3 - t}\,. \tag{11b}$$

Note that in spite of the form of equations (11) the derivatives are *not* accelerations, since the *x*-separations are held constant while the time is varied.

As before (equation 6),

$$S_{AB} = L_A \cdot (t - t_1) + L_B \cdot (t_3 - t) \tag{12}$$

where we insert dots to make clear that the parentheses containing time differences are multipliers, not arguments of the Lagrange functions. Find the time for minimum action by setting the derivative of S_{AB} equal to zero.

$$\frac{dS_{\mathrm{AB}}}{dt} = 0 = \frac{\partial L_{\mathrm{A}}}{\partial v_{\mathrm{A}}}\frac{dv_{\mathrm{A}}}{dt}\cdot\left(t - t_1\right) + L_{\mathrm{A}} + \frac{\partial L_{\mathrm{B}}}{\partial v_{\mathrm{B}}}\frac{dv_{\mathrm{B}}}{dt}\cdot\left(t_3 - t\right) - L_{\mathrm{B}} \tag{13}$$

Substitute from equations (11) into equation (13) and rearrange the result:

$$\frac{\partial L_{\mathrm{A}}}{\partial v_{\mathrm{A}}} v_{\mathrm{A}} - L_{\mathrm{A}} = \frac{\partial L_{\mathrm{B}}}{\partial v_{\mathrm{B}}} v_{\mathrm{B}} - L_{\mathrm{B}} \tag{14}$$

Because the expression for the action (2) is additive, equation (14) is valid on every segment of the worldline and identifies the quantity $E = v\,\partial L/\partial v - L$ as a constant of the motion. Substitute expression (1) for the Lagrangian into the expression for E and carry out the partial derivatives to show that the constant of the motion E is total energy.

$$E = 2T - \left(T - V\right) = T + V = \text{ constant} \tag{15}$$

V. Conclusions

The derivations discussed in this paper and the extension to multiple degrees of freedom discussed in the Appendix can replace many applications of the calculus of variations with simpler calculus derivations. This permits the introduction of Lagrange's equations early in the study of mechanics.

One of us (Tuleja) has successfully employed these derivations and applied the resulting Lagrange equations with a small group of talented high school students. The excitement and enthusiasm of these students leads us to hope that others will undertake trials with larger numbers and a greater variety of students.

APPENDIX: Extension to Multiple Degrees of Freedom

We discuss here Lagrange's equations for a system with multiple degrees of freedom, without pausing to detail the usual conditions assumed in the derivations, since these can be found in standard advanced mechanics texts.[7]

Let a given mechanical system be described by the following Lagrangian:

$$L = L(q_1, q_2, \ldots, q_s, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_s, t) \tag{16}$$

where the q are independent **generalized coordinates** and the dot over a q indicates a derivative with respect to time. The subscript s is the number of degrees of freedom of the system. Generalized coordinates q_1, q_2, and q_3, for example, could be the x, y, and z or the r, θ, and φ coordinates of one particle in a multi-particle system. Note that we have generalized to a Lagrangian that is an explicit function of time t. Specifying values of all variables in (16) defines what is called a **configuration** of the system. The **action** S summarizes the evolution of the system as a whole from an initial configuration to a final configuration, along what might be called a "worldline through multidimensional spacetime." Symbolically we write:

$$S = \int\limits_{\substack{\text{initial configuration} \\ \text{to final configuration}}} L\left(q_1, q_2, \ldots q_s, \dot{q}_1, \dot{q}_2, \ldots \dot{q}_s, t\right) dt \tag{17}$$

The generalized principle of least action demands that the value of *S* be a minimum for the actual evolution of the system symbolized in (17). We make an argument analogous to that in Section III for the one-dimensional motion of a single particle moving in a potential. If the principle of least action holds for the entire trajectory through the intermediate configurations (17), it also holds for an infinitesimal change in configuration anywhere during this process.

Let the system pass through three infinitesimally close configurations in the ordered sequence 1, 2, 3 such that all generalized coordinates remain fixed except for any single coordinate *q* at configuration 2. Then the increment of action from configuration 1 to configuration 3 can be considered to be a function of the single variable *q*. As a consequence, for *each* of the *s* degrees of freedom we can make an argument formally identical to that carried out from equation (3) through equation (9). Repeated *s* times, once for each generalized coordinate q_i, this derivation leads to *s* scalar Lagrange equations that describe the motion of the system:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \qquad i = 1, 2, 3, \ldots s \tag{18}$$

The inclusion of time explicitly in the Lagrangian (16) does not affect these derivations, because in each the time coordinate is held fixed.

Suppose that the Lagrangian (16) is *not* a function of a given coordinate q_k. Then equation (18) tells us that the corresponding **generalized momentum** $\partial L / \partial \dot{q}_k$ is a constant of the motion.

If the Lagrangian (16) is *not* an explicit function of time, then a derivation formally equivalent to that carried out in Section IV (with time again as the single variable) shows that $E = \left(\sum \dot{q}_i \, \partial L / \partial \dot{q}_i\right) - L$ is a constant of the motion of the system, which can be interpreted as the conservation of energy *E* of the entire system.

If the Lagrangian (16) depends explicitly on time, then this derivation goes beyond the conservation of energy to the equation $dE / dt = -\partial L / \partial t$.

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[1] Chapter 1 in L. D. Landau and E. M. Lifshitz, *Mechanics* (Butterworth-Heinemann, Oxford, 1976); Chapter 1 in Gerald J. Sussman and Jack Wisdom, *Structure and Interpretation of Classical Mechanics,* (MIT Press, 2001).

[2] The derivation presented in this paper is a modification of the finite difference technique employed by Euler in his path breaking 1744 work, *The Method of Finding Plane Curves that Show Some Property of Maximum and Minimum*, ... Full references and Euler's original treatment can be found in Herman H. Goldstine, *A history of the calculus of variations from the 17th through the 19th century,* Springer-Verlag, New York, 1980, Chapter 2. Cornelius Lanczos presents an abbreviated version of Euler's original derivation using contemporary mathematical symbolism in *The Variational Principles of Mechanics* (Dover Publications, New York, 1986), pages 49-54.

[3] R. P. Feynman, R. B. Leighton and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1964), Vol. 2, Chap. 19.

[4] Reference 3, page 19-8 or in more detail J. Hanc, S. Tuleja, M. Hancova: "Simple derivation of Newtonian mechanics from the principle of least action" (accepted for publication in the *American Journal of Physics*).

[5] There is no particular reason to use the midpoint of the segment in the Lagrangian of equation (2). In Riemann integrals we can use any point on the given segment. For example, all results in this paper will be the same using the coordinates of either end of each segment instead of the coordinates of the midpoint. Repositioning this point can be the basis of an exercise to test student understanding of the derivations in this paper.

[6] The zero derivative most often leads to the worldline of *minimum* action. It is possible also to have zero derivative at an inflection point or saddle point in the action (or the multidimensional equivalent in configuration space). So the most general — but rather esoteric — term for our basic law is the *principle of stationary action.*

[7] Texts in reference 1 and Herbert Goldstein, Charles Poole, and John Safko, *Classical Mechanics*, third edition (Addison-Wesley, 2002);