I. INTRODUCTION

“It is increasingly clear that the symmetry group of nature is the deepest thing that we understand about nature today” (Steven Weinberg). Many of us have heard statements such as for each symmetry operation there is a corresponding conservation law. The conservation of momentum is related to the homogeneity of space. Invariance under translation in time means that the law of conservation of energy is valid. Such statements come from Noether’s theorem, one of the most amazing and useful theorems in physics.

When the German mathematician Emmy Noether proved her theorem,2,3 she uncovered the fundamental justification for conservation laws. This theorem tells us that conservation laws follow from the symmetry properties of nature. Symmetries (called “principles of simplicity” in Ref. 1) can be regarded as a way of stating the most fundamental properties of nature. Symmetries limit the possible forms of new physical laws. The deep connection between symmetry and conservation laws requires the existence of a minimum principle in nature: the principle of least action. In classical mechanics, symmetry arguments are developed using high level mathematics. On the other hand, the corresponding physical ideas often are much easier to understand than the mathematical derivations.

In this paper we give an elementary introduction to the relation between symmetry arguments and conservation laws, as mediated by the principle of least action. We shall use only elementary calculus so that our approach can be used in introductory university physics classes.

Because the paper deals mainly with symmetry, it is important how we define or characterize this concept in the framework of introductory physics. We adopt Feynman’s simple description of symmetry from his lectures on physics,4 which says that anything is symmetrical if one can subject it to a certain operation and it appears exactly the same after the operation.

Like Feynman, we will concentrate on symmetry in physical laws. The question is what can be done to a physical law so that this law remains the same. Noether’s theorem derives conservation laws from symmetries under the assumption that the principle of least action governs the motion of a particle in classical mechanics. This principle can be phrased as “The action is a minimum for the path (worldline) taken by the particle,”5 which leads to the reformulation of our basic question about symmetry: What changes can we make in the worldline that do not lead to changes in either the magnitude or the form of the action?

We will explore and apply symmetry operations to the action along an infinitesimally small path segment. Because the action is additive, conclusions reached about a path segment apply to the entire path. The simplest examples of symmetry show the independence of the action on the difference in some quantity such as position, time, or angle.6 When such a symmetry exists, Noether’s theorem tells us that a physical quantity corresponding to this symmetry is a constant of the motion that does not change along the entire path of the particle.7 The existence of such a constant implies a conservation law, which we then identify.

Section II briefly describes our software that helps students study the action and its connection to conservation laws. Section III analyzes four examples of symmetry operations: translation in space and time, rotation through a fixed angle, and symmetry under uniform linear motion, namely the Galilean transformation. The first three symmetries lead to three conservation laws: momentum, energy, and angular momentum. Section IV extends the analysis to symmetry in relativity, showing that these conservation laws exist in that realm. Moreover, for uniform linear motion the symmetry argument applies more elegantly to the Lorentz transformation than to the Galilean transformation.

In the following we often talk about variations or changes in the action. Consistent with standard practice, we will only be interested in variations representing infinitesimal first-order changes in the action. To keep the arguments simple, we also assume that the particle’s invariant mass $m$ (rest mass) does not change during the motion to be studied.

II. SOFTWARE

We start with the well-known definition of action for a particle of mass $m$ that moves from some initial position at time $t_1$ to some final position at time $t_2$:

$$S = \int_{t_1}^{t_2} (KE - PE) dt,$$

or equivalently

$$S = \int_{t_1}^{t_2} (KE - PE) dt.$$
Action

Table Endpoints Grid Finding

\begin{array}{|c|c|c|c|}
\hline
\text{position} & \text{velocity} & \text{acceleration} \\
\hline
\text{KE} & \text{PE} & \text{gravity} \\
\hline
\text{1 kg} & \text{253.8} & \text{-1.287.9} \\
\hline
\end{array}

Here \( KE_{av} \) denotes the time averaged kinetic energy and \( PE_{av} \) the time averaged potential energy between \( t_1 \) and \( t_2 \). We use the notation \( KE \) and \( PE \) as symbols for kinetic and potential energies, respectively, because they are more mnemonic than the traditional symbols \( T \) and \( V \).

Action is not a familiar quantity\(^8\) for many students, so we employ an interactive computer program\(^9\) to help them develop an intuition about the nature of the action and the principle of least action. By using an interactive computer display, the student cannot only explore the operation of the principle of least action, but also study the relation between this principle and conservation laws in specific cases (Fig. 1). In carrying out this manipulation, the student naturally works with the central concepts of a worldline (a graph of the time dependence of a particle’s position) and an event (a point on a worldline). Unlike the trajectory in space, the worldline specifies completely the motion of a particle. For background on the symmetry properties of nature, we suggest that our students read a selection from Ref. 10.

III. SYMMETRY AND CONSERVATION LAWS IN NEWTONIAN MECHANICS

A. Translation in space

We first examine the symmetry related to translation in space. When we perform an experiment at some location and then repeat the same experiment with identical equipment at another location, then we expect the results of the two experiments to be the same. So the physical laws should be symmetrical with respect to space translation.

As a simple example, consider the action of a free particle (in zero potential or uniform potential) moving along the \( x \) axis between two events \( 1 \{ t_1, x_1 \} \) and \( 2 \{ t_2, x_2 \} \) infinitesimally close to one another along its worldline. Because the worldline section is considered to be straight, the particle moves at constant velocity \( v = (x_2 - x_1)/(t_2 - t_1) \) and therefore with a constant kinetic energy \((1/2)mv^2\). According to Eq. (1b), the action along this straight segment in zero potential is (the consideration for uniform potential is analogous)

\[
S_{\text{for segment}} = \frac{1}{2} m \left( \frac{x_2 - x_1}{t_2 - t_1} \right)^2.
\]

If we change the positions of both observed events by a fixed displacement \( a \), the action remains unchanged (invariant), because the value of the action depends only on the difference between the positions: \( x_2 + a - (x_1 + a) = x_2 - x_1 \). The principle of least action is symmetrical with respect to a fixed displacement of the position. Noether’s theorem implies that this symmetry is connected with a conservation law. In the following, we demonstrate that the conservation law related to symmetry under space translation is conservation of momentum.

1. Principle of least action and momentum

Think of the motion of a free particle along the \( x \) axis. To explore the connection between the principle of least action and the conservation of momentum, we take advantage of the additive property of the action to require that the action along an arbitrary infinitesimal section of the true worldline have a minimal value.\(^11\) Thus we consider three successive infinitesimally close events, 1, 2, and 3 on the particle’s worldline and approximate a real worldline by two connected straight segments, \( A \) and \( B \) (see Fig. 2).

Because we are considering translation in space, we fix the first and last events, 1 and 3, and change the space coordinate \( x_2 \) of the middle event 2 so as to minimize the value of the total action \( S \). This minimum condition corresponds to a zero value of the derivative of \( S \) with respect to \( x_2 \):

\[
\frac{dS}{dx_2} = 0.
\]

Because the action is an additive quantity, the total action equals the sum of the actions for segments \( A \) and \( B \), so 

\[ S = S(A) + S(B). \]

If we use Eq. (2), we can write

\[
S(A) + S(B) = \int_{x_1}^{x_2} S_{\text{action}} dx + \int_{x_2}^{x_3} S_{\text{action}} dx.
\]

Fig. 1. The use of software helps students study the action along a worldline for a particle moving vertically in a gravitational field (as shown) or in other conservative potentials. The user clicks on events to create a worldline and then drags the events to minimize the action, which the computer continuously calculates and displays. The computer also displays a table of energy, momentum, or other quantities that demonstrate conservation of these quantities. Students discover that for the worldline of minimum action, momentum is conserved for the motion of a free particle and that in a gravitational field total energy is conserved.
We could continue and add other segments. The effect is obtained if we first change the position of event 1:

\[ x \rightarrow x^* \]

and the worldline momentum. Again consider three infinitesimally close events on the worldline that describes the particle motion. For all these segments the momentum will have the same value, which yields the conservation law of momentum.

We shift the worldline through a fixed infinitesimal displacement \( a \). The new worldline \( x^* \) remains unchanged and does not depend on the parameter \( a \). Thus the change in action with respect to the displacement \( a \) is zero:

\[ \Delta_a S = S(x^*; a) - S(x; a) = 0. \]  

The application of the derivatives in Eq. (4) yields the identical result for a free particle as Eq. (5), but this time as a result of spatial translation of the entire incremental worldline segment. Thus the left-hand side of Eq. (10) can also be interpreted as the momentum at event 1 and the right-hand side as the momentum at event 3.

The preceding considerations can be applied to the entire worldline \( x(t) \). We did not specify the location of the segments \( A \) and \( B \). Therefore, an arbitrary number of additional segments can be added between them. Then we shift the segments as before (see Fig. 4). By the same analysis we conclude that the momentum for segment \( A \) (effectively the momentum at event 1) has the same value as for segment \( B \) (effectively at event 3). The arbitrariness of position of these segments on the worldline means that the value of the momen-
momentum remains constant at every event on the worldline. Thus, in classical mechanics, the symmetry of spatial translation means that momentum is conserved for a free particle.

The invariance of the action with respect to translation in space is also called the homogeneity of space, which means that all points in space are equivalent. In other words, it does not matter where an experiment is performed. Therefore, we can state that the law of momentum conservation results from the homogeneity of space.

### B. Translation in time

It is easy to envision the symmetry related to translation in time. Repeating an experiment on identical initial systems yields the same result when the two experiments are separated by a lapse of time. Our conclusion is that physical laws should not change with translation in time.

Again we will show the relation of translation in time symmetry to a relevant conservation law. We start with an expression for the action of a particle moving in the $x$ direction along an infinitesimally small worldline segment in a potential field described by $PE(x)$. As in Sec. III A the action for this segment can be written according to Eq. (1b) as

$$S_{\text{for segment}} = \frac{1}{2} m \left( \frac{x_2 - x_1}{t_2 - t_1} \right)^2 \left( \frac{PE(x_1 + x_2)}{2} \right) (t_2 - t_1),$$

where the potential energy is evaluated at the average position along the segment. Now suppose that we translate the time $t$ by an amount $\tau$. It is easy to see that the action will not change, because only the difference of the time, $t_2 + \tau - (t_1 + \tau) = t_2 - t_1$, appears in the equation for the action. So the action is symmetrical with respect to a fixed displacement of time $t$. What conservation law is related to this time symmetry? We will show that it is conservation of energy.

We follow the same line of reasoning as for the case of translation in space, but now we fix all position and time coordinates with the exception of $t_2$. Think of a particle that moves along the $x$ axis in the potential field with potential energy $PE(x)$. To simplify the algebra, we denote space and time differences by

$$\Delta x_A = x_2 - x_1, \quad \Delta x_B = x_3 - x_2, \quad \Delta t_A = t_2 - t_1, \quad \Delta t_B = t_3 - t_2.$$  

According to Eqs. (11) and (12), the values of the actions $S(A)$ and $S(B)$ for segments $A$ and $B$ are equal to

$$S(A) = \frac{1}{2} m \left( \frac{\Delta x_A^2}{\Delta t_A} \right) PE \left( \frac{x_2 + x_1}{2} \right) \Delta t_A,$$  \hspace{1cm} (13a)

$$S(B) = \frac{1}{2} m \left( \frac{\Delta x_B^2}{\Delta t_B} \right) PE \left( \frac{x_3 + x_2}{2} \right) \Delta t_B.$$  \hspace{1cm} (13b)

The principle of least action leads to the following condition for the total action $S$:

$$\frac{dS}{dt_2} = \frac{d[S(A) + S(B)]}{dt_2} = 0.$$  \hspace{1cm} (14)

If we substitute Eq. (13) into Eq. (14), differentiate, and rearrange the terms, we obtain

$$\frac{1}{2} m \left( \frac{\Delta x_A^2}{\Delta t_A} \right) + PE \left( \frac{x_2 + x_1}{2} \right) \Delta t_A = \frac{1}{2} m \left( \frac{\Delta x_B^2}{\Delta t_B} \right) + PE \left( \frac{x_3 + x_2}{2} \right) \Delta t_B.$$  \hspace{1cm} (15)

The expressions on both sides of Eq. (15) are sums of average kinetic and potential energies. For infinitesimally close events, Eq. (15) gives an equality for the instantaneous values $(1/2)mv_A^2 + PE_A = (1/2)mv_B^2 + PE_B$, and expresses the conservation of mechanical energy.

Next we carry out an argument that translates all three times $t_1$, $t_2$, and $t_3$ by the same amount $\tau$, similar to the way we translated positions for the momentum case. Equations (6)–(8) apply to the present case as well, and also Eq. (9) when the derivatives are taken with respect to time rather than position. Then the result of the temporal translation is an equation similar to Eq. (10):

$$- \frac{dS}{dt_1} = \frac{dS}{dt_3},$$  \hspace{1cm} (16)

which yields Eq. (15) multiplied by $(-1)$. We again obtain conservation of energy, but this time as a result of symmetry under time translation. For infinitesimally close events, the left-hand side of Eq. (16) also can be interpreted as the negative of the total energy at event 1 and the right-hand side as the negative of the energy at event 3. The energy is a constant of the motion for the entire worldline $x(t)$. Similar to the last paragraph of Sec. III A, we can say that the symmetry of translation in time, or in other words the homogeneity of time, implies conservation of energy.

### C. Rotation through a fixed angle

We now trace the implications of another symmetry, symmetry under rotation in space. If we rotate an experimental setup through a fixed angle, the experiment will yield the same result. If this symmetry were not true, a laboratory in New York would not be able to verify what is measured in another laboratory in Los Angeles. Indeed, repeating the experiment in New York must lead to the same results as the earth rotates. So physical laws should remain invariant with respect to rotation.

We use polar coordinates to determine which conservation law corresponds to this symmetry and consider the planar motion of a particle in a spherically symmetric potential field of energy $PE(r)$. As before, we consider the expression for the action along the infinitesimal segment. The definition (1b) shows that the action is equal to

Fig. 4. Following the same analysis as before, we conclude that the momentum at event 1 is the same as at event 3. The events 1 and 3 can be chosen arbitrarily. The arbitrariness of position of these events on the worldline implies the same value of momentum at every point (event) along the whole worldline of the moving object.
The increment $\Delta s$ is the length of a path segment traveled by the particle during the time interval $\Delta t$ and $r_{av}$ is the average position of the particle on this segment.

Consider three infinitesimally close points on the real path of a particle and approximate the real path by a broken line consisting of two infinitesimally small segments $A$ and $B$ (Fig. 5). (In this case we do not display a worldline because it would require curves in three-dimensional space–time.) To find the required expression for the action in polar coordinates, we use the Pythagorean theorem. The infinitesimal lengths $\Delta s_A$ and $\Delta s_B$ of segments $A$ and $B$ are

\[
\Delta s_A^2 = \Delta r_A^2 + (r_A \Delta \phi_A)^2,
\]

\[
\Delta s_B^2 = \Delta r_B^2 + (r_B \Delta \phi_B)^2,
\]

where $\Delta r_A = r_2 - r_1$, $\Delta \phi_A = \phi_2 - \phi_1$, $\Delta r_B = r_3 - r_2$, and $\Delta \phi_B = \phi_3 - \phi_2$. If we substitute Eq. (18) into Eq. (17), we find values of the action for segments $A$, $B$:

\[
S(A) = \frac{1}{2} \cdot \frac{m}{\Delta t_A} \frac{\Delta r_A^2 + r_A^2 \Delta \phi_A^2}{\Delta t_A} - PE(r_{av}) \Delta t_A,
\]

\[
S(B) = \frac{1}{2} \cdot \frac{m}{\Delta t_B} \frac{\Delta r_B^2 + r_B^2 \Delta \phi_B^2}{\Delta t_B} - PE(r_{av}) \Delta t_B.
\]

Once again, note that the action for these two segments depends only on the difference in the $\phi$ coordinate, and not on the $x$ coordinate itself. As before, we conclude that neither $S(A)$ nor $S(B)$ will change as we increase all $\phi$ coordinates by a fixed angle $\Phi$, because $\phi_2 + \Phi - (\phi_1 + \Phi) = \phi_2 - \phi_1$. As a result, the motion of the particle is symmetrical with respect to a fixed change in angle $\phi$. Conservation of angular momentum which arises from this symmetry is derived as follows.

The condition of stationary action $S$ is expressed as:

\[
\frac{dS}{d\phi_2} = \frac{d[S(A) + S(B)]}{d\phi_2} = 0.
\]

We substitute Eq. (19) into Eq. (20), differentiate and do some rearrangement and obtain:

\[
\frac{r_A^2 \Delta \phi_A}{\Delta t_A} = \frac{r_B^2 \Delta \phi_B}{\Delta t_B}.
\]  

Equation (21) represents conservation of angular momentum $L$, so $L_A = L_B$. The rate of change of the angle is the angular velocity $\omega$. Thus Eq. (21) can be expressed as $m r_A^2 \omega_A = m r_B^2 \omega_B$.

A derivation dealing with the fixed change in the angle coordinate for all three events, similar to those of the previous cases of translations in space and time, yields

\[
\frac{dS}{d\phi_1} = \frac{dS}{d\phi_3},
\]

which immediately implies conservation of angular momentum (21). Moreover, the left-hand side of Eq. (22) can be interpreted as the angular momentum at point 1 and the right-hand side as the angular momentum at point 3. Angular momentum is conserved for the entire path. The result is that symmetry under rotation through a fixed angle implies conservation of angular momentum.

The condition that physical laws remain invariant with respect to rotation through a fixed angle is called the isotropy of space. That is, space has the same properties in every direction. Therefore conservation of angular momentum results from the isotropy of space.

D. Galilean transformation

Finally, we present a simple example of an interesting and very important symmetry: symmetry under uniform linear motion, known in classical mechanics as Galileo’s principle of relativity. We will be surprised to find that the classical action is not invariant under a Galilean transformation.

Consider again a free particle moving along the $x$ axis between closely adjacent events 1 and 2 as observed in a laboratory frame, where the action takes the form (2). The (slowly moving) rocket observer, moving with a velocity $v_{rel}$ with respect to the laboratory, calculates the particle’s action given by the same equation

\[
S'_{\text{for segment}} = \frac{1}{2} \cdot \frac{m}{t_2 - t_1} \frac{(x_2 - x_1)^2}{t_2 - t_1}.
\]

Here we use primes for rocket coordinates, not for the derivative. If we apply the Galilean transformation

\[
x' = x - v_{rel}t, \quad t' = t
\]

for the rocket coordinates to Eq. (23), we obtain the following form of the action $S'$ in the laboratory frame:

\[
S'_{\text{for segment}} = \frac{1}{2} \cdot \frac{m}{t_2 - t_1} (x_2 - x_1)^2 - v_{rel} m (x_2 - x_1) + \frac{1}{2} m v_{rel}^2 (t_2 - t_1).
\]

This form of action is not the same as Eq. (2). The action is not invariant under a Galilean transformation. Which action, $S$ in Eq. (2) or $S'$ in Eq. (25), governs the motion of the particle in the laboratory? Or is the Galilean transformation incorrect? According to Appendix A everything is consistent. The two actions $S$ and $S'$ differ by a function $F(x, t) = -v_{rel} mx$
+ \frac{1}{2}mv_{rel}^2$, so the mechanical laws are the same as determined by using $S$ as they are by using $S'$.

If we use slightly more general considerations, but reasoning similar to that employed previously, we can demonstrate that the corresponding conservation law to Galilean transformation (24) is related to the uniform motion of the center of mass.

IV. SYMMETRY AND CONSERVATION LAWS IN RELATIVITY

A. Action in relativity

We have shown that the classical action is not symmetrical with respect to uniform linear motion, but all laws of motion remain unchanged under a Galilean transformation. We believe that this asymmetry for the principle of least action is not accidental, but rather results from the fact that the Galilean transformation and Newton’s laws are only approximate laws of motion. Symmetry under uniform linear motion is a basic assumption of Einstein’s special relativity.

We consider the same free particle, but now we use the special theory of relativity. The action for linear segment between 1 and 2 has the form:

$$S_{\text{for segment}} = -mc^2 \left(1 - \frac{v^2}{c^2}\right)^{1/2} \Delta t,$$

(26)

where $c$ is the velocity of light, $\Delta t = t_2 - t_1$, and $v = (x_2 - x_1)/(t_2 - t_1)$. It can be seen from Eq. (26) that Newtonian mechanics is a special case of relativistic mechanics in the low-velocity limit ($v \ll c$):

$$S \approx -mc^2 \left(1 - \frac{1}{2} \frac{v^2}{c^2}\right) \Delta t = \frac{1}{2} mv^2 \Delta t - mc^2 \Delta t.$$

(27)

According to Appendix A, if we take $F(x,t) = -mc^2 t$, Eq. (27) will give the same laws of motion for a free particle as the classical Newtonian action in Eq. (2).

B. Lorentz transformation

Now we outline the symmetry argument connected to the relativistic Lorentz transformation which has the form ($c = 1$):

$$x = \gamma(x' + v_{rel} t'), \quad t = \gamma(t' + v_{rel} x'),$$

(28)

where $\gamma = 1/(1 - v^2_{rel})^{1/2}$. Here $v_{rel}$ has the same meaning as in Sec. III D. We express the action (26) along a segment of the worldline:

$$S_{\text{for segment}} = -m\left[(t_2 - t_1)^2 - (x_2 - x_1)^2\right]^{1/2}.$$

(29)

The expression in the square root is the particle’s proper time (wristwatch time) between the two events, which is easily verified to be an invariant under the Lorentz transformation. Hence the relativistic action is symmetrical under a transformation connected to uniform linear motion.

Noether’s theorem can be used also in relativity. The same procedure used in Sec. III can be repeated in special relativity to yield the laws of conservation of relativistic energy, momentum, and angular momentum:

$$\frac{m \Delta t}{\Delta \tau} = E_{\text{relativistic}} = \text{constant},$$

$$\frac{m v^2 \Delta \varphi}{\Delta \tau} = L_{\text{relativistic}} = \text{constant},$$

(30)

where $\Delta \tau$ is the particle’s proper time. As for the Lorentz transformation, there also exists a corresponding conservation law, but its derivation goes beyond the scope of this paper.

We see that the theory of relativity eliminates the asymmetry of the action under translation. The invariance of the action under all the transformations we have considered makes the theory of relativity a more beautiful and elegant theory than the Newtonian theory of classical mechanics.

If one uses the correct expression for the action (or proper time), the constants of motion also can be derived for general relativity without complicated or advanced mathematics.

V. SUMMARY

We have discussed the connection between symmetries and conservation laws provided by Noether’s theorem using only elementary calculus. This approach can be used to help familiarize students with the powerful consequences of symmetry in the physical world. In addition, students can see a unified and systematic approach to all the conservation laws, mediated by Noether’s theorem and the principle of least action.

All our considerations can be easily generalized to three dimensions. We note that all symmetries in this paper are one-parameter transformations, which provide the central conservation laws using the most common form of Noether’s theorem related to the invariance of the Lagrangian (see Appendix B). Reference 17 and the pedagogically oriented Refs. 18 and 19 give clear, elegant, and more mathematically precise (but much more mathematically oriented) applications of Noether’s theorem to particle dynamics.

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APPENDIX A: THE ADDITION OF CERTAIN TERMS TO THE ACTION HAS NO EFFECT ON THE LAWS OF MOTION

Think of two expressions for the action $S(12)$ and $S^*(12)$ for a given worldline between any two events 1 and 2 in space–time. Suppose that these two expressions are related to each other as

$$S^*(12) = S(12) + F(2) - F(1),$$

(31)

where $F$ is an arbitrary function that depends only on the space and time coordinates of a given event. For example, $F(1)$ could be the value of $F$ at the event 1. Then laws of
motion are the same for both forms of action. Why?

We answer this question by repeating the same procedure as for earlier symmetries, starting with three events 1, 2, and 3. If we apply Eq. (31), we obtain the following equations relating action $S$ and $S^*$ for segment 1–2 and 2–3:

$$
S^*(12) = S(12) + F(2) - F(1), \quad (32a)
$$

$$
S^*(23) = S(23) + F(3) - F(2). \quad (32b)
$$

The total action $S^*(123)$ is the sum of (32a) and (32b):

$$
S^*(123) = S(123) + F(3) - F(1). \quad (33)
$$

The two total actions $S^*$ and $S$ in Eq. (33) differ only in the difference in $F$ at the fixed events 3 and 1. If we change the space or time coordinate (generally $u_2$) of the middle event 2, this difference remains constant. So the minima of $S$ and $S^*$ yield the same position of event 2, or in other words, the first derivatives of $S$ and $S^*$ with respect to $u_2$ are the same (all other variables being fixed):

$$
\frac{dS^*}{du_2} = \frac{dS}{du_2}. \quad (34)
$$

According to Eq. (34), the principle of least action for $S^*$ gives the same particle’s path as in the case of $S$. The laws of motion are unchanged if an additive constant (the difference in an arbitrary function between final position and initial position of a particle) is added to the action.\(^\text{21}\)

**APPENDIX B: NOETHER’S THEOREM AND THE LAGRANGIAN**

Noether’s theorem determines the connection between constants of the motion and conditions of invariance of the action under different kinds of symmetry. The function KE-PE in Newtonian mechanics is called the Lagrangian and is denoted by the symbol $L$. So we can write $S_{\text{for segment}} = \Delta S = L\Delta t$. (Do not confuse the symbol $L$ for the action with the symbol $L$ for angular momentum used in Sec. III C.) If we discuss symmetry transformations such that time is transformed identically, $t^* = t$, or transformations involving a uniform time translation, $t^* = t + \tau$, where $\Delta t = \Delta t^*$, then the invariance of the Lagrangian implies the invariance of the action. Therefore, most textbooks state Noether’s theorem as: for each symmetry of the Lagrangian, there is a corresponding conserved quantity.
dent projects dealing with a special topic on the principle of least action in a semester quantum mechanics course for future teachers of physics at the Faculty of Science, P. J. Safárik University, Kosice, Slovakia. To obtain our materials and corresponding software, see (http://leastaction.topcities.com) (the mirror site (http://www.LeastAction.host.sk)) or see Edwin Taylor’s website: (http://www.eftaylor.com/leastaction.html), which also includes our newest, continually updated and expanded materials.


SYMMETRY AS INDIFFERENCE

Without meaning to appear sacrilegious, I like to think of symmetries as representing aspects of nature that its Creator didn’t care about. If the Creator cared about something, I feel sure that there would be some mechanism for letting us know that it had changed. But a symmetry is precisely something that you can change without knowing it has changed. Thus, in my opinion, in a certain sense the quantity being changed is unimportant. Symmetries are principles of indifference.


Submitted by Alan DeWeerd.