Symmetries and conservation laws: Consequences of Noether's theorem

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We derive conservation laws from symmetry operations using the principle of least action. These derivations, which are examples of Noether's theorem, require only elementary calculus and are suitable for introductory physics. We extend these arguments to the transformation of coordinates due to uniform motion to show that a symmetry argument applies more elegantly to the Lorentz transformation than to the Galilean transformation.

I. INTRODUCTION

"It is increasingly clear that the symmetry group of nature is the deepest thing that we understand about nature today." (Steven Weinberg).[1] Many of us have heard statements like: For each symmetry operation there is a corresponding conservation law. The conservation of momentum is related to the homogeneity of space. Invariance under translation in time means that the law of conservation of energy is valid. Such statements come from one of the most amazing and useful theorems in physics, known as *Noether's theorem*.

When the German mathematician Emmy Noether proved her theorem,[2,3] she uncovered the fundamental justification for conservation laws. This theorem tells us that conservation laws follow from the symmetry properties of nature. Symmetries (called "principles of simplicity" in Ref. 1) can be regarded as a way of stating the deepest properties of nature. Symmetries limit the possible forms of new physical laws. However, the deep connection between symmetry and conservation laws requires the existence of a minimum principle in nature: the principle of least action. In classical mechanics, symmetry arguments are developed using high level mathematics. On the other hand, the corresponding physical ideas are often much easier to understand than mathematical ones.

In this paper we give an elementary introduction to and explanation of the relation between symmetry arguments and central conservation laws, as mediated by the principle of least action. We shall use only elementary calculus, so that our considerations can be used in introductory university physics classes.

Because the paper deals mainly with symmetry, it is important how we define or characterize this concept in the framework of introductory physics. We have decided to take Feynman's simple description of symmetry from his lectures on physics[4] which says that anything is symmetrical if one can subject it to a certain operation and it appears exactly the same after the operation.

Like Feynman, we will concentrate on symmetry in physical laws. The question is what can be done to a physical law so that this law remains the same? Noether's theorem derives

Paper accepted in American Journal of Physics (Received 30 December 2002; accepted 23 May 2003)

conservation laws from symmetries under the assumption that the principle of least action is the basic law that governs the motion of a particle in classical mechanics. This principle can be phrased as follows: "The action is a minimum for the path (worldline) taken by the particle."[5], which leads to the reformulation of our basic question about symmetry: What changes can we make in the worldline that do not lead to changes in either the magnitude or the form of the action?

We will explore and apply symmetry operations to the action along an infinitesimally small path segment. Because the action is additive, conclusions reached about a path segment apply to the entire path. The simplest examples of symmetry show the independence of the action on the difference in some quantity such as position, time, or angle.[6] When such a symmetry exists, Noether's theorem tells us that a physical quantity corresponding to this symmetry is a constant of the motion that does not change along the entire path of the particle.[7] The existence of such a constant implies a conservation law, which we then need to identify.

Section II briefly describes our software that helps students study action and its connection to conservation laws. Section III analyzes four examples of symmetry operations: translation in space and time, rotation through a fixed angle, and symmetry under uniform linear motion, namely the Galilean transformation. The first three symmetries lead to three conservation laws: momentum, energy, and angular momentum. Section IV extends the analysis to symmetry in relativity, showing that these conservation laws exist in that realm. Moreover, for uniform linear motion the symmetry argument applies more elegantly to the Lorentz transformation than to the Galilean transformation.

In the following we often talk about variations or changes in the action. Consistent with standard practice, we will be only interested in variations representing infinitesimal first-order changes in the action. To keep the arguments simple, we also assume that the particle's invariant mass m ("rest mass") does not change during the motion to be studied.

II. SOFTWARE

We start with the well-known definition of action for a particle of mass m that moves from some initial position at time t_1 to some final position at time t_2:

$$S = \int_{t_1}^{t_2} \left(\mathrm{KE} - \mathrm{PE}\right) \mathrm{d}t\ , \tag{1a}$$

or equivalently

$$S = (\mathrm{KE}_{\mathrm{av}} - \mathrm{PE}_{\mathrm{av}})\ (t_2 - t_1). \tag{1b}$$

Here $\mathrm{KE}_{\mathrm{av}}$ denotes the time averaged kinetic energy and $\mathrm{PE}_{\mathrm{av}}$ the time averaged potential energy between t_1 and t_2. We use the notation KE and PE as symbols for kinetic and potential energies respectively, because they are more mnemonic than the traditional symbols T and V.

Action is not a familiar quantity[8] for many students, so we employ an interactive computer program[9] to help them develop an intuition about action and the principle of least action. By using an interactive computer display, the student can not only explore the operation of the principle of least action, but also study the relation between this principle and conservation laws in specific cases (Fig. 1). In carrying out this manipulation, the student naturally works with the central concepts of the worldline (a graph of the time dependence of a particle's position) and an event (a point on a worldline). Unlike the path (trajectory in space), the worldline specifies completely the motion of a particle. For background in the symmetry properties of nature, students read a selection from Ref. 10.

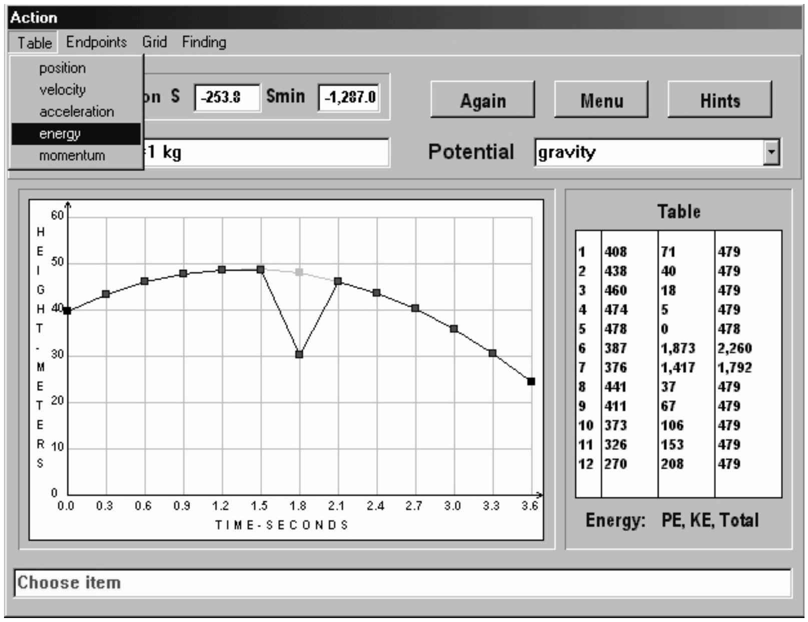

FIG. 1. The use of software helps students study the action along a worldline for a particle moving vertically in a gravitational field (as shown) or in other conservative potentials. The user clicks on events to create a worldline and then drags the events to minimize the action, which the computer continuously calculates and displays. The computer also displays a table of energy, momentum or other quantities that demonstrate conservation of these quantities. Students discover that for the worldline of minimum action, momentum is conserved for the motion of a free particle and that in a gravitational field total energy is conserved.

III. SYMMETRY AND CONSERVATION LAWS IN NEWTONIAN MECHANICS

A. Translation in space

We first examine the symmetry related to translation in space. When we perform an experiment at some location and then repeat the same experiment with identical equipment at another location, then we expect the results of the two experiments to be the same. So the physical laws should be symmetrical with respect to space translation.

As a simple example, consider the action of a free particle (in zero potential or uniform potential) moving along the x-axis between two events 1 $[t_1, x_1]$ and 2 $[t_2, x_2]$ infinitesimally close to one another along its worldline. Because the worldline section is considered to be straight, the particle moves at constant velocity $v = (x_2 - x_1)/(t_2 - t_1)$ and therefore with a constant kinetic energy $(1/2)mv^2$. According to Eq. (1b), the action along this straight segment in zero potential is (the consideration for uniform potential is analogous)

$$S_{\text{for segment}} = \frac{1}{2} m \frac{(x_2 - x_1)^2}{(t_2 - t_1)} . \tag{2}$$

If we change the positions of both observed events by a fixed displacement a, the action remains unchanged (invariant), because the value of the action depends only on the difference between the positions: $x_2 + a - (x_1 + a) = x_2 - x_1$. The principle of least action is symmetrical with respect to a fixed displacement of the position. Noether's theorem implies that this symmetry is connected with some conservation law. In the following we demonstrate that the conservation law related to symmetry under space translation is conservation of momentum.

1. Principle of least action and momentum

Think of the motion of a free particle along x-axis. To explore the connection between the principle of least action and the conservation of momentum, we take advantage of the additive property of the action to require that the action along an arbitrary infinitesimal section of the true worldline have a minimal value.[11] Thus we consider three successive infinitesimally close events, 1, 2, and 3 on the particle's worldline and approximate a real worldline by two connected straight segments, A and B (see Fig. 2).

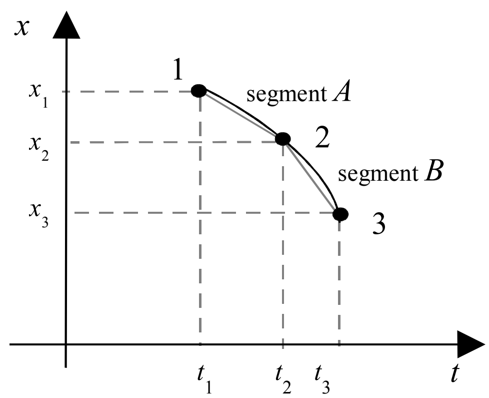

FIG. 2. Segment of the worldline of a particle that passes through three infinitesimally close events, for which every smooth curve can be approximated by two connected straight segments.

Because we are considering translation in space, we fix the first and last events 1 and 3 and change the space coordinate x_2 of the middle event 2 so as to minimize the value of the total action S. This minimum condition corresponds to a zero value of the derivative of S with respect to x_2:

$$\frac{\mathrm{d}S}{\mathrm{d}x_2} = 0 . \tag{3}$$

Because the action is an additive quantity, the total action equals the sum of the actions for segment A and B, so $S = S(A) + S(B)$. If we use Eq. (2), we can write

$$S = \frac{1}{2} m \frac{(x_2 - x_1)^2}{(t_2 - t_1)} + \frac{1}{2} m \frac{(x_3 - x_2)^2}{(t_3 - t_2)} . \tag{4}$$

If we perform in Eq. (4) the derivative indicated in Eq. (3), we derive the condition:

$$m\frac{(x_2 - x_1)}{(t_2 - t_1)} = m\frac{(x_3 - x_2)}{(t_3 - t_2)}. \tag{5}$$

The expression on the left side of Eq. (5) is the momentum p_A for segment A while the expression on the right side is the momentum p_B for segment B, so $p_A = p_B$. We could continue and add other segments C, D, E … to cover the entire worldline that describes the particle motion. For all these segments the momentum will have the same value, which yields the conservation law of momentum. The action for this free particle depends only on the change of the coordinate x and the result of this dependence is the conservation of the particle's momentum.

However, this derivation uses only the displacement of one event on the worldline. Therefore we have not yet demonstrated the relation between the conservation of momentum and the symmetry of translation in space in which all three events are displaced.

2. Symmetry and the conservation of momentum

Now we show the straightforward relation between the symmetry of translation in space and conservation of momentum. Again consider three infinitesimally close events on the worldline $x(t)$ of the free particle shown in Fig. 3 (the extension to the entire worldline will be discussed later).

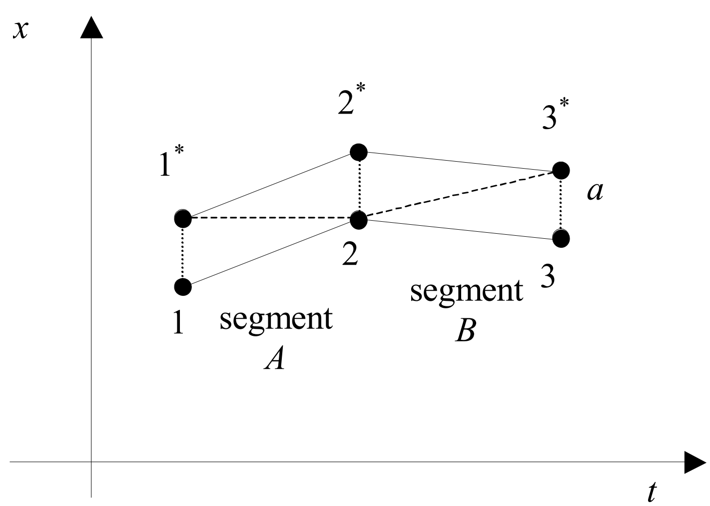

FIG. 3. Three infinitesimally close events 1, 2, 3 on the actual worldline. We shift this worldline through a fixed infinitesimal displacement a. An arbitrary displacement can be composed from a sequence of such infinitesimal displacements.

We shift the worldline $x(t)$ so that every event changes its position by a fixed infinitesimal displacement a. The new events create a shifted worldline that we indicate by an asterisk: $x^*(t)$. As pointed out previously, the form of the action for $x^*(t)$ remains unchanged and does not depend on the parameter a. Thus the change in action with respect to the displacement a is zero:

$$\Delta_a S \equiv S(1^*2^*3^*) - S(123) \equiv 0. \tag{6}$$

Note that the worldline $x^*(t)$ is just as valid as the original one. Therefore the worldline $x^*(t)$ also obeys the principle of least action. In translating from $x(t)$ to $x^*(t)$ we do not need to shift all the events simultaneously. The same effect is obtained if we first change the position of event 1 (in Fig. 3 only coordinate x_1 changes, which creates the worldline 1^*23), then event 3 (only x_3 changes, which creates 1^*23^*) and finally event 2 (only x_2 changes, which creates $1^*2^*3^*$). The total change in action for displacement a can be written as:

$$\Delta_a S = \Delta S_{1\to 1^*} + \Delta S_{3\to 3^*} + \Delta S_{2\to 2^*}\,, \tag{7}$$

where $\Delta S_{1\to 1^*}$, $\Delta S_{2\to 2^*}$, $\Delta S_{3\to 3^*}$ denotes the changes in the action after the shifts in the corresponding events.

Equation (6) tells us that $\Delta_a S$ is always zero. The final change $\Delta S_{2\to 2^*}$ must also be zero, from the principle of least action applied to the new worldline. Hence Eqs. (6) and (7) give:

$$-\Delta S_{1\to 1^*} = \Delta S_{3\to 3^*}\,. \tag{8}$$

If we now calculate the changes in the action in Eq. (8), we obtain the conservation law of momentum. Because the displacement a is infinitesimal, we can write:

$$\Delta S_{1\to 1^*} \equiv S(1^*23) - S(123) = \frac{\mathrm{d}S}{\mathrm{d}x_1}a \tag{9a}$$

$$\Delta S_{3\to 3^*} \equiv S(1^*23^*) - S(1^*23) = \frac{\mathrm{d}S}{\mathrm{d}x_3}a\,. \tag{9b}$$

If we substitute Eq. (9) into Eq. (8) and use the fact that the fixed infinitesimal displacement a is arbitrary, we have:[12]

$$-\frac{\mathrm{d}S}{\mathrm{d}x_1} = \frac{\mathrm{d}S}{\mathrm{d}x_3}\,. \tag{10}$$

The application of the derivatives in Eq. (10) to the expression for the action in Eq. (4) yields the identical result for a free particle as Eq. (5), but this time as a result of spatial translation of the entire incremental worldline segment. Thus the left side of Eq. (10) can also be interpreted as the momentum at event 1 and the right side as the momentum at event 3.

The preceding considerations can be applied to the entire worldline $x(t)$. We did not specify the location of the segments A and B. Therefore an arbitrary number of additional segments can be added between them. Then we shift segments as before (see Fig. 4). By the same analysis we conclude that the momentum for segment A (effectively the momentum at event 1) has the same value as for segment B (effectively at event 3). Arbitrariness of position of these segments on the worldline means that the value of the momentum remains constant at every event on the worldline. Thus, in classical mechanics, the symmetry of spatial translation means that momentum is conserved for a free particle.

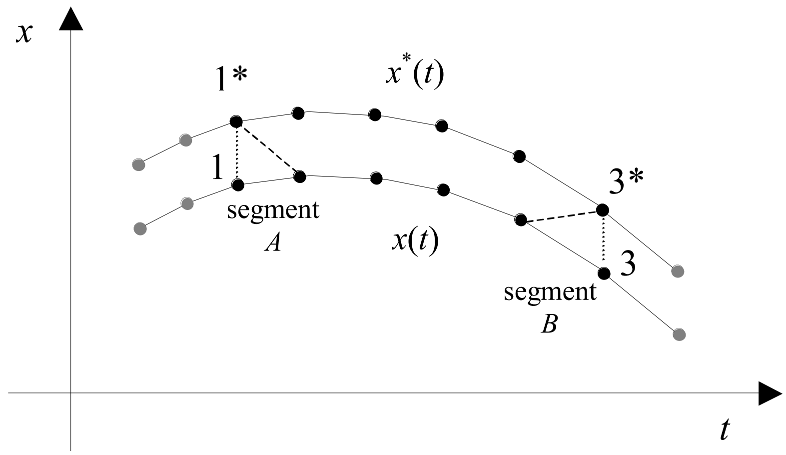

FIG. 4. Following the same analysis as before, we conclude that the momentum at event 1 is the same as at event 3. The events 1 and 3 can be chosen arbitrarily. The arbitrariness of position of these events on the worldline implies the same value of momentum at every point (event) along the whole worldline of the moving object.

The invariance of the action with respect to translation in space is also called the homogeneity of space, which means that all points in space are equivalent as the origin of our reference frame. In other words, it does not matter where an experiment is performed. Therefore we can briefly state that the law of momentum conservation results from the homogeneity of space.

B. Translation in time

It is easy to envision the symmetry related to translation in time. Repeating an experiment on identical initial systems yields the same result when the two experiments are separated by a lapse of time. Our conclusion is that physical laws should not change with translation in time.

Again we will show the relation of translation in time symmetry to a relevant conservation law. We start with an expression for the action of a particle moving in the x-direction along an infinitesimally small worldline segment in a potential field described by PE(x). As in Sec. IIIA the action for this segment can be written according to Eq. (1b) as

$$S_{\text{for segment}} = \frac{1}{2} m \frac{(x_2 - x_1)^2}{(t_2 - t_1)} - \mathrm{PE}\left(\frac{x_1 + x_2}{2}\right)(t_2 - t_1), \tag{11}$$

where the potential energy is evaluated at the average position along the segment. Now suppose that we translate the time t by an amount τ. It is easy to see that the action will not change, because only the difference of the time, $t_2 + \tau - (t_1 + \tau) = t_2 - t_1$, appears in the equation for the action. So the action is symmetrical with respect to a fixed displacement of time t. What conservation law is related to this time symmetry? We will show that it is conservation of energy.

We follow the same line of reasoning as for the case of translation in space, but now we fix all position and time coordinates with the exception of t_2. Think of a particle that moves along the x-axis in the potential field with potential energy PE(x). To simplify the algebra, we denote space and time differences by:

$$\begin{aligned}\Delta x_A = x_2 - x_1 \;\; ; \;\; \Delta x_B = x_3 - x_2 \\ \Delta t_A = t_2 - t_1 \;\; ; \;\; \Delta t_B = t_3 - t_2\end{aligned} \; . \tag{12}$$

According to Eqs. (11) and (12), the values of the actions $S(A)$ and $S(B)$ for segments A and B are equal to:

$$S(A) = \frac{1}{2} m \frac{\Delta x_A{}^2}{\Delta t_A} - \mathrm{PE}\left(\frac{x_2 + x_1}{2}\right)\Delta t_A \tag{13a}$$

$$S(B) = \frac{1}{2} m \frac{\Delta x_B{}^2}{\Delta t_B} - \mathrm{PE}\left(\frac{x_3 + x_2}{2}\right)\Delta t_B \, . \tag{13b}$$

The principle of least action leads to the following condition for the total action S:

$$\frac{\mathrm{d}S}{\mathrm{d}t_2} = \frac{\mathrm{d}\left[S(A) + S(B)\right]}{\mathrm{d}t_2} = 0 \, . \tag{14}$$

If we substitute Eq. (13) into Eq. (14), differentiate, and rearrange, we obtain:

$$\frac{1}{2} m \frac{\Delta x_A{}^2}{\Delta t_A{}^2} + \mathrm{PE}\left(\frac{x_2 + x_1}{2}\right) = \frac{1}{2} m \frac{\Delta x_B{}^2}{\Delta t_B{}^2} + \mathrm{PE}\left(\frac{x_3 + x_2}{2}\right). \tag{15}$$

The expressions on both sides of Eq. (15) are sums of average kinetic and potential energies. For infinitesimally close events, Eq. (15) gives an equality for the instantaneous values $(1/2)mv_A^2 + \mathrm{PE}_A = (1/2)mv_B^2 + \mathrm{PE}_B$, and expresses the conservation of mechanical energy.

Next we carry out an argument that translates all three times t_1, t_2, and t_3 by the same amount τ, similar to the way we translated positions for the momentum case. Equations (6), (7) and (8) apply to the present case as well, and also Eq. (9) when the derivatives are taken with respect to time rather than position. Then the result of the temporal translation is an equation similar to Eq. (10):

$$-\frac{\mathrm{d}S}{\mathrm{d}t_1} = \frac{\mathrm{d}S}{\mathrm{d}t_3} \, , \tag{16}$$

which yields Eq. (15) multiplied by (−1). We again obtain conservation of energy, but this time as a result of symmetry under time translation. For infinitesimally close events, the left side of Eq. (16) also can be interpreted as the negative of the total energy at event 1 and the right side as the negative of the energy at event 3. The energy is a constant of the motion for the entire worldline $x(t)$. Similarly as in the last paragraph of Sec. IIIA, we can say that the symmetry of translation in time, or in other words the homogeneity of time, implies conservation of energy.

C. Rotation through a fixed angle

We now trace the consequences of another symmetry, symmetry under rotation in space. If we rotate an experimental setup through a fixed angle, the experiment will yield the same result. If this symmetry were not true, a laboratory in New York would not be able to verify what is measured in another laboratory in Los Angeles. Indeed, repeating the experiment in New York must lead to the same results as the earth rotates. So physical laws should remain invariant with respect to rotation.

We use polar coordinates to determine which conservation law corresponds to this symmetry and consider the planar motion of a particle in a spherically symmetric potential field of energy PE(r). As before, we consider the expression for the action along the infinitesimal segment. The definition (1b) shows that the action is equal to:

$$S_{\text{for segment}} = \frac{1}{2} m \frac{\Delta s^2}{\Delta t} - \text{PE}(r_{\text{av}})\Delta t \,. \tag{17}$$

The increment Δs is the length of a path segment traveled by the particle during the time interval Δt and r_{av} is the average position of the particle on this segment.

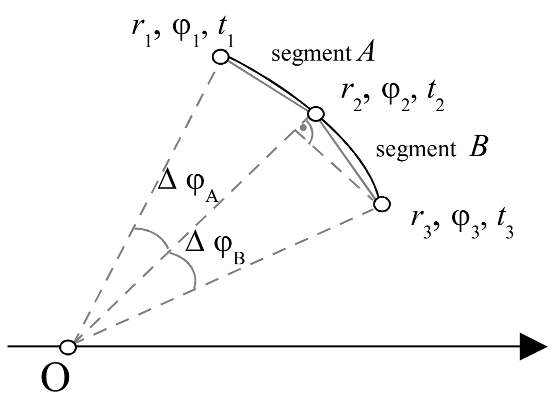

FIG.5 Path segment of planar motion with three infinitesimally close points whose positions are described by polar coordinates. The radius r_A (r_B) represents the average position of the particle on segment A (B). All coordinates of the points 1, 2, 3 are fixed with the exception of the angle coordinate φ_2, which we vary to satisfy the principle of least action

Consider three infinitesimally close points on the real path of a particle and approximate the real path by a once-broken line consisting of two infinitesimally small segments A and B (Fig. 5). (In this case we do not display a worldline because it would require curves in three-dimensional spacetime.) To find the required expression for the action in polar coordinates, we use the Pythagorean theorem. The infinitesimal lengths Δs_A and Δs_B of segments A and B are

$$\begin{aligned} \Delta s_A{}^2 &= \Delta r_A{}^2 + (r_A \Delta\varphi_A)^2 \\ \Delta s_B{}^2 &= \Delta r_B{}^2 + (r_B \Delta\varphi_B)^2 \end{aligned} \tag{18}$$

where $\Delta r_A = r_2 - r_1$, $\Delta\varphi_A = \varphi_2 - \varphi_1$, $\Delta r_B = r_3 - r_2$, and $\Delta\varphi_B = \varphi_3 - \varphi_2$. If we substitute Eq. (18) into Eq. (17), we find values of the action for segments A, B:

$$S(A) = \frac{1}{2} m \frac{\Delta r_A{}^2 + r_A^2 \Delta\varphi_A{}^2}{\Delta t_A} - \text{PE}(r_A)\Delta t_A \tag{19a}$$

$$S(B) = \frac{1}{2} m \frac{\Delta r_B{}^2 + r_B^2 \Delta\varphi_B{}^2}{\Delta t_B} - \text{PE}(r_B)\Delta t_B \,. \tag{19b}$$

Once again, note that the action for these two segments depends only on the difference in the φ-coordinate, and not on the φ-coordinate itself. As before, we conclude that neither $S(A)$ nor $S(B)$ will change as we increase all φ-coordinates by a fixed angle Φ, because $\varphi_2 + \Phi - (\varphi_1 + \Phi) = \varphi_2 - \varphi_1$. As a result, the motion of the particle is symmetrical with respect to a fixed change in angle φ. Conservation of angular momentum which arises from this symmetry is derived as follows.

The condition of stationary action S is expressed as:

$$\frac{\mathrm{d}S}{\mathrm{d}\varphi_2} = \frac{\mathrm{d}\left[S(A) + S(B)\right]}{\mathrm{d}\varphi_2} = 0\,. \tag{20}$$

Substituting Eq. (19) into Eq. (20), differentiating and doing some rearrangement, we obtain:

$$m\frac{r_A^2 \Delta\varphi_A}{\Delta t_A} = m\frac{r_B^2 \Delta\varphi_B}{\Delta t_B}\,. \tag{21}$$

Equation (21) represents the conservation law of the angular momentum L, so $L_A = L_B$. The rate of change of the angle is the angular velocity ω. Thus Eq. (21) can be expressed as $mr_A^2\omega_A = mr_B^2\omega_B$.

A derivation similar to those of the previous cases of translations in space and time yields

$$-\frac{\mathrm{d}S}{\mathrm{d}\varphi_1} = \frac{\mathrm{d}S}{\mathrm{d}\varphi_3}\,, \tag{22}$$

which immediately implies conservation of angular momentum (21). Moreover, the left side of Eq. (22) can be interpreted as the angular momentum at point 1 and the right side as the angular momentum at point 3. Angular momentum is conserved for the entire path. The result is that symmetry under rotation through a fixed angle implies conservation of angular momentum.

The condition that physical laws remain invariant with respect to rotation through a fixed angle is called the isotropy of space. That is, space has the same properties in every direction. Therefore conservation of angular momentum results from the isotropy of space.

D. Galilean transformation

Finally, we present a simple example of an interesting and very important symmetry: symmetry under uniform linear motion, known in classical mechanics as Galileo's principle of relativity. We will be surprised to learn that the classical action is not invariant under a Galilean transformation.

Consider again a free particle moving along the x-axis between closely adjacent events 1 and 2 as observed in a laboratory frame, where the action takes the form (2). The (slowly-moving) rocket observer, moving with a velocity v_{rel} with respect to the laboratory, calculates particle's action given by the same equation

$$S'_{\text{for segment}} = \frac{1}{2}m\frac{(x_2' - x_1')^2}{t_2' - t_1'}\,. \tag{23}$$

Here we use primes for rocket coordinates, not for the derivative. If we apply the Galilean transformation

$$\begin{aligned} x' &= x - v_{rel} t \\ t' &= t \end{aligned}, \tag{24}$$

for the rocket coordinates to Eq. (23), we obtain the following form of the action S' in the laboratory frame:

$$S'_{\text{for segment}} = \frac{1}{2} m \frac{(x_2 - x_1)^2}{t_2 - t_1} - v_{rel} m (x_2 - x_1) + \frac{1}{2} m v_{rel}^2 (t_2 - t_1)\,. \tag{25}$$

This form of action is not the same as Eq. (2). The action is not invariant under a Galilean transformation. Which action, S in Eq. (2) or S' in Eq. (25), governs the motion of the particle in the laboratory? Or is the Galilean transformation incorrect? According to Appendix A everything is all right. The two actions S and S' differ by a function that depends only on the coordinates of a given event $F(x,t) = -v_{rel} m x + \frac{1}{2} m v_{rel}^2 t$, so the mechanical laws are the same as determined by using S as they are by using S'.

If we use slightly more general considerations, but reasoning similar to that employed previously,[13] we can demonstrate that the corresponding conservation law to Galilean transformation (24) is related to the uniform motion of the center of mass.

IV. SYMMETRY AND CONSERVATION LAWS IN RELATIVITY

A. Action in Relativity

We have shown that the classical action is not symmetrical with respect to uniform linear motion, but all laws of motion remain unchanged under a Galilean transformation. We believe that this asymmetry for the principle of least action is not accidental, but rather results from the fact that the Galilean transformation and Newton's laws are only approximate laws of motion. Symmetry under uniform linear motion is a basic assumption of Einstein's special relativity.

We consider the same free particle, but now we use the special theory of relativity. The action for linear segment between 1 and 2 has the form:[14]

$$S_{\text{for segment}} = -mc^2 \left(1 - \frac{v^2}{c^2}\right)^{1/2} \Delta t\,, \tag{26}$$

where c is the velocity of light, $\Delta t = t_1 - t_2$, and $v = (x_2 - x_1)/(t_2 - t_1)$. It can be seen from Eq. (26) that Newtonian mechanics is a special case of relativistic mechanics in the low-velocity limit ($v \ll c$):

$$S \approx -mc^2 \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) \Delta t = \frac{1}{2} m v^2 \Delta t - mc^2 \Delta t \tag{27}$$

According to Appendix A, if we take $F(x,t) = -mc^2 t$, Eq. (27) will give the same laws of motion for a free particle as the classical Newtonian action in Eq. (2).

B. Lorentz transformation

Now we outline the symmetry argument connected to the relativistic Lorentz transformation which has the form (c = 1):

$$\begin{aligned} x &= \gamma(x' + v_{rel} t') \\ t &= \gamma(t' + v_{rel} x') \end{aligned} , \qquad (28)$$

where $\gamma = 1/\left(1 - v_{rel}^2\right)^{1/2}$. Here v_{rel} has the same meaning as in Sec. IIID. We express the action (26) along a segment of the worldline:

$$S_{\text{for segment}} = -m\left[(t_2 - t_1)^2 - (x_2 - x_1)^2\right]^{1/2}. \qquad (29)$$

The expression in the square root is the particle's proper time (wristwatch time) between the two events, which is easily verified to be an invariant under the Lorentz transformation. Hence the relativistic action is symmetrical under a transformation connected to uniform linear motion.

Noether's theorem can be used also in relativity. The same procedure used in Sec. III, can be repeated in special relativity to yield the laws of conservation of relativistic energy, momentum and angular momentum:

$$\begin{aligned} \frac{m\Delta x}{\Delta\tau} &= p_{relativistic} = \text{constant} \\ \frac{m\Delta t}{\Delta\tau} &= E_{relativistic} = \text{constant} \\ \frac{mr^2\Delta\varphi}{\Delta\tau} &= L_{relativistic} = \text{constan} \end{aligned}$$

(30)

where $\Delta\tau$ is the particle's proper time. As for the Lorentz transformation, there also exists a corresponding conservation law, but its derivation goes beyond the scope of this paper.[15]

We see that the theory of relativity eliminates the asymmetry of the action under translation. The invariance of the action under all the transformations we have considered makes the theory of relativity a more beautiful and elegant theory than the Newtonian theory of classical mechanics.

If one uses the correct expression for the action (or proper time), the constants of motion also can be derived for general relativity without complicated or advanced mathematics.[16]

V. SUMMARY

We have discussed the connection between symmetries and conservation laws provided by Noether's theorem using only elementary calculus. This approach can be used to help familiarize students with the powerful consequences of symmetry in the physical world. In addition, students can see a unified and systematic approach to all the conservation laws, mediated by Noether's theorem and the principle of least action.

All our considerations can be easily generalized to three dimensions. We note that all symmetries in this paper are one-parameter transformations, which provide the central conservation laws using the most common form of Noether's theorem related to the invariance of the Lagrangian (see Appendix B). Reference 17 and the pedagogically oriented Refs. 18 and 19 give clear, elegant, and more mathematically precise (but much more mathematically oriented) applications of Noether's theorem to particle dynamics.

ACKNOWLEDGMENTS

This paper was written after we read Taylor and Wheeler's general relativity book[16] sent to us along with other materials by Taylor, who also made very helpful suggestions for this paper. The authors also wish to thank Nilo C. Bobillo Ares for helpful advice. Slavko Chalupka provided important discussion and encouragement and gave us the opportunity to teach an experimental course in quantum mechanics using some of these ideas.[20]

APPENDIX A: THE ADDITION OF CERTAIN TERMS TO THE ACTION HAS NO EFFECT ON THE LAWS OF MOTION

Think of two expressions for the action $S(12)$ and $S^*(12)$ for a given worldline between any two events 1 and 2 in spacetime. Suppose that these two expressions are related to each other in the following way:

$$S^*(12) = S(12) + F(2) - F(1), \tag{31}$$

where F is an arbitrary function that depends only on the space and time coordinates of a given event. For example, $F(1)$ could be the value of F at the event 1. Then laws of motion are the same for both forms of action. Why?

We answer this question by repeating the same procedure as for earlier symmetries, starting with three events 1, 2, 3. If we apply Eq. (31), we obtain the following equations relating action S and S^* for segment 1-2 and 2-3:

$$S^*(12) = S(12) + F(2) - F(1) \tag{32a}$$

$$S^*(23) = S(23) + F(3) - F(2)\,. \tag{32b}$$

The total action $S^*(123)$ is the sum of (32a) and (32b):

$$S^*(123) = S(123) + F(3) - F(1)\,. \tag{33}$$

The two total actions S^* and S in Eq. (33) differ only in the difference in F at the fixed events 3 and 1. If we change the space or time coordinate (generally u_2) of the middle event 2, this difference remains constant. So the minima of S and S^* yield the same position of event 2, or in other words, the first derivatives of S and S^* with respect to u_2 are the same (all other variables being fixed):

$$\frac{\mathrm{d}S^*}{\mathrm{d}u_2} = \frac{\mathrm{d}S}{\mathrm{d}u_2} \tag{34}$$

According to Eq. (34), the principle of least action for S^* gives the same particle's path as in the case of S. The laws of motion are unchanged if an additive constant (the difference in an arbitrary function between final position and initial position of a particle) is added to the action.[21]

APPENDIX B: NOETHER'S THEOREM AND THE LAGRANGIAN

Noether's theorem determines the connection between constants of the motion and conditions of invariance of the action under different kinds of symmetry. The function KE – PE in Newtonian mechanics is called the Lagrangian and is denoted by the symbol L. So we can write $S_{\text{for segment}} \equiv \Delta S = L\ \Delta t$ (Do not confuse the symbol L for the action with the symbol L for angular momentum used in Sec. IIIC.) If we discuss symmetry transformations such that time is transformed identically, $t^* = t,$ or transformations involving a uniform time translation:, $t^* = t + \tau$, where $\Delta t = \Delta t^*$, then the invariance of the Lagrangian implies the invariance of the action. Therefore, most textbooks state Noether's theorem as: for each symmetry of the Langrangian, there is a corresponding conserved quantity.

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[1]R. P. Feynman and S. Weinberg, *Elementary Particles and the Laws of Physics* (Cambridge University Press, 1999), p.73.

[2]In reality, there are two Noether's theorems and their converses. The first one refers to the invariance of action with respect to a group of symmetries where symmetry transformations depend analytically on finitely many arbitrary parameters. The second theorem deals with the invariance of action with respect to such group where transformations depend on arbitrary functions and their derivatives instead of on the arbitrary parameters. Our paper considers one-parameter symmetry transformations. Therefore it is connected with the first theorem. See E. Noether, "Invariante Variationsprobleme," Nachr. v. d. Ges. d. Wiss. zu Göttingen, Math-phys. Klasse (1918), pp. 235-257; English translation by M. A. Tavel, "Invariant variation problem," Transport Theory and Statistical Mechanics **1** (3), 183-207 (1971); all available at <http://www.physics.ucla.edu/~cwp/Phase2/Noether,_Amalie_Emmy@861234567.html>

[3]N. Byers, "E. Noether's discovery of the deep connection between symmetries and conservation laws," in Israel Mathematical Conference Proceedings **12**, 67-82 (1999).

[4]R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1963), Vol. I, Chap. 11, p. 11-1 or Chap. 52, p. 52-1.

[5]More accurately, the principle says that a particle moves along that path for which the action has a stationary value. So it is frequently and correctly called the principle of stationary action; See more in I.M. Gelfand, S.V. Fomin, *Calculus of variations*, (Prentice-Hall, New Jersey, 1963), Sec. 32.2 or D. J. Morin, <http://www.courses.fas.harvard.edu/~phys16/handouts/textbook/ch5.pdf >, Chap. 5.

[6]Generally such a quantity is called a cyclic or ignorable coordinate; H. Goldstein, *Classical Mechanics* (Addison-Wesley, New York, 1970), p. 48 or Ref. 5.

[7]Every quantity that depends on position coordinates and velocities and whose value does not change along actual trajectories is called a constant of the motion.

[8]We recommend a more detailed described procedure for introducing action in J. Hanc, S. Tuleja, and M. Hancova, "Simple derivation of Newtonian mechanics from the principle of least action", Am. J. Phys. **70**, (4), 386-391 (2003).

[9]The idea of using computers comes from E. F. Taylor. See E. F. Taylor, S. Vokos, J. M. O ´Meara, and N. S. Thornber, “Teaching Feynman’s sum over paths quantum theory,” Comp. Phys. **12** (2), 190-199 (1998) or E. F. Taylor, *Demystifying Quantum Mechanics,* <http://www.eftaylor.com>. Our software is based on Taylor’s.

[10]R. P. Feynman, *The Character of Physical Law* (Random House Inc., New York, 1994), Chap. 4.

[11]Ref. 4, Vol. II, Chap. 19, p. 19-8 or the more detailed discussion in Ref. 8

[12]Strictly speaking, in these and the following cases we should use the more traditional notation of partial instead of total derivatives. But in all cases it is clear which coordinates are variable and which are fixed.

[13] In that case there is necessary to consider the invariance of the action up to an additive constant (the difference in any arbitrary function between final position and initial position of a particle), which will give conservation of motion of the center of mass. See also Refs. 19 or 17.

[14]The relativistic formula for the action is given in Ref.4, Vol. II, Chap. 19. We use the concept of invariance of mass that is used by E. F. Taylor and J. A. Wheeler in, *Spacetime Physics: Introduction to Special Relativity* (W. H. Freeman, New York, 1992), 2nd ed.

[15]The conservation law corresponding to the Lorentz transformation is derived in L. D. Landau and E. M. Lifshitz, *The Classical Theory of Field s* (Pergamon press, London, 1975), Vol. 2, pp. 41–42.

[16]E. F. Taylor and J. A. Wheeler, *Exploring Black Holes: An Introduction to General Relativity* (Addison-Wesley Longman, NY, 2000), Chaps. 1 and 4; also available at <http://www.eftaylor.com>. The authors use a very similar, easy, and effective variational method.

[17] P. Havas and J. Stachel, “Invariances of approximately relativistic Lagrangians and the center of mass theorem. I,” Phys. Rev. **185** (5), 1636-1647 (1969).

[18]N. C. Bobillo-Ares, “Noether’s theorem in discrete classical mechanics,” Am. J. Phys. **56** (2), 174-177 (1988).

[19]C. M. Giordano and A. R. Plastino, “Noether’s theorem, rotating potentials, Jacobi’s integral of motion,” Am. J. Phys. **66** (11), 989-995 (1998).

[20]The substance of this article was used by the authors as subjects for student projects dealing with a special topic on the principle of least action in a semester quantum mechanics course for future teachers of physics at the Faculty of Science, P.J. Safarik University, Kosice, Slovakia. To obtain our materials and corresponding software, see <http://leastaction.topcities.com> (the mirror site <http://www.LeastAction.host.sk>) or see Edwin Taylor’s website: <http://www.eftaylor.com/leastaction.html>, which also includes our newest, continually updated and expanded materials.

[21]L. D. Landau and E. M. Lifshitz, *Mechanics* (Butterworth-Heinemann, Oxford, 1976), Sec. 1.2.