CHAPTER 7

MOMENERGY

Every physical quantity is represented by a geometric object.

Theme of Herman Weyl

7.1 MOMENERGY: TOTAL CONSERVED IN A COLLISION

momentum conserved.
energy conserved.
***momenergy* conserved!**

Paradoxically, few examples of motion are more complicated than a collision, and few are simpler. The complication shows nowhere more clearly than in the slow-motion videotape of the smashup of two automobiles. Millisecond by millisecond the fender of one colliding car deforms another fraction of a centimeter. Millisecond by millisecond the radiator grille of the other car bends inward a little more on the way to total collapse: steel against steel, force against force, crumpling surface against crumpling surface. What could be more complex?

Smashup complicated?

For the drivers of the colliding cars the experience is shattering. They are hardly aware of noise and complicated damage. A single impression overpowers their senses: the inevitability of the crash. Call it what we will—inertia, momentum, the grip of spacetime on mass—something is at work that drives the two vehicles together as the frantic drivers jam their brake pedals down, locking the wheels as the cars slither over the glassy ice, crash into one another, then slide apart.

Smashup is simple!

Does mass lose its inertia during the collision? No. Inertia does its best to keep each car going as it was, to keep its **momentum** constant in magnitude and direction. Momentum: we can think of it loosely as an object's will to hold its course, to resist deflection from its appointed way. The higher the object's momentum, the more violently it hits whatever stands in its way. But the momentum of a single object is not all-powerful. The two vehicles exchange momentum. But spacetime insists and demands that whatever momentum one car gains the other car must lose. Regardless of all complications of detail and regardless of how much the momentum of any one

Momentum conservation simplifies description object may change, the combined momentum of the two objects remains constant: the total is unchanged in the collision. A like statement applies to energy, despite a conversion of energy of motion into heat energy and fender crumpling.

Energy too is conserved

A collision thus manifests a wonderful simplicity: the combination of the motion-descriptive quantities (momentum and energy) of the two colliding bodies does not change. That combination is identical before and after the collision. In a word, it is *conserved*. This conserved combination we call **momentum-energy** or, more briefly, **momenergy** (defined more carefully in Section 7.2). We will use the two terms interchangeably in this book.

Momenergy is conserved!

A collision cannot be elevated from mere talk to numbers without adopting, directly or indirectly, the principle of conservation of momentum and energy. In the enterprise of identifying the right numbers, using them, and understanding them, no concept is more powerful than what relativity smilingly holds forth: momenergy.

Wait a minute. Apparently you are going to find new expressions for momentum and energy, then combine them in some way to form a unity: momenergy. But I have three complaints. (1) What is wrong with what good old-fashioned secondary school physics textbooks give us, the Newtonian expressions for momentum — $p_{Newton} = mv_{conv}$ *— and kinetic energy* $K_{Newton} = \frac{1}{2}mv_{conv}^2$ *— where* v_{conv} *is expressed in conventional units, say meters/second? (2) Momentum and energy do not even have the same units, as these formulas make clear. How can you combine quantities with different units? (3) Momentum and energy are different things entirely; why try to combine them at all?*

Take your questions in order.

1. **Newtonian Expressions:** Only for slow-moving particles do we get correct results when we use Newtonian expressions for momentum and energy. For particle speeds approaching that of light, however, total energy and momentum of an isolated system, as Newton defined momentum and energy, are *not* conserved in a collision. In contrast, when momentum and energy are defined relativistically, then total momentum and total energy of particles in an isolated system *are* conserved, no matter what their observed speeds.
2. **Units:** It is easy to adopt identical units for momentum and energy. As a start we adopt identical units for space and time. Then the speed of a particle is expressed in unit-free form, v, in meters of distance per meter of light-travel time (Section 2.8). This choice of units, which we have already accepted earlier in this book, gives even Newtonian expressions for momentum — $p_{Newton} = mv$ — and kinetic energy — $K_{Newton} = \frac{1}{2}mv^2$ — the same unit: **mass.** These are not relativistic expressions, but they do agree in their units, and agree in units with the correct relativistic expressions.
3. **Momentum and Energy Different:** Yes, of course, momentum and energy are different. Space and time are different too, but their combination, spacetime, provides a powerful unification of physics. Space and time are put on an equal footing, but their separate identities are maintained. Same for momenergy: We will see that its "space part" is momentum, its "time part" energy. We will also discover that its magnitude is the mass of the particle, reckoned using the good ol', ever-lovin', familiar minus sign: $m^2 = E^2 - p^2$.

Thus relativity offers us a wonderful unity. Instead of three separate motion-descriptive quantities — momentum, energy, and mass — we have a single quantity: momenergy.

7.2 MOMENERGY ARROW

a spacetime arrow pointing along the worldline

What lies behind the name *momentum–energy (momenergy)?* What counts are its properties. We most easily uncover three central properties of momenergy by combining everyday observation with momenergy's essential feature: Total momenergy is conserved in any collision.

First, think of two pebbles of different sizes moving with the same velocity toward the windshield of a speeding car. One bounces off the windshield without anyone noticing; the other startles the occupants and leaves a scratch. Five times the mass? Five times the punch-delivering capacity! Five times the momenergy. Momenergy, in other words, is proportional to mass.

Momenergy of particle proportional to its mass

Second, momentum-energy of a particle depends on its direction of travel. A pebble coming from the front takes a bigger chip out of the windshield than a pebble of equal mass and identical speed glancing off the windshield from the side. Therefore momenergy is not measurable by a mere number. It is a directed quantity. Like an arrow of a certain length, it has magnitude and direction.

Momenergy a directed quantity

Our experience with the unity of spacetime leads us to expect that the momenergy arrow will have three parts, corresponding to three space dimensions, plus a fourth part corresponding to time. In what follows we find that momenergy is indeed a four-dimensional arrow in spacetime, the **momenergy 4-vector** (Box 7-1). Its three "space parts" represent the momentum of the object in the three chosen space directions. Its "time part" represents energy. The unity of momentum and energy springs from the unity of space and time.

Momenergy a 4-vector

In what direction does the momenergy 4-vector of a particle point? It points in the "same direction in spacetime" as the worldline of the particle itself (Figure 7-1). There is no other natural direction in which it *can* point! Spacetime itself has no structure that indicates or favors one direction rather than another. Only the motion of the particle itself gives a preferred direction in spacetime. The particle moves from one event to a nearby event along its worldline. In so doing, it undergoes a **spacetime displacement,** small changes in the three space positions along with an accompanying small advance in the time. The spacetime displacement has four parts: it is a 4-vector. The momenergy arrow points in the direction of another arrow, the arrow of the particle's spacetime 4-vector displacement. Momenergy runs parallel to worldline!

Particle momenergy points along its worldline

Compare the worldline of an individual particle in spacetime with a single straw in a great barn filled with hay. This particular straw has a direction, an existence, and a meaning independent of any measuring method imagined by humans who stack the hay or by mice that live in it. Similarly, in the rich trelliswork of worldlines that course through spacetime, the arrowlike momenergy of the particle has an existence and definiteness independent of the choice—or even use—of any free-float frame of reference (Section 5.9).

Momenergy independent of reference frame

No frame of reference? Then no clock available to time motion from here to there! Or rather no clock except one that the particle itself carries, its own wristwatch that records proper time. Proper time for what? Proper time for spacetime displacement between two adjacent events on the worldline of the particle. Proper time provides the only natural way to clock the rate of motion of the particle; that is the third and final feature of momenergy.

Particle wristwatch logs time for momenergy

In brief, the momenergy of a particle is a 4-vector: Its magnitude is proportional to its *mass,* it points in the direction of the particle's *spacetime displacement,* and it is reckoned using the *proper time* for that displacement. How are these properties combined to form momenergy? Simple! Use the recipe for Newtonian momentum: mass times displacement divided by time lapse for that displacement. Instead of

BOX 7-1

WHAT IS A 4-VECTOR?

A vector is a mathematical object that has both *magnitude* and *direction*. The meanings of the terms *magnitude* and *direction*, however, differ between one geometry and another. Mathematics offers many geometries. The two geometries important to us in this book are Euclidean geometry and Lorentz geometry.

Euclidean geometry defines **3-vectors** located in 3-dimensional space. Let a speeding particle emit two sparks. The particle's spatial displacement from first spark to second spark is a 3-vector. Each of the **three components** (northward, eastward, and upward) of this 3-vector displacement has a value larger or smaller, depending on the orientation of the coordinate system chosen. In contrast, the magnitude of the displacement — the distance traveled (computed as the square root of the *sum* of the squares of the three components of displacement) — has the same value in all coordinate systems.

Lorentz geometry defines **4-vectors** located in 4-dimensional spacetime. Construct the 4-vector spacetime displacement from the three space components supplemented by the time component, the time between sparks emitted by the speeding particle. Each of these **four components** (including time) has a value larger or smaller, depending on the choice of free-float frame of reference from which it is measured. The square of the separation in time between the two sparks as so measured, diminished by the square of the separation in space in the chosen frame, yields the square of the spacetime interval between the two events. This interval has the same value in all free-float frames. It is also the proper time, the time between the two sparks read directly on the particle's wristwatch.

Newtonian mechanics combines (in various ways) time and mass of the particle with Euclidean 3-vector displacement of the particle to yield additional 3-vectors that describe particle motion: velocity, momentum, acceleration. Each 3-vector has magnitude and direction. The values of the three components of each 3-vector depend on the orientation of the chosen coordinate system. But for each 3-vector quantity, the 3-vector itself is the same, both in magnitude and direction in space, no matter what Euclidean coordinate system we choose. Every 3-vector exists even in the absence of any coordinate system at all! That is why the analysis of Newtonian mechanics can proceed in all its everyday applications independent of choice of coordinate system.

Relativistic mechanics combines (in various ways) proper time and mass of the particle with Lorentz 4-vector displacement of the particle to yield additional 4-vectors that describe particle motion. Central among these is the particle's **momentum – energy 4-vector,** or **momenergy.** Values of the four components of the momenergy 4-vector differ as measured in different free-float frames in relative motion. But the momenergy 4-vector itself is the same, both in magnitude (mass!) and direction in spacetime, no matter what the frame. The momenergy 4-vector of a particle exists even in the absence of any reference frame at all! That is why the analysis of relativistic mechanics can proceed in all its power independent of choice of free-float frame of reference.

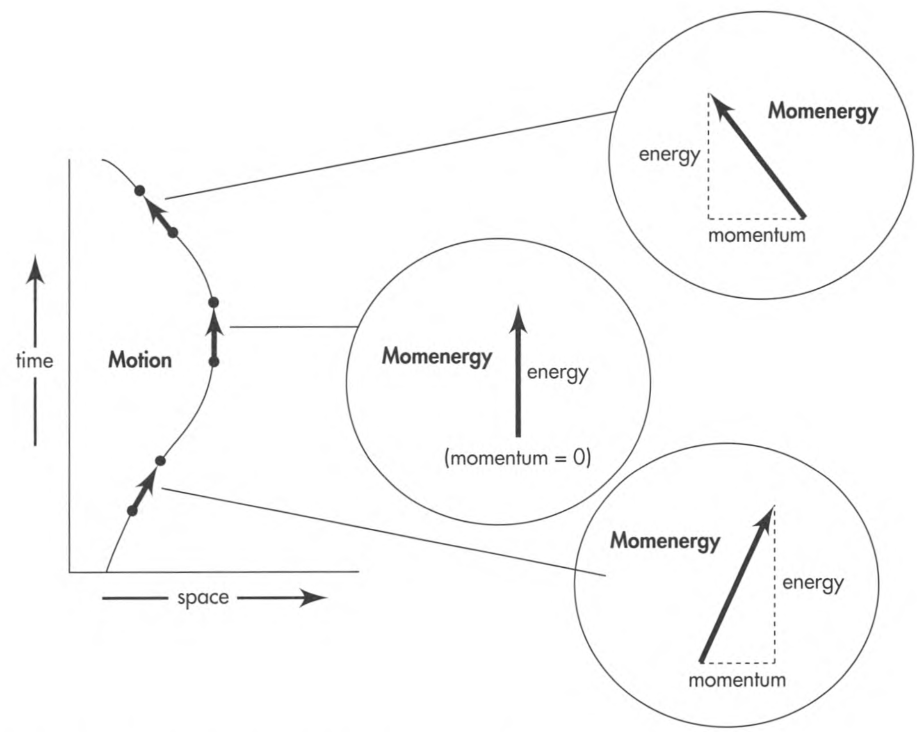


FIGURE 7-1. ***Motion and momenergy seen as identically directed arrows.*** *The momenergy arrow is carried along the worldline with the particle. Under action of a force, the particle traces out a curved worldline. The momenergy arrow — its constant magnitude equal to the mass of the particle — continually alters its tilt to point in the same direction in spacetime as the worldline. (For the special case shown here, the particle moves in* x *and* t, *but not in* y *or* z.*)*

Newtonian displacement in space, use Einstein's displacement in spacetime; instead of Newton's "universal time," use Einstein's proper time.

The result expresses the momenergy 4-vector in terms of the spacetime displacement 4-vector:

Particle momenergy
Magnitude: Mass
Direction: Along spacetime displacement

$$(\textbf{momenergy}) = (\text{mass}) \times \frac{\textbf{(spacetime displacement)}}{(\text{proper time for that displacement})} \qquad (7\text{-}1)$$

In any given free-float frame, the momentum of the particle is the three "space parts" of the momenergy and the particle's energy is the "time part." This expression for momenergy is simple, and it works — works as employed in the law of conservation of momenergy: Total momenergy before reaction equals total momenergy after reaction. Investigators have observed and analyzed more than a million collisions, creations, transformations, decays, and annihilations of particles and radiation. They have failed to discover a single violation of the relativistic law of conservation of momenergy.

To arrive at a formula as important as (7-1) so painlessly may at first sight create doubts. These doubts have to be dismissed. Fact is, there is no room for any alternative — as we see by going step by step through the factors in this equation.

Statement 1: *m* units of mass pursuing a given motion carry *m* times the momenergy of one unit of mass. Reasoning: *m* identical objects racing along side by side carry *m* times the momentum and *m* times the energy — and therefore *m* times the momenergy — of an object of unit mass.

Statement 2: Momenergy points in the same direction in spacetime as worldline. Reasoning: Where else *can* it point? Even the slightest difference in direction between momenergy and direction of motion along the worldline would bear witness to some crazy asymmetry in spacetime, for which no experiment in field-free space has ever given the slightest evidence.

Momenergy formula justified

Statement 3: The spacetime displacement between one event on the worldline and a nearby event on it specifies the direction of that worldline. Reasoning: The very concept of direction implies that there exists a segment, AB, of the worldline short enough to be considered straight. And to fix the direction of this spacetime displacement AB, it suffices to know the location of any two events, A and B, on this short segment.

Statement 4: Worldline direction—and therefore momenergy—is independent of the magnitude of the spacetime displacement. Reasoning: To pick an event B' on the worldline half as far from A as B along the short straight segment—thus to cut in half the spacetime displacement—makes no change in the direction of the worldline, therefore no change in the direction of the momenergy, therefore no change in the momenergy itself.

Statement 5: The unit 4-vector (spacetime displacement)/(proper time for that displacement) defines and measures the direction of the worldline displacement and therefore the direction of the momenergy 4-vector. Reasoning: What matters is not spacetime displacement individually, not proper time individually, but only their ratio. This ratio is the only directed quantity available to us to describe the *rate of motion* of the particle through spacetime.

The spacetime displacement, AB, has a magnitude equal to the interval (or proper time or wristwatch time) the particle requires to pass from A to B. That is why the ratio in question is a unit 4-vector.

Proper time provides the only natural way, the only frame-independent means, to clock the particle. If instead we should incorrectly put frame time into the denominator—frame time measured by the array of clocks in a particular free-float frame—the value of this time would differ from one frame to another. Divided into the spacetime displacement, it would typically not yield a unit vector. The vector's magnitude would differ from one frame to another. Therefore we must use in the denominator the proper time to go from A to B, a proper time identical to the magnitude of the spacetime displacement AB in the numerator.

Statement 6: The momenergy 4-vector of the particle is

$$\textbf{(momenergy)} = (\text{mass}) \times \frac{\textbf{(spacetime displacement)}}{(\text{proper time for that displacement})} \tag{7-1}$$

Reasoning: There is no other frame-independent way to construct a 4-vector that lies along the worldline and has magnitude equal to the mass.

Unit of momenergy: mass

Units: In this book, as in more and more present-day writing, space and time appear in the same unit: meter. Numerator and denominator on the right side of equation (7-1) have the unit of meter. Therefore their quotient is unit-free. As a result, the right side of the equation has the same unit as the first factor: mass. So the left side, the momenergy arrow, must also have the unit of mass. As the oneness of spacetime is emphasized by measuring space and time in the same unit, so the oneness of momenergy is clarified by measuring momentum and energy in the same unit: mass. Table 7-1 at the end of the chapter compares expressions for momentum and energy in units of mass with expressions in conventional units.

You say that the equation for momenergy is

$$(\text{momenergy}) = (\text{mass}) \times \frac{(\textbf{spacetime displacement})}{(\text{proper time for that displacement})}$$

I thought that "spacetime displacement" was the interval, which is the proper time. I know, however, that I am wrong, because if spacetime displacement and proper time were the same, then the numerator and denominator of the fraction would cancel, and momenergy would simply equal mass. Surely you would have told us of such simplicity. What have I missed?

It is easy to confuse a vector—or a 4-vector—with its magnitude.

In the expression for momenergy, the spacetime displacement is a 4-vector (Box 7-1). In the laboratory frame this displacement 4-vector has four components, $\{dt, dx, dy, dz\}$. In a free-float rocket moving in an arbitrary direction, the displacement 4-vector has four components, $\{dt', dx', dy', dz'\}$, typically different, respectively, from those in the laboratory frame.

A vector in space (a 3-vector) has not only a magnitude but also a direction independent of any coordinate system. ("Which way did they go?" "That-a-way!" —pointing.) Similarly, the spacetime displacement has a magnitude and direction in spacetime independent of any reference frame. This **spacetime direction** distinguishes the 4-vector displacement (the numerator above) from its magnitude, which is the proper time for that displacement (the denominator). This proper time (interval) can be observed directly: it is the time lapse read off the wristwatch carried by the particle while it undergoes the spacetime displacement.

In summary the fraction

Unit 4-vector along worldline

$$\frac{(\textbf{spacetime displacement})}{(\text{proper time for that displacement})}$$

has a numerator that is a 4-vector. This 4-vector numerator has the same *magnitude* as the denominator. The resulting fraction is therefore a *unit 4-vector* pointing along the worldline of the particle. This unit 4-vector determines the *direction* of the particle's momenergy in spacetime. And the magnitude of the momenergy? It is the mass of the particle, the first term on the right of the expression at the top of this page. In brief, the momenergy of a particle is 4-vector of magnitude m pointing along its worldline in spacetime. This description is independent of reference frame.

7.3 MOMENERGY COMPONENTS AND MAGNITUDE

space part: momentum of the object
time part: energy of the object
magnitude: mass of the object

Break down momenergy for examination

Accidents of history have given us not one word, momenergy, but two words, momentum and energy, to describe mass in motion. Before Einstein, mass and motion were described not in the unified context of spacetime but in terms of space and time separately, as that division shows itself in some chosen free-float frame. Often we still think in those separated terms. But the single concept *spacetime location* of an event unites the earlier two ideas of its position in space and the time of its happening. In the

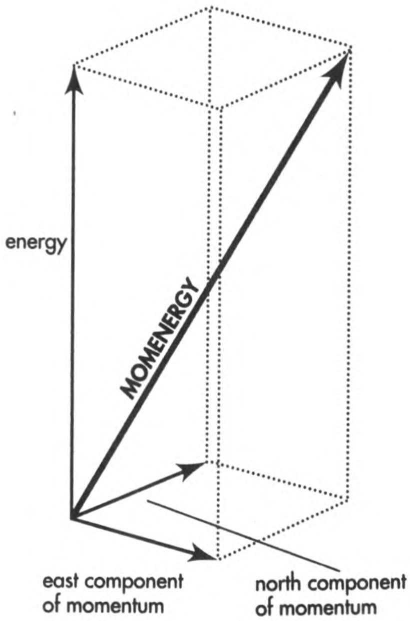


FIGURE 7-2. ***Momenergy arrow of a moving object translated into the language of momentum and energy, shown for the special case in which upward momentum (vertical momentum) equals zero.*** *The momenergy arrow itself has an existence and direction (in that great haystack of worldlines and events that we call spacetime) independent of the choice, or even presence, of any free-float frame. In contrast, separate measures of momentum and energy* do *depend on choice of frame. They point parallel to, that is in the same direction as, the corresponding space and time directions of the chosen frame itself. See Figures 7-3 and 7-4 for a still more revealing representation of the proportion between momenergy and its components.*

same way we combine momentum and energy of a moving object into the single idea of momenergy arrow. Having assembled it, we now break momenergy down again, seeking new insight by examining its separate parts.

The unity of momenergy dissolves—in our thinking—into the separateness of momentum and energy when we choose a free-float frame, say the laboratory. In that laboratory frame the spacetime separation between two nearby events on the worldline of a particle resolves itself into four different separations: one in laboratory time and one in each of three perpendicular space directions, such as north, east, and upward. With each spacetime separation goes a separate part, a separate portion, a separate **component** of momenergy in the laboratory free-float frame (Figure 7-2).

The "space parts" of momenergy of a particle are its three components of momentum relative to a chosen frame. Their general form is not strange to us—mass times a velocity component. The "time part," however, is new to us, foreshadowing important insights into the nature of energy (Section 7.5). The four components are

$$\begin{pmatrix}\text{eastward}\\ \text{component}\\ \text{of}\\ \text{momenergy}\end{pmatrix} = \begin{pmatrix}\text{eastward}\\ \text{component}\\ \text{of}\\ \text{momentum}\end{pmatrix} = (\text{mass}) \times \frac{(\text{eastward displacement})}{(\text{proper time for that displacement})}$$

$$\begin{pmatrix}\text{northward}\\ \text{component}\\ \text{of}\\ \text{momenergy}\end{pmatrix} = \begin{pmatrix}\text{northward}\\ \text{component}\\ \text{of}\\ \text{momentum}\end{pmatrix} = (\text{mass}) \times \frac{(\text{northward displacement})}{(\text{proper time for that displacement})}$$

$$\begin{pmatrix}\text{upward}\\ \text{component}\\ \text{of}\\ \text{momenergy}\end{pmatrix} = \begin{pmatrix}\text{upward}\\ \text{component}\\ \text{of}\\ \text{momentum}\end{pmatrix} = (\text{mass}) \times \frac{(\text{upward displacement})}{(\text{proper time for that displacement})}$$

$$\begin{pmatrix}\text{time}\\ \text{component}\\ \text{of}\\ \text{momenergy}\end{pmatrix} = (\text{energy}) = (\text{mass}) \times \frac{(\text{time displacement})}{(\text{proper time for that displacement})}$$

The calculus version of these equations is deliciously brief. Here, as in Section 6.2, tau (τ) stands for proper time:

Momenergy components of particle in a given frame

$$\begin{aligned} E &= m\frac{dt}{d\tau} \\ p_x &= m\frac{dx}{d\tau} \\ p_y &= m\frac{dy}{d\tau} \\ p_z &= m\frac{dz}{d\tau} \end{aligned} \tag{7-2}$$

The components of the momenergy 4-vector we now have before us in simple form, but how much is the absolutely-number-one measure of this physical quantity, its magnitude? This magnitude we reckon as we figure the magnitude of any Lorentz 4-vector: magnitude squared is the difference of squares of the time part and the space part:

$$
\begin{aligned}
&(\text{magnitude of momenergy arrow})^2\\
&\quad = (\text{energy})^2 - (\text{east momentum})^2 - (\text{north momentum})^2 - (\text{up momentum})^2\\
&\quad = \mathrm{E}^2 - (p_x)^2 - (p_y)^2 - (p_z)^2\\
&\quad = m^2 \frac{(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2}{(d\tau)^2} = m^2 \frac{(d\tau)^2}{(d\tau)^2} = m^2
\end{aligned}
$$

In brief, the magnitude of the momenergy 4-vector, or its square,

$$(\text{magnitude of momenergy arrow})^2 = E^2 - p^2 = m^2 \tag{7-3}$$

Magnitude of momenergy 4-vector: mass!

is identical with the particle mass, or its square. Moreover, this mass is a quantity characteristic of the particle and totally independent of its state of motion.

It's worthwhile to translate this story into operational language. Begin with a particle that is at rest. Its 4-vector of energy and momentum points in the pure timelike direction, all energy, no momentum. Let an accelerator boost that particle. The particle acquires momentum. The space component of the 4-vector, originally zero, grows to a greater and greater value. In other words, the momenergy 4-vector tilts more and more from the "vertical," that is, from a purely timelike direction. However, its magnitude remains totally unchanged, at the fixed value m. In conse-

SAMPLE PROBLEM 7-1

MASS

The energy and momentum components of a particle, measured in the laboratory, are

$$
\begin{aligned}
E &= 6.25 \text{ kilograms}\\
p_x &= 1.25 \text{ kilograms}\\
p_y = p_z &= 2.50 \text{ kilograms}
\end{aligned}
$$

What is the value of its mass?

SOLUTION

We obtain a value for mass using equation (7-3):

$$
\begin{aligned}
m^2 &= E^2 - (p_x)^2 - (p_y)^2 - (p_z)^2\\
&= [(6.25)^2 - (1.25)^2 - (2.50)^2 - (2.50)^2] \text{ (kilograms)}^2\\
&= [39.06 - 1.56 - 6.25 - 6.25] \text{ (kilograms)}^2\\
&= [39.06 - 14.06] \text{ (kilograms)}^2\\
&= 25.00 \text{ (kilograms)}^2
\end{aligned}
$$

Hence

$$m = 5.0 \text{ kilograms}$$

quence, the time component of that 4-vector, that is, the energy of the particle, undergoes a systematic alteration.

If the geometry of spacetime were Euclidean, this ever-growing tilt, this continuing rotation of the direction of the arrow of momenergy, would cause the vertical or time component to become ever shorter. However, spacetime is not Euclidean. It is Lorentzian, as appears in the minus sign in the equation for momenergy magnitude m: $m^2 = E^2 - p^2$. With momenergy magnitude, or particle mass m, being constant, and momentum p ever growing, Lorentz geometry itself tells us that the ever-growing tilt, the ever-larger momentum value, p, causes the time component of the momenergy — the energy E — not to shorten, as in a Euclidean spacetime, but to lengthen as the acceleration proceeds:

$$E = (m^2 + p^2)^{1/2} = \text{an increasing function of momentum, } p$$

This marvelously simple relation between energy and momentum, full of geometric as well as physical content, has by now been tried and verified in so many thousands of experiments of such varied kinds that it counts today as battle-tested.

Energy, momentum, and mass, expressed so far in the language of algebra, let themselves be displayed even more clearly in the language of pictures. Only one obstacle stands in the way. The paper is Euclidean and the vertical leg of a right triangle typically is shorter than the hypotenuse. In contrast, spacetime is Lorentzian, and the timelike dimension (the energy) is typically longer than the "hypotenuse" (the mass). We are indebted to our colleague William A. Shurcliff for a way to have our cake and eat it too, a device to employ Euclidean paper and yet display Lorentzian length. How? By laying over the hypotenuse of the Euclidean triangle a fat line or **handle** of length adjusted to the appropriate Lorentzian magnitude (Figure 7-3). The length of the handle represents the invariant value of the particle mass. This length remains the same, whatever the values of energy and momentum, values that differ as the particle is observed from one or another frame of reference in relative motion.

Figure 7-3 shows a few of the infinitely many different values of energy and momentum that one and the same particle can have as measured in different free-float frames. Each arrow, being depicted on a Euclidean sheet of paper, necessarily appears with an apparent length that increases with slope or particle speed. The handle on the arrow, by contrast, has the length appropriate to Lorentz geometry. This length represents particle mass, $m = 20$, a quantity independent of particle speed. The momenergy 4-vector of a material particle is always timelike. Why timelike? Because the momenergy 4-vector lies in the same spacetime direction as the worldline of the particle (Section 7-2). The events along the worldline have a timelike relationship: Time displacement between events is greater than the space displacement. One

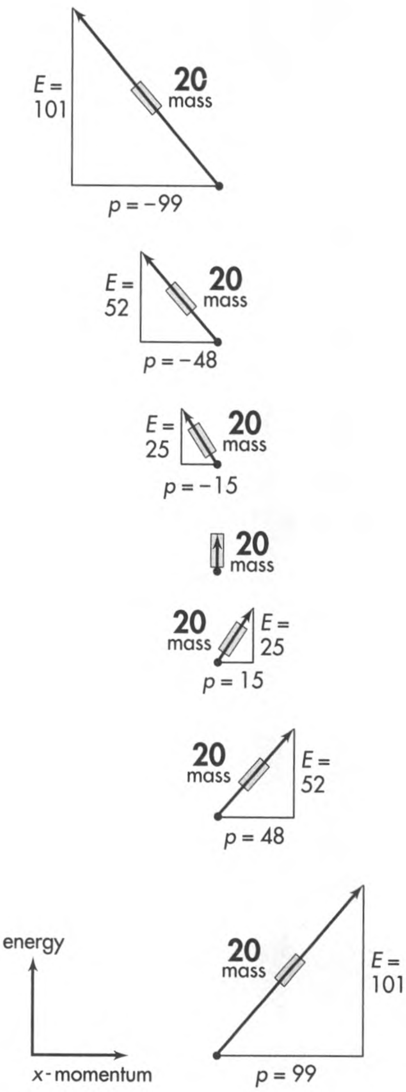


FIGURE 7-3. ***Different views of one and the same momenergy 4-vector of a particle in seven different free-float frames.*** *The* y- *and* z-*components of momentum are assumed to equal zero, and frames are chosen to give integer values for energy and* x-*momentum components. The mass of the particle equals 20 units as reckoned in every free-float frame:* $m^2 = E^2 - p^2$. *This invariant value of the mass is shown by the thick "handle" on each vector. For a frame in which the particle is at rest (center diagram), the energy is equal to the mass and the handle covers the vector.*

Does the momenergy 4-vector for this particle require for its existence any reference frame? No one would laugh more at such a misapprehension than the particle! The momenergy 4-vector has an existence in spacetime independent of any clocks and measuring rods. We, however, wish to assign to this 4-vector an energy and momentum. For that purpose we do require one or another free-float frame.

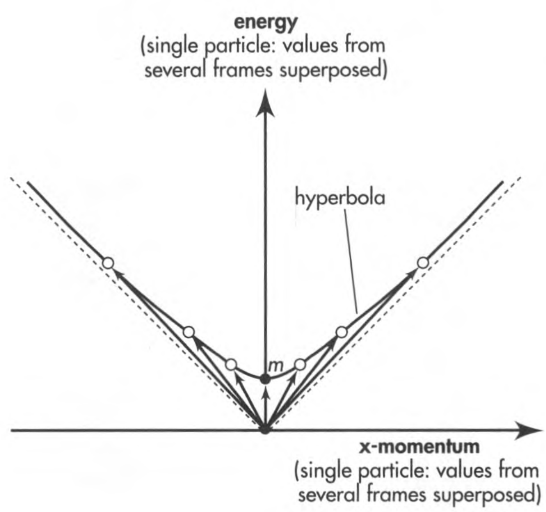


FIGURE 7-4. ***Momenergy 4-vector for the single particle of Figure 7-3 as observed in seven free-float frames, these plots then superposed on a composite momenergy diagram.*** *Frames are chosen so that* y- *and* z-*components of momentum equal zero. Locus of the tips of the arrows traces out a hyperbola. The central short vertical arrow pointing to the dot labeled* m *represents momenergy as measured in the particle rest frame. In this frame momentum has value zero and energy — "rest energy" — equals the mass of the particle. For clarity, the handles have been omitted from the 4-vectors, which all have identical invariant magnitude* $m = 20$.

consequence is that the particle moves at less than the speed of light in every possible free-float frame.

E, p, m of particle in different frames related by hyperbola

The equation $E^2 - p^2 = m^2 =$ (constant) is the formula for a hyperbola. Figure 7-4 generates this hyperbola by superposing on the same figure spacetime vectors that represent energy and momentum of the same particle in different free-float frames. For visual clarity the handles are omitted from these 4-vectors. However, each momenergy 4-vector has the same magnitude, equal to the particle mass, $m = 20$.

7.4 MOMENTUM: "SPACE PART" OF MOMENERGY

simply use proper time instead of Newton's so-called "universal" time

Newton called momentum "quantity of motion." The expressions for momentum that spacetime physics gives us, the last three equations in (7-2), seem at first sight to distinguish themselves by a trivial difference from the expressions for momentum given to us long ago by Newton's followers:

$$p_{x\,\text{Newton}} = m\frac{dx}{dt}, \quad p_{y\,\text{Newton}} = m\frac{dy}{dt}, \quad p_{z\,\text{Newton}} = m\frac{dz}{dt} \qquad \text{[valid for low velocity]}$$

Newtonian versus relativistic expressions for momentum

That difference? Today, proper time $d\tau$ between nearby events on the worldline of the particle. Laboratory time, in older days, when the concepts of proper time and interval were unknown. The percentage difference between the two, trivial or even negligible under everyday circumstances, becomes enormous when the speed of the object approaches the speed of light.

We explore most simply the difference between relativistic and Newtonian predictions of momentum by analyzing a particle that travels with speed v in the x-direction only. Then the relation between displacement of this particle and its speed is $x = vt$. For small displacements, for example between two nearby spark events on the worldline, this becomes, in the mathematical limit of interest in calculus notation, $dx = vdt$.

The proper time between the two nearby sparks is always less than the laboratory time:

$$\begin{aligned} d\tau &= [(d\tau)^2]^{1/2} = [(dt)^2 - (dx)^2]^{1/2} = [(dt)^2 - (vdt)^2]^{1/2} \\ &= (dt)(1 - v^2)^{1/2} = \frac{dt}{\gamma} \end{aligned} \qquad (7\text{-}4)$$

where gamma, $\gamma = 1/(1 - v^2)^{1/2}$ is the time stretch factor (Section 5.8). This figure for the interval, or proper time, between the two nearby sparks we now substitute into equations (7-2) in order to learn how the relativistic expressions for energy and momentum depend on particle speed:

$$\begin{aligned} E &= m\frac{dt}{d\tau} = \frac{m}{(1 - v^2)^{1/2}} = m\gamma \\ p_x &= m\frac{dx}{d\tau} = \frac{m\,(dx/dt)}{(1 - v^2)^{1/2}} = \frac{mv_x}{(1 - v^2)^{1/2}} = mv_x\gamma \end{aligned} \qquad (7\text{-}5)$$

Low speed: Newton and Einstein agree on value of momentum

The momentum expression is the same as for Newtonian mechanics—mass m times velocity (dx/dt)—except for the factor $(1 - v^2)^{1/2}$ in the denominator. That factor we can call 1 when the speed is small. For example, a commercial airliner moves through the air at approximately one millionth of the speed of light. Then the factor $(1 - v^2)^{1/2}$ differs from unity by only five parts in 10^{12}. Even for an alpha particle (helium nucleus) ejected from a radioactive nucleus with approximately 5 percent of the speed of light, the correction to the Newtonian figure for momentum is only a little more than one part in a thousand. Thus for low speeds the momentum expressed in equation (7-5) reduces to the Newtonian version.

High speed: Relativity reveals much larger momentum

At a speed close to that of light, however, the particle acquires a momentum enormous compared with the Newtonian prediction. The unusually energetic cosmic-ray protons mentioned at the end of Section 5.8 crossed the Milky Way in 30 seconds of their own time, but a thousand centuries or 3×10^{12} seconds of Earth time. The ratio $dt/d\tau$ between Earth time and proper time is thus 10^{11}. That is also the ratio between the correct relativistic value of the protons' momentum and the Newtonian prediction.

Units: Both Newtonian and relativistic expressions for momentum contain speed, a ratio of distance to time. From the beginning we have measured distance and time in the same unit, for example meter. Therefore the ratio of distance to time is unit-free. In Section 2.8, we expressed speed as a dimensionless quantity, the fraction of light speed:

Unit of momentum: mass

$$v = \frac{\text{(meters of distance covered by particle)}}{\text{(meters of time required to cover that distance)}}$$
$$= \frac{\text{(particle speed in meters/second)}}{\text{(speed of light in meters/second)}} = \frac{v_{\text{conv}}}{c} \qquad (7\text{-}6)$$

In terms of speed v (called beta, β, by some authors), Newtonian and relativistic expressions for the magnitude of the momentum have the forms

$$p_{\text{Newton}} = mv \qquad \text{[valid for low speed]} \quad (7\text{-}7)$$
$$p = mv/(1 - v^2)^{1/2} \qquad \text{[good at any speed]} \quad (7\text{-}8)$$

More Units: In order to convert momentum in units of mass to momentum in conventional units, such as kilogram meters/second, multiply expressions (7-6), (7-7), and (7-8) by the speed of light c and use the subscript "conv" for "conventional":

$$p_{\text{conv Newton}} = p_{\text{Newton}}\, c = mvc = m(v_{\text{conv}}/c)\, c = mv_{\text{conv}} \qquad \text{[low speed]} \quad (7\text{-}9)$$

Conversion to conventional momentum units

$$p_{\text{conv}} = pc = \frac{mvc}{[1 - v^2]^{1/2}} = \frac{m(v_{\text{conv}}/c)c}{[1 - (v_{\text{conv}}/c)^2]^{1/2}}$$
$$= \frac{mv_{\text{conv}}}{[1 - (v_{\text{conv}}/c)^2]^{1/2}} \qquad \text{[any speed]} \quad (7\text{-}10)$$

Thus conversion from momentum in units of mass to momentum in conventional units is always accomplished by multiplying by the conversion factor c. This is true whether the expression for momentum being converted is Newtonian or relativistic. Table 7-1 at the end of the chapter summarizes these comparisons.

7.5 ENERGY: "TIME PART" OF MOMENERGY

energy has two parts: rest energy (= mass) plus kinetic energy

What about the "time part" of the momentum-energy of a particle—the part we have called its energy? This is certainly a strange-looking beast! As measured in a particular free-float frame, say the laboratory, this time component as given in equation (7-5) is

$$E = m\frac{dt}{d\tau} = \frac{m}{(1 - v^2)^{1/2}} = m\gamma \qquad \text{(7-11)}$$

Relativistic expression for energy

Compare this with the Newtonian expression for kinetic energy, using K as the symbol for kinetic energy:

$$K_{\text{Newton}} = \frac{1}{2}\, mv^2 \qquad \text{[valid for low speed]} \quad \text{(7-12)}$$

How does the relativistic expression for energy, equation (7-11), compare with the Newtonian expression for kinetic energy (7-12)? To answer this question, first look at the behavior of these two expressions when particle speed equals zero. The Newtonian kinetic energy goes to zero. In contrast, at zero speed $1/(1 - v^2)^{1/2} = 1$ and the relativistic value for energy becomes equal to mass of the particle,

$$E_{\text{rest}} = m \qquad \text{(7-13)}$$

Rest energy of a particle equals its mass

where E_{rest} is called **rest energy of the particle.** Rest energy of a particle is simply its mass. So the relativistic expression for energy does not go to zero at zero speed, while the Newtonian expression for kinetic energy does go to zero.

Is this an irreconcilable difference? The Newtonian formula does not contain an expression for rest energy, equal to the mass of the particle. But here is the distinction: The relativistic expression gives the value for *total energy* of the particle, while the Newtonian expression describes *kinetic energy* only (valid for low speed). However, in Newtonian mechanics any constant potential energy whatever can be added to the energy of a particle without changing the laws that describe its motion. One may think of the zero-speed limit of the relativistic expression for energy as providing this previously undetermined constant.

When we refer to energy of a particle we ordinarily mean total energy of the particle. As measured in a frame in which the particle is at rest, this total energy equals rest energy, the mass of the particle. As measured from frames in which the particle moves, total energy includes not only rest energy but also kinetic energy.

This leads us to define kinetic energy of a particle as energy above and beyond its rest energy:

$$\text{(energy)} = \text{(rest energy)} + \text{(kinetic energy)}$$

Kinetic energy defined

or

$$E = m + K \qquad \text{(7-14)}$$

SAMPLE PROBLEM 7-2

MOTION IN THE X-DIRECTION

An object of mass 3 kilograms moves 8 meters along the x-direction in 10 meters of time as measured in the laboratory. What is its energy and momentum? Its rest energy? Its kinetic energy? What value of kinetic energy would Newton predict for this object? Using relativistic expressions, verify that the velocity of this object equals its momentum divided by its energy.

SOLUTION

From the statement of the problem:

$$\begin{aligned} m &= 3 \text{ kilograms} \\ t &= 10 \text{ meters} \\ x &= 8 \text{ meters} \\ y &= 0 \text{ meters} \\ z &= 0 \text{ meters} \end{aligned}$$

From this we obtain a value for the speed:

$$v = \frac{x}{t} = \frac{8 \text{ meters of distance}}{10 \text{ meters of time}} = 0.8$$

Use v to calculate the factor $1/(1 - v^2)^{1/2}$ in equation (7-8):

$$\frac{1}{(1 - v^2)^{1/2}} = \frac{1}{(1 - (0.8)^2)^{1/2}} = \frac{1}{(1 - 0.64)^{1/2}} = \frac{1}{(0.36)^{1/2}} = \frac{1}{0.6} = \frac{5}{3}$$

From equation (7-11) the energy is

$$E = m/(1 - v^2)^{1/2} = (3 \text{ kilograms})\,(5/3) = 5 \text{ kilograms}$$

From equation (7-8) momentum has the magnitude

$$p = mv/(1 - v^2)^{1/2} = (5/3) \times (3 \text{ kilograms}) \times 0.8 = 4 \text{ kilograms}$$

Rest energy of the particle just equals its mass:

$$E_{\text{rest}} = m = 3 \text{ kilograms}$$

From equation (7-15) kinetic energy K equals total energy minus rest energy:

$$K = E - m = 5 \text{ kilograms} - 3 \text{ kilograms} = 2 \text{ kilograms}$$

The Newtonian prediction for kinetic energy is

$$K_{\text{Newton}} = \frac{1}{2} mv^2 = \frac{1}{2} \times 3 \times (0.8)^2 = 0.96 \text{ kilogram}$$

which is a lot smaller than the correct relativistic result. Even at the speed of light, the Newtonian prediction would be $K_{\text{Newton}} = 1.5$ kilogram, whereas relativistic value would increase without limit.

SAMPLE PROBLEM 7-2

Equation (7-16) says that velocity equals the ratio (magnitude of momentum)/(energy):

$$v = \frac{p}{E} = \frac{4 \text{ kilograms}}{5 \text{ kilograms}} = 0.8$$

This is the same value as reckoned directly from the given quantities.

From this comes the relativistic expression for kinetic energy K:

$$K = E - E_{\text{rest}} = E - m = \frac{m}{(1 - v^2)^{1/2}} - m = m\left[\frac{1}{(1 - v^2)^{1/2}} - 1\right] \quad \text{(7-15)}$$

Box 7-2 elaborates the relation between this expression and the Newtonian expression (7-12). Notice that if we divide the respective sides of the momentum equation (7-8) by corresponding sides of the energy equation (7-11), the result gives particle speed:

$$v = \frac{p}{E} \quad \text{(7-16)}$$

We could have predicted this directly from the first figure in this chapter, Figure 7-1. Speed v is the tilt (slope) of the worldline from the vertical: (space displacement)/(time for this displacement). Momenergy points along the worldline, with space component p and time component E. Therefore momenergy slope p/E equals worldline slope v.

Still More Units: In order to convert energy in units of mass to energy in conventional units, such as joules, multiply the expressions above by the square of light speed, c^2, and use subscript "conv":

Conversion to conventional energy units

$$E_{\text{conv}} = Ec^2 = \frac{mc^2}{[1 - (v_{\text{conv}}/c)^2]^{1/2}} \quad \text{[good at any speed]} \quad \text{(7-17)}$$

$$E_{\text{conv rest}} = mc^2 \quad \text{[particle at rest]} \quad \text{(7-18)}$$

$$K_{\text{conv}} = (E - E_{\text{rest}})c^2 = mc^2\left[\frac{1}{[1 - (v_{\text{conv}}/c)^2]^{1/2}} - 1\right] \quad \text{[good at any speed]} \quad \text{(7-19)}$$

$$K_{\text{conv Newton}} = \frac{1}{2}mv^2c^2 = \frac{1}{2}m\left(\frac{v_{\text{conv}}}{c}\right)^2 c^2 = \frac{1}{2}mv_{\text{conv}}^2 \quad \text{[low speed only]} \quad \text{(7-20)}$$

Thus conversion from energy in units of mass to energy in conventional units is always accomplished by multiplying by conversion factor c^2. This is true whether the expression for energy being converted is Newtonian or relativistic. Table 7-1 at the end of the chapter summarizes these comparisons.

Equation (7-18) is the most famous equation in all physics. Historically, the factor c^2 captured the public imagination because it witnessed to the vast store of energy available in the conversion of even tiny amounts of mass to heat and radiation. The units of mc^2 are joules; the units of m are kilograms. However, we now recognize that joules and kilograms are units different only because of historical accident. The

SAMPLE PROBLEM 7-3

MOMENERGY COMPONENTS

For each of the following cases, write down the four components of the momentum–energy 4-vector in the given frame in the form $[E, p_x, p_y, p_z]$. Each particle has mass m.

a. A particle moves in the positive x-direction in the laboratory with kinetic energy equal to three times its rest energy.

b. The same particle is observed in a rocket in which its kinetic energy equals its mass.

c. Another particle moves in the y-direction in the laboratory frame with momentum equal to twice its mass.

d. Yet another particle moves in the negative x-direction in the laboratory with total energy equal to four times its mass.

e. Still another particle moves with equal x, y, and z momentum components in the laboratory and kinetic energy equal to four times its rest energy.

SOLUTION

a. Total energy of the particle equals rest energy m plus kinetic energy $3m$. Therefore its total energy E equals $E = m + 3m = 4m$. The particle moves along the x-direction, so $p_y = p_z = 0$ and $p_x = p$, the total momentum. Substitute the value of E into the equation $m^2 = E^2 - p^2$ to obtain

$$p^2 = E^2 - m^2 = (4m)^2 - m^2 = 16m^2 - m^2 = 15m^2$$

Hence $p_x = (15)^{1/2}m$.

In summary, the components of the momenergy 4-vector are

$$[E, p_x, p_y, p_z] = [4m, (15)^{1/2}m, 0, 0]$$

Of course the magnitude of this momenergy 4-vector equals the mass of the particle m—true whatever its speed, its energy, or its momentum.

b. In this rocket frame, total energy—rest energy plus kinetic energy—has the value $E = 2m$. As before, $p^2 = E^2 - m^2 = (2m)^2 - m^2 = 4m^2 - m^2 = 3m^2$. Hence $p_x = 3^{1/2}m$ and components of the 4-vector are $[E, p_x, p_y, p_z] = [2m, 3^{1/2}m, 0, 0]$.

c. In this case $p_x = p_z = 0$ and $p_y = p = 2m$. Moreover, $E^2 = m^2 + p^2 = m^2 + (2m)^2 = 5m^2$. So, finally, $[E, p_x, p_y, p_z] = [5^{1/2}m, 0, 2m, 0]$.

d. We are given directly that $E = 4m$, the same as in part **a**, except here the particle travels in the negative x-direction so has negative x-momentum. Hence:

$$[E, p_x, p_y, p_z] = [4m, -(15)^{1/2}m, 0, 0]$$

e. Total energy equals $E = 5m$. All momentum components have equal value, say

$$p_x = p_y = p_z = P$$

In this case we use the full equation that relates energy, momentum, and mass:

$$(p_x)^2 + (p_y)^2 + (p_z)^2 = 3P^2 = E^2 - m^2 = (5m)^2 - m^2 = 24m^2$$

or $P^2 = 8m^2$ and hence $[E, p_x, p_y, p_z] = [5m, 8^{1/2}m, 8^{1/2}m, 8^{1/2}m]$.

BOX 7-2

ENERGY IN THE LOW-VELOCITY LIMIT

Energy at relativistic speeds and energy at everyday speeds: How are expressions for these two cases related?

Energy in Terms of Momentum: In the limit of velocities low compared with the speed of light, the relativistically accurate expression for energy $E = (m^2 + p^2)^{1/2}$ reduces to $E = m + p^2/(2m) +$ corrections. To see why and how, and to estimate the corrections, it is convenient to work in dimensionless ratios. Thus we focus on the accurate expression in the form $E/m = [1 + (p/m)^2]^{1/2}$, or even simpler, $y = [1 + x]^{1/2}$, and on the approximation to this result, in the form

$$E/m = 1 + (1/2)\,(p/m)^2 + \text{corrections, or } y = 1 + (1/2)\,x + \text{corrections}$$

Example: $x = 0.21$. Then our approximation formula gives $y = (1.21)^{1/2} = 1 + 0.105 +$ a correction. The accurate result is $y = 1.100$, which is the square root of 1.21. In other words, the correction is negative and extremely small: correction $= -0.005$.

Energy in Terms of Velocity: In the limit of velocities low compared with the speed of light, the relativistically accurate expression for energy $E = m/(1 - v^2)^{1/2}$, reduces to $E = m + (1/2)mv^2 +$ corrections. It is convenient again to work in dimensionless ratios. Thus we focus on the accurate expression in the form $E/m = [1 - v^2]^{-1/2}$, or even simpler, $y = [1 - x]^{-1/2}$, and on the approximation to this result, in the form

$$E/m = 1 + (1/2)\,v^2 + \text{corrections, or } y = 1 + (1/2)\,x + \text{corrections}$$

Example: $x = 0.19$. Then our approximation formula gives $y = 1 + (1/2)\,0.19 +$ a correction $= 1.095 +$ a correction. The accurate result is $y = [1 - 0.19]^{-1/2} = (0.81)^{-1/2} = (0.9)^{-1} = 1.1111\ \ldots$ In other words, the correction is positive and small: correction $= +0.01611$.

Another example: A jet plane. Take its speed to be exactly $v = 10^{-6}$. That speed, according to our approximation, brings with it a fractional augmentation of energy, a kinetic energy per unit mass, equal to $(1/2)v^2 = 5 \times 10^{-13}$ or 0.000 000 000 000 5. In contrast, the accurate expression $E/m = [1 - v^2]^{-1/2}$ gives the result $E/m = 1.000\,000\,000\,000\,500\,000\,000\,009\,375\,000\,000\,000\ \ldots$ The 5 a little less than halfway down the length of this string of digits is no trifle, as anyone will testify who has seen the consequences of the crash of a jet plane into a skyscraper. However, the 9375 further down the line is approximately a million million times smaller and totally negligible in its practical consequences.

In brief, low speed gives rise to a kinetic energy which, relative to the mass, is given to good approximation by $(1/2)\,v^2$ or by $(1/2)\,(p/m)^2$. Moreover, the same one or other unit-free number (a "fraction" because it is small compared to unity) automatically reveals to us the order of magnitude of the fractional correction we would have had to make in this fraction itself if we were to have insisted on a perfectly accurate figure for the kinetic energy.

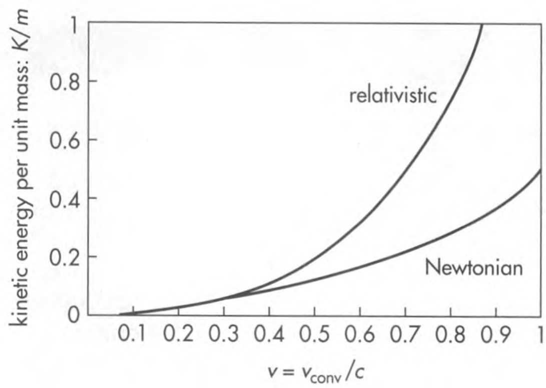


FIGURE 7-5. *Kinetic energy as a function of speed, as predicted by relativity [equation (7-19), valid for all speeds] and by Newtonian mechanics [equation (7-20), valid for low speeds only].*

conversion factor c^2, like the factor of conversion from seconds to meters or miles to feet, can today be counted as a detail of convention rather than as a deep new principle.

Energy: Time part of momenergy 4-vector
Mass: Magnitude of that 4-vector

Central to an understanding of the equation $E_{\text{rest}} = m$ or its equivalent $E_{\text{conv rest}} = mc^2$ is the subscript "rest." Energy is not the same as mass! Energy is only the time part of the momenergy 4-vector. Mass is the magnitude of that 4-vector. The energy of an object, expressed in conventional units, has the value mc^2 *only* when that object is observed from a frame in which it is at rest. Observed from all other free-float frames, the energy of the object is greater than its rest energy, as shown by equation (7-17).

Figure 7-5 compares relativistic and Newtonian predictions for kinetic energy per unit mass as a function of speed. At low speeds the values are indistinguishable (left side of the graph). When a particle moves with high speed, however, so that the factor $1/(1 - v^2)^{1/2}$ has a value much greater than one, relativistic and Newtonian expressions do not yield at all the same value for kinetic energy (right side of the graph). Then one must choose which expression to use in analyzing collisions and other high-speed phenomena. We choose the relativistic expression because it leads to the same value of the total energy of an isolated system before and after any interaction between particles in the system—it leads to conservation of total energy of the system.

Relativity: All forms of energy automatically conserved

All this talk of reconciliation at low speeds obscures an immensely powerful feature of the relativistic expression for total energy of an isolated system of particles. Total energy is conserved in *all* interactions among particles in the system: elastic and inelastic collisions as well as creations, transformations, decays, and annihilations of particles. In contrast, total kinetic energy of a system calculated using the Newtonian formula for low-speed interactions is conserved only for *elastic* collisions. Elastic collisions are *defined* as collisions in which kinetic energy is conserved. In collisions that are not elastic, kinetic energy transforms into heat energy, chemical energy, potential energy, or other forms of energy. For Newtonian mechanics of low-speed particles, each of these forms of energy must be treated separately: Conservation of energy must be invoked as a separate principle, as something beyond Newtonian analysis of mechanical energy.

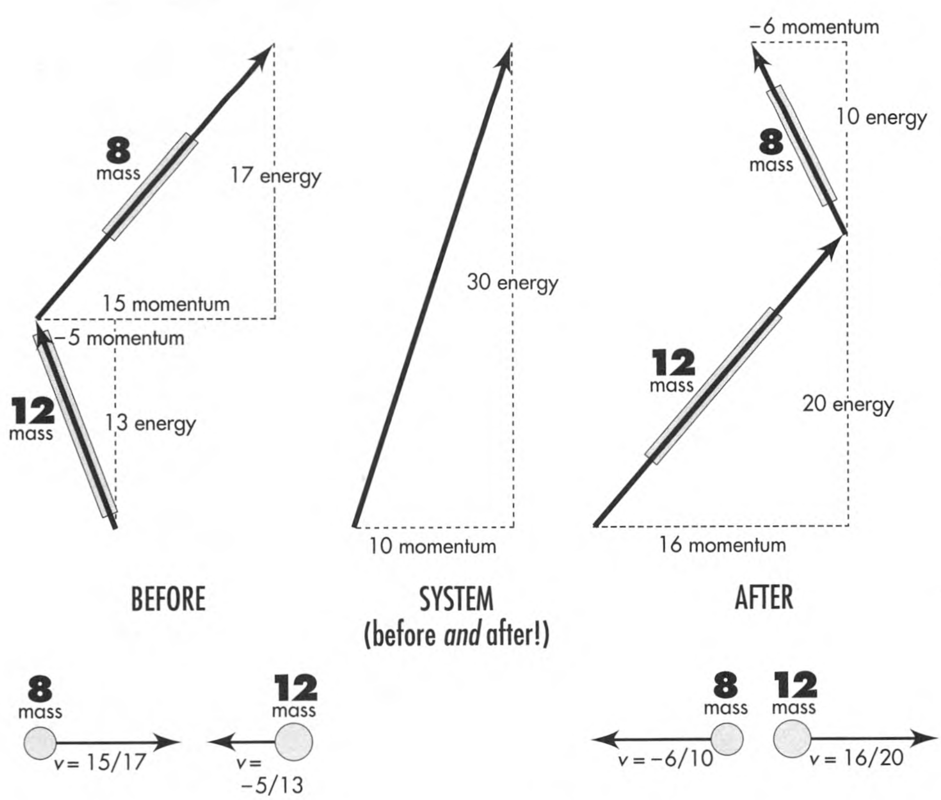
In relativity, all these energies are included automatically in the single time component of total momenergy of a system—total energy—which is always conserved for an isolated system. Chapter 8 discusses more fully the momenergy of a system of particles and the effects of interactions between particles on the energy and mass of the system.

7.6 CONSERVATION OF MOMENERGY AND ITS CONSEQUENCES

total momenergy of an isolated system of particles is conserved

Momenergy puts us at the heart of mechanics. The relativity concept momenergy gives us the indispensable tool for mastering every interaction and transformation of particles.

What does it mean in practice to say in this language of momenergy components that the punch given to particle *A* by particle *B* in a collision is exactly equal in magnitude and opposite in spacetime direction to the punch given to *B* by *A*? That gain in momenergy of *A* is identical to loss of momenergy by *B*? That the sum of separate momenergies of *A* and *B*—this sum itself regarded as an arrow in spacetime, the arrow of *total momenergy* (Figure 7-6)—has the same magnitude and direction after the encounter that it had before? Or, in brief, how does the **principle of conservation of momenergy** translate itself into the language of components in a



Momenergy of a system of particles

FIGURE 7-6. ***Conservation of total momenergy in a collision. Before:*** *The lighter 8-unit mass, moving right with 15/17 light speed, collides with the slower and heavier 12-unit mass moving left (with 5 units of momentum to the left and 13 units of energy).* ***System:*** *Arrow of total momenergy of the* system *of two particles. Combined momentum of the colliding particles has value* $-5 + 15 = 10$ *units rightward. Combined energy of the two equals* $13 + 17 = 30$ *units. The total system momenergy is conserved.* ***After:*** *One of many possible outcomes of this collision: The 8-unit mass bounces back leftward after collision, but the punch that it provided has reversed the direction of motion and increased the speed of the heavier 12-unit mass. The handle of the momenergy arrow of each particle gives the true magnitude of that momenergy, figured in the Lorentz geometry of the real physical world, as contrasted to the length of that 4-vector as it appears in the Euclidean—and therefore misleading—geometry of this sheet of paper. The scale of magnitudes in this figure is different from that of Figure 7-3.*

given free-float frame? Answer: Each *component* of the momenergy vector, when added together for particles A and B, has the same value after the collision as before the collision. In other words,

Energy of system conserved

$$\begin{aligned}
\begin{pmatrix}\text{energy of } A\\ \text{before the}\\ \text{encounter}\end{pmatrix} + \begin{pmatrix}\text{energy of } B\\ \text{before the}\\ \text{encounter}\end{pmatrix} &= \begin{pmatrix}\textbf{total}\text{ energy}\\ \textbf{before}\text{ the}\\ \text{encounter}\end{pmatrix}\\
&= \begin{pmatrix}\textbf{total}\text{ energy}\\ \textbf{after}\text{ the}\\ \text{encounter}\end{pmatrix}\\
&= \begin{pmatrix}\text{energy of } A\\ \text{after the}\\ \text{encounter}\end{pmatrix} + \begin{pmatrix}\text{energy of } B\\ \text{after the}\\ \text{encounter}\end{pmatrix}
\end{aligned}$$

called *conservation of the time part of momenergy*. Add to this three statements about the three space components of momenergy, of which the first one reads,

BOX 7-3

INVARIANT? CONSERVED? CONSTANT?

Is the speed of light a constant? An invariant? Is mass conserved in a collision? Is it an invariant? A constant? Many terms from everyday speech are taken over by science and applied to circumstances far beyond the everyday. The three useful adjectives *invariant, conserved,* and *constant* have distinct meanings in relativity .

Invariant

In relativity a quantity is invariant if it has the same value when measured by observers in different free-float frames—frames in relative motion. First among relativistic invariants is the speed of light: It has the same value when reckoned using data from the laboratory latticework of recording clocks as when figured using data from the rocket latticework. A second central invariant is the interval between two events: All inertial observers agree on the interval (proper time or proper distance). A third mighty invariant is the mass of a particle. There are many other invariants, every one with its special usefulness.

Some very important quantities do *not* qualify as invariants. The time between two events is not an invariant. It differs as measured by observers in relative motion. Neither is the distance between events an invariant. It too differs from one frame to another. Neither the energy nor the momentum of a particle is an invariant.

Conserved

A quantity is conserved if it has the same value before and after some encounter or does not change during some interaction. The total momenergy of an isolated system of particles is conserved in an interaction among the particles. In a given free-float frame this means that the total energy is conserved. So is each component of total momentum. The magnitude of total momenergy of a system — the mass of that system — is also conserved in an interaction. On the other hand, the sum of the individual masses of the

$$
\begin{aligned}
&\begin{pmatrix}\text{eastward component}\\ \text{of momentum of } A\\ \text{before the encounter}\end{pmatrix} + \begin{pmatrix}\text{eastward component}\\ \text{of momentum of } B\\ \text{before the encounter}\end{pmatrix}\\
&\qquad = \begin{pmatrix}\text{eastward component}\\ \text{of } \textbf{total} \text{ momentum}\\ \textbf{before} \text{ the encounter}\end{pmatrix}\\
&\qquad = \begin{pmatrix}\text{eastward component}\\ \text{of } \textbf{total} \text{ momentum}\\ \textbf{after} \text{ the encounter}\end{pmatrix}\\
&\qquad = \begin{pmatrix}\text{eastward component}\\ \text{of momentum of } A\\ \text{after the encounter}\end{pmatrix} + \begin{pmatrix}\text{eastward component}\\ \text{of momentum of } B\\ \text{after the encounter}\end{pmatrix}
\end{aligned}
$$

Momentum of system conserved

called *conservation of the space part of momenergy.* Figure 7-6 illustrates the conservation of momenergy in a recoil collision between two particles. Momentum is laid out

constituent particles of a system ordinarily is *not* conserved in a relativistic interaction. (For examples, see Chapter 8.)

Constant

Something that is constant does not change with time. The speed of the Great Pyramid with respect to the rock plateau of Giza is constant — equal to zero, or at least less than one millimeter per millennium. This speed may be constant, but it is not an invariant: As observed from a passing rocket, the Great Pyramid moves with blinding speed! Is the speed of the Great Pyramid conserved? Conserved during what encounter? There is no *before* or *after* to which the term "conserved" can refer. The term "conserved" simply does not apply to the speed of the Great Pyramid.

It is true that the speed of light in a vacuum is constant — it does not change with time. It is also true, but an entirely different statement, that the speed of light is an invariant — has the same value measured by different observers in uniform relative motion. It is true that total momenergy of an isolated system is constant — does not change with time. It is also true, but an entirely different statement, that total momenergy of an isolated system is conserved in a collision or interaction among particles in that system.

> When anyone hears the word *invariant, conserved,* or *constant,* she is well-advised to listen for the added phrase *with respect to,* which should always be expressed or implied. Usually (but not always) *constant* means *with respect* to the passage of time. *Conserved* usually (but not always) means *with respect to* a collision or interaction. *Invariant* can have at least as many meanings as there are geometries to describe Nature: In Euclidean geometry, *distance* is invariant as measured *with respect to* relatively rotated coordinate axes. In Lorentz geometry, *interval* and *mass* are invariants as measured *with respect to* free-float frames in relative motion. The full meaning of the word *invariant* or *conserved* or *constant* depends on the condition under which this property is invoked.

right and left on the page; energy is marked off vertically. The left diagram shows two particles before collision and their momentum-energy vectors. The right diagram shows the corresponding display after the collision.

Momenergy of system conserved!

The center diagram shows total momenergy of the **system** of two particles. The momenergy vectors of the two particles *before* the collision add up to this total; the momenergy vectors of the two particles *after* the collision add up to the same total. Total momenergy of the system has the same value after as before: it is *conserved* in the collision.

Well, you've done it again: You've given us a powerful tool that seems impossible to visualize. How can one think about this momenergy 4-vector, anyway? Can you personally picture it in your mind's eye?

We can *almost* visualize the momenergy arrow, by looking at Figure 7-6 for example. There momentum and energy *components* of a given momenergy vector have their correct relative values. And the *direction* of the momenergy arrow in spacetime is correctly represented in the diagram.

However, the *magnitude* of this arrow — mass of the particle — does *not* correspond to its length in the momenergy diagram. This is because mass is reckoned from the *difference* of squares of energy and momentum, whereas length of a line on the Euclidean page of a book is computed from the *sum* of squares of horizontal and vertical dimensions. The handle or thickened region on the typical arrow and the big, boldface number for mass remind us of the failure — the lie — that results from trying to represent momenergy on such a page.

To observe a given momenergy 4-vector first from one free-float frame, then from another, and then from another (Figure 7-3) is to see the apparent direction of the arrow changing. The change in frame brings with it changes in the energy and momentum components. However, magnitude does not change. Mass does not change. To examine the momenergy 4-vector of a particle in different frames is to gain improved perspective on what momenergy is and does.

See if this analogy helps: The momentum – energy 4-vector is like a tree. The tree has a location for its base and for its tip whether or not we choose this, that, or the other way to measure it. The shadow the tree casts on the ground, however, depends upon the tilt of the tree and the location of Sun in the sky.

Likewise, momenergy of a particle as it passes through a given event on its worldline has a magnitude and direction, a fixity in spacetime, independent of any choice we make of free-float frame from which to observe and measure it. No means of reporting momenergy is more convenient for everyday purposes than separate specification of momentum and energy of the object in question in some chosen free-float frame. Those two quantities separately, however, are like the shadow of the tree on the ground. As Sun rises the shadow shortens. Similarly the momentum of a car or spaceship depends on the frame in which we see it. In one frame, terrifying. In another frame, tame. In a comoving frame, zero momentum, as the tree's shadow disappears when Sun lies in exactly that part of the sky to which the tilted tree points. In such a special frame of reference, the time component of an object's momenergy — that is, its energy — takes on its minimum possible value, which is equal to the mass itself of that object. However, in whatever free-float frame we observe it, the arrow of momenergy clings to the same course in spacetime, maintains the same length, manifests the same mass.

7.7 SUMMARY

momenergy of an object unifies energy, momentum, and mass

The **momenergy 4-vector** of a particle equals its mass multiplied by the ratio of its spacetime displacement to proper time—wristwatch time—for that displacement (Section 7.2):

$$\begin{pmatrix}\text{momenergy}\\ \text{4-vector}\end{pmatrix} = (\text{mass})\left(\frac{\begin{array}{c}\text{spacetime}\\ \text{displacement}\\ \text{4-vector}\end{array}}{\begin{array}{c}\text{proper time}\\ \text{for that}\\ \text{displacement}\end{array}}\right) \tag{7-1}$$

Momenergy of a particle is a 4-vector. It possesses magnitude equal to the particle's mass. The momenergy at any given event in the motion of the particle points in the direction of the worldline at that event (Section 7.2).

The momenergy of a particle has an existence independent of any frame of reference.

The terms momenergy, momentum, and energy, as we deal with them in this book, all have a common unit: mass. In older times mass, momentum, and energy were all conceived of as different in nature and therefore were expressed in different units. The conventional units are compared with mass units in Table 7-1.

The magnitude of the momenergy 4-vector of a particle is reckoned from the difference of the squares of energy and momentum components in any given frame (Section 7.3):

$$m^2 = E^2 - (p_x)^2 - (p_y)^2 - (p_z)^2$$

or, more simply,

$$m^2 = E^2 - p^2 = (E')^2 - (p')^2 \tag{7-3}$$

Mass m of the particle is an *invariant,* has the same numerical value when computed using energy and momentum components in the laboratory frame (unprimed components) as in any rocket frame (primed components).

In a given inertial frame, the momenergy 4-vector of a particle has four **components.** Three space components describe the momentum of the particle in that frame (Sections 7.3 and 7.4):

$$\begin{aligned} p_x &= m\frac{dx}{d\tau}\\ p_y &= m\frac{dy}{d\tau}\\ p_z &= m\frac{dz}{d\tau}\end{aligned} \tag{7-2}$$

The magnitude of the momentum can be expressed as the factor $1/(1 - v^2)^{1/2}$ times the Newtonian expression for momentum mv. The result is

$$p = mv/(1 - v^2)^{1/2} \tag{7-8}$$

The "time part" of the momenergy 4-vector in a given inertial frame equals energy of the particle in that frame (Sections 7.3 and 7.5):

$$E = m\frac{dt}{d\tau} = \frac{m}{(1 - v^2)^{1/2}} \qquad \text{(7-2), (7-11)}$$

For a particle at rest, the energy of the particle has a value equal to its mass:

$$E_{\text{rest}} = m \qquad \text{(7-13)}$$

For a moving particle, the energy combines two parts: rest energy—equal to mass of the particle—plus the additional kinetic energy K that the particle has by virtue of its motion:

$$E = E_{\text{rest}} + K = m + K \qquad \text{(7-14)}$$

From these equations comes an expression for **kinetic energy:**

$$K = E - m = m\left[\frac{1}{(1 - v^2)^{1/2}} - 1\right] \qquad \text{(7-15)}$$

The momenergy 4-vector derives from **conservation** its power to analyze particle interactions. Conservation states that the total momenergy 4-vector of an isolated system of particles is conserved, no matter how particles in the system interact with one another or transform themselves. This conservation law holds independent of choice of the free-float frame in which we employ it (Section 7.6).

In any given inertial frame, conservation of total momenergy of an isolated system breaks apart into four conservation laws:

1. Total energy of the system before an interaction equals total energy of the system after the interaction.

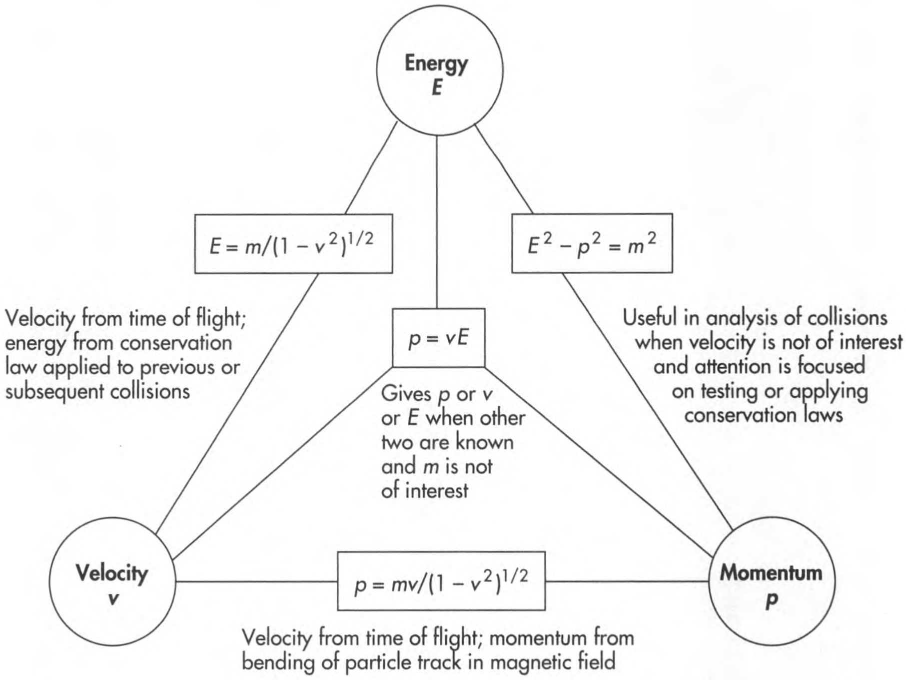


FIGURE 7-7. *Formulas that relate momentum, energy, mass, and velocity of an object, and notes about their uses in analyzing experiments. In this diagram,* p *is the magnitude of the momentum.*

TABLE 7-1

QUANTITIES RELATING TO MOMENERGY

	In units of mass (for example, E and p both in kilograms; x, y, z, t, τ in meters)	Reference equations	In conventional units (for example, E_{conv} in joules, p_{conv} in kilogram meters/second; t_{conv} in seconds)
Energy	$E = m\frac{dt}{d\tau} = \frac{m}{(1-v^2)^{1/2}}$	(7-2, 5, 11, 17)	$E_{conv} = \frac{mc^2}{[1-(v_{conv}/c)^2]^{1/2}}$
Rest energy	$E_{rest} = m$	(7-13, 18)	$E_{conv\ rest} = mc^2$
Kinetic energy	$K = m\left(\frac{1}{(1-v^2)^{1/2}} - 1\right)$	(7-15, 19)	$K_{conv} = mc^2\left(\frac{1}{[1-(v_{conv}/c)^2]^{1/2}} - 1\right)$
Momentum	$p = \frac{mv}{(1-v^2)^{1/2}}$	(7-8, 10)	$p_{conv} = \frac{mv_{conv}}{[1-(v_{conv}/c)^2]^{1/2}}$
Momentum components	$p_x = m\frac{dx}{d\tau} = \frac{mv_x}{(1-v^2)^{1/2}}$	(7-2, 5)	$p_{x\ conv} = \frac{mv_{x\ conv}}{[1-(v_{conv}/c)^2]^{1/2}}$
	$p_y = m\frac{dy}{d\tau} = \frac{mv_y}{(1-v^2)^{1/2}}$	(7-2, 5)	$p_{y\ conv} = \frac{mv_{y\ conv}}{(1-(v_{conv}/c)^2)^{1/2}}$
	$p_z = m\frac{dz}{d\tau} = \frac{mv_z}{(1-v^2)^{1/2}}$	(7-2, 5)	$p_{z\ conv} = \frac{mv_{z\ conv}}{(1-(v_{conv}/c)^2)^{1/2}}$
Mass	$m^2 = E^2 - p^2$	(7-3)	$m^2c^4 = E^2_{conv} - p^2_{conv}\ c^2$
Particle speed	$v = \frac{p}{E}$	(7-16)	$v_{conv} = \frac{p_{conv}\ c^2}{E_{conv}}$
Newtonian low-speed limit			
Kinetic energy	$K_{Newton} = \frac{1}{2}mv^2$	(7-12, 20)	$K_{conv\ Newton} = \frac{1}{2}mv^2_{conv}$
Momentum	$p_{Newton} = mv$	(7-7, 9)	$p_{conv\ Newton} = mv_{conv}$
Momentum components	$p_{x\ Newton} = mv_x$		$p_{x\ conv\ Newton} = mv_{x\ conv}$
	$p_{y\ Newton} = mv_y$		$p_{y\ conv\ Newton} = mv_{y\ conv}$
	$p_{z\ Newton} = mv_z$		$p_{z\ conv\ Newton} = mv_{z\ conv}$

2. Total x-momentum of the system is the same before and after the interaction.
3. Total y-momentum of the system is the same before and after the interaction.
4. Total z-momentum of the system is the same before and after the interaction.

In this chapter we have developed expressions that relate energy, momentum, mass, and velocity. Which of these expressions is useful depends upon circumstances and the system we are trying to analyze. Figure 7-7 summarizes these equations and circumstances under which they may be useful. Table 7-1 compares energy and momentum in units of mass and in conventional units.

ACKNOWLEDGMENT

The authors are grateful to William A. Shurcliff for the idea of a "handle" that displays on a Euclidean page the invariant magnitude of a particle's momenergy 4-vector—a magnitude equal to the mass of the particle.

CHAPTER 7 EXERCISES

PRACTICE

7-1 momenergy 4-vector

For each of the following cases, write down the four components of the momentum–energy (momenergy) 4-vector in the given frame in the form $[E, p_x, p_y, p_z]$. Assume that each particle has mass m. You may use square roots in your answer.

a A particle moves in the positive x-direction in the laboratory with total energy equal to five times its rest energy.

b Same particle as observed in a frame in which it is at rest.

c Another particle moves in the z-direction with momentum equal to three times its mass.

d Yet another particle moves in the negative y-direction with kinetic energy equal to four times its mass.

e Still another particle moves with total energy equal to ten times its mass and x-, y-, and z-components of momentum in the ratio 1 to 2 to 3.

7-2 system mass

Determine the mass of the system of particles shown in Figure 7-6. Is this system mass equal to the sum of the masses of the individual particles in the system? Does the mass of this system change as a result of the interaction? Does the momenergy 4-vector of the system change as a result of the interaction? (In Chapter 8 there is a lot more discussion about the mass of a system of particles.)

7-3 much ado about little

Two freight trains, each of mass 5×10^6 kilograms (5000 metric tons) travel in opposite directions on the same track with equal speeds of 42 meters/second (about 100 miles/hour). They collide head on and come to rest.

a Calculate in milligrams the kinetic energy for each train $(1/2)mv^2$ before the collision. (Newtonian expression OK for 100 mph!) (1 milligram $= 10^{-3}$ gram $= 10^{-6}$ kilogram)

b After the collision, the mass of the trains plus the mass of the track plus the mass of the roadbed has increased by what number of milligrams? Neglect energy lost in the forms of sound and light.

7-4 fast protons

Each of the protons described in the table emits a flash of light every meter of its own (proper) time $d\tau$. Between successive flash emissions, each proton travels a distance given in the left column. Complete the table. Take the rest energy of the proton to be equal to 1 GeV $= 10^9$ eV and express momentum in the same units. Hints: Avoid calculating or using the speed v in relativistic particle problems; it is too close to unity to distinguish between protons of radically different energies. An accuracy of two significant fig-

EXERCISE 7-4

FAST PROTONS

Lab distance Δx traveled between flashes (meters)	Momentum $mdx/d\tau$ (GeV)	Energy (GeV)	Time stretch factor γ	Lab time between flashes (meters)
0				
0.1				
1				
5				
10				
10^3				
10^6				

ures is fine; don't give more. Recall: $E^2 - p^2 = m^2$ and $E = mdt/d\tau = m\gamma$ [note tau!].

PROBLEMS

7-5 Lorentz transformation for momenergy components

The rocket observer measures energy and momentum components of a particle to have the values E' and p_x', p_y', and p_z'. What are the corresponding values of energy and momentum measured by the laboratory observer? The answer comes from the Lorentz transformation, equation (L-10) in the Special Topic following Chapter 3.

The moving particle emits a pair of sparks closely spaced in time as measured on its wristwatch. The rocket latticework of clocks records these emission events; so does the laboratory latticework of clocks. The rocket observer constructs components of particle momentum and energy, equation (7-2), from knowledge of particle mass m, the spacetime displacements dt', dx', dy', and dz' derived from the event recordings, and the proper time $d\tau$ computed from these spacetime components. Laboratory momenergy components come from transforming the spacetime displacements. The Lorentz transformation, equation (L-10), for incremental displacements gives

$$\begin{aligned} dt &= v\gamma dx' + \gamma dt' \\ dx &= \gamma dx' + v\gamma dt' \\ dy &= dy' \\ dz &= dz' \end{aligned}$$

a Multiply both sides of each equation by the invariant mass m and divide through by the invariant proper time $d\tau$. Recognizing the components of the momenergy 4-vector in equation (7-2), show that the transformation equations for momenergy are

$$\begin{aligned} E &= v\gamma p'_x + \gamma E' \\ p_x &= \gamma p'_x + v\gamma E' \\ p_y &= p'_y \\ p_z &= p'_z \end{aligned}$$

b Repeat the process for particle displacements dt, dx, dy, and dz recorded in the laboratory frame to derive the inverse transformations from laboratory to rocket.

$$\begin{aligned} E' &= -v\gamma p_x + \gamma E \\ p'_x &= \gamma p_x - v\gamma E \\ p'_y &= p_y \\ p'_z &= p_z \end{aligned}$$

7-6 fast electrons

The Two-Mile Stanford Linear Accelerator accelerates electrons to a final kinetic energy of 47 GeV (47×10^9 electron-volts; one electron-volt $= 1.6 \times 10^{-19}$ joule). The resulting high-energy electrons are used for experiments with elementary particles. Electromagnetic waves produced in large vacuum tubes ("klystron tubes") accelerate the electrons along a straight pipelike structure 10,000 feet long (approximately 3000 meters long). Take the rest energy of an electron to be $m \approx 0.5$ MeV $= 0.5 \times 10^6$ electron-volts.

a Electrons increase their kinetic energy by approximately equal amounts for every meter traveled along the accelerator pipe as observed in the laboratory frame. What is this energy gain in MeV/meter? Suppose the Newtonian expression for kinetic energy were correct. In this case how far would the electron travel along the accelerator before its speed were equal to the speed of light?

b In reality, of course, even the 47-GeV electrons that emerge from the end of the accelerator have a speed v that is less than the speed of light. What is the value of the difference $(1 - v)$ between the speed of light and the speed of these electrons as measured in the laboratory frame? [Hint: For v very near the value unity, $1 - v^2 = (1 + v)(1 - v) \approx 2(1 - v)$.] Let a 47-GeV electron from this accelerator race a flash of light along an evacuated tube straight through Earth from one side to the other (Earth diameter 12,740 kilometers). How far ahead of the electron is the light flash at the end of this race? Express your answer in millimeters.

c How long is the "3000-meter" accelerator tube as recorded on the latticework of rocket clocks moving along with a 47-GeV electron emerging from the accelerator?

7-7 super cosmic rays

The Haverah Park extensive air shower array near Leeds, England, detects the energy of individual cosmic ray particles indirectly by the resulting shower of particles this cosmic ray creates in the atmosphere. Between 1968 and 1987 the Haverah Park array detected more than 25,000 cosmic rays with energies greater than 4×10^{17} electron-volts, including 5 with an energy of approximately 10^{20} electron-volts. (rest energy of the proton $\approx 10^9$ electron-volts $= 1.6 \times 10^{-10}$ joule)

a Suppose a cosmic ray is a proton of energy 10^{20} electron-volts. How long would it take this proton to cross our galaxy as measured on the proton's wristwatch? The diameter of our galaxy is approximately

10^5 light-years. How many centuries would this trip take as observed in our Earth-linked frame?

b The research workers at Haverah Park find no evidence of an upper limit to cosmic ray energies. A proton must have an energy of how many times its rest energy for the diameter of our galaxy to appear to it Lorentz-contracted to the diameter of the proton (about 1 femtometer, which is equal to 10^{-15} meters)? How many metric tons of mass would have to be converted to energy with 100-percent efficiency in order to give a proton this energy? One metric ton equals 1000 kilograms.

Reference: M. A. Lawrence, R. J. O. Reid, and A. A. Watson, *Journal of Physics G: Nuclear and Particle Physics,* Volume 17, pages 733-757 (1991).

7-8 rocket nucleus

A radioactive decay or "inverse collision" is observed in the laboratory frame, as shown in the figure.

Suppose that $m_A = 20$ units, $m_C = 2$ units, and $E_C = 5$ units.

a What is the total energy E_A of particle A?

b From the conservation of energy, find the total energy E_D (rest plus kinetic) of particle D.

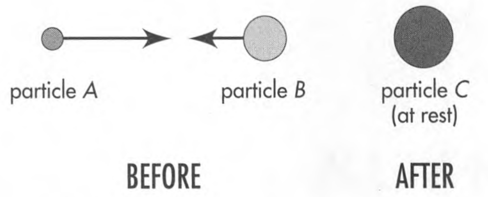
c Using the expression $E^2 - p^2 = m^2$ find the momentum p_C of particle C.

d From the conservation of momentum, find the momentum p_D of particle D.

e What is the mass m_D of particle D?

f Does $m_C + m_D$ after the collision equal m_A before the collision? Explain your answer.

g Draw three momenergy diagrams for this reaction similar to those of Figure 7-6: BEFORE, SYSTEM, and AFTER. Plot positive and negative momentum along the positive and negative horizontal direction, respectively, and energy along the vertical direction. On the AFTER diagram draw the momenergy vectors for particles C and D head to tail so that they add up to the momenergy vector for the system. Place labeled mass handles on the arrows in all three diagrams, including the arrow for the system.



EXERCISE 7-9. *Two particles collide to form a third at rest in the laboratory frame.*

7-9 sticky collision

An inelastic collision is observed in the laboratory frame, as shown in the figure. Suppose that $m_A = 2$ units, $E_A = 6$ units, $m_C = 15$ units.

a From the conservation of energy, what is the energy E_B of particle B?

b What is the momentum p_A of particle A? Therefore what is the momentum p_B of particle B?

c From $m^2 = E^2 - p^2$ find the mass m_B of particle B.

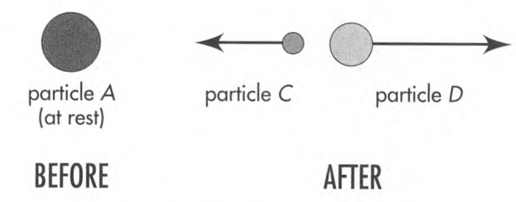
d Quick guess: Is the mass of particle C after the collision less than or greater than the sum of the masses of particles A and B before the collision? Validate your guess from the answer to part **c**.

7-10 colliding putty balls

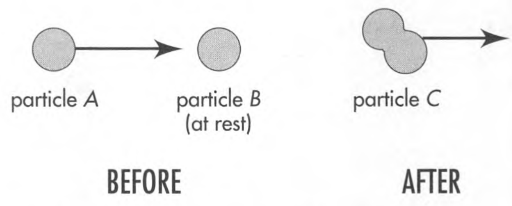
A ball of putty of mass m and kinetic energy K streaks across the frozen ice of a pond and hits a second identical ball of putty initially at rest on the ice. The two stick together and skitter onward as one unit. Referring to the figure, find the mass of the combined particle using parts **a**–**e** or some other method.

a What is the total energy of the system before the collision? Keep the kinetic energy K explicitly, and don't forget the rest energies of both particles A and B. Therefore what is the total energy E_C of particle C after the collision?

b Using the equation $m^2 = E^2 - p^2 = (m + K)^2 - p^2$ find the momentum p_A of particle A before the collision. What is the total momentum of the system before the collision? Therefore what is the momentum p_C of particle C after the collision?



EXERCISE 7-8. *Radioactive decay of a particle.*



EXERCISE 7-10. *Two putty balls stick together.*

c Again use the equation $m^2 = E^2 - p^2$ to find the mass m_C of particle C. Show that the result satisfies the equation

$$m_C^2 = (2m)^2 + 2mK = (2m)^2\left(1 + \frac{K}{2m}\right)$$

d Examine the result of part **c** in two limiting cases. (1) The value of m_C in the Newtonian low-velocity limit in which kinetic energy is very much less than mass: $K/m << 1$. Is this what one expects from everyday living? (2) What is the value of m_C in the highly relativistic limit in which $K/m >> 1$? What is the upper limit on the value of m_C? **Discussion:** Submicroscopic particles moving at extreme relativistic speeds rarely stick together when they collide. Rather, their collision often leads to creation of additional particles. See Chapter 8 for examples.

e **Discussion question:** Are the results of part c changed if the resulting blob of putty rotates, whirling like a dumbbell about its center as it skitters along?

7-11 limits of Newtonian mechanics

a One electron-volt (eV) is equal to the increase of kinetic energy that a singly charged particle experiences when accelerated through a potential difference of one volt. One electron-volt is equal to 1.60×10^{-19} joules. Verify the rest energies of the electron and the proton (masses listed inside the back cover) in units of million electron-volts (MeV).

b The kinetic energy of a particle of a given velocity v is not correctly given by the expression $1/2\ mv^2$. The error

$$\frac{\begin{pmatrix}\text{relativistic expression}\\ \text{for kinetic energy}\end{pmatrix} - \begin{pmatrix}\text{Newtonian expression}\\ \text{for kinetic energy}\end{pmatrix}}{\begin{pmatrix}\text{Newtonian expression}\\ \text{for kinetic energy}\end{pmatrix}}$$

is one percent when the Newtonian kinetic energy has risen to a certain fraction of the rest energy. What fraction? Hint: Apply the first three terms of the binomial expansion

$$(1 + z)^n = 1 + nz + \frac{1}{2}n(n-1)\,z^2 + \ldots$$

to the relativistic expression for kinetic energy, an accurate enough approximation if $|z| << 1$. Let this point— where the error is one percent—be arbitrarily called the "limit of Newtonian mechanics." What is the speed of the particle at this limit? At what kinetic energy does a proton reach this limit (energy in MeV)? An electron?

c An electron in a modern color television tube is accelerated through a voltage as great as 25,000 volts and then directed by a magnetic field to a particular pixel of luminescent material on the inner face of the tube. Must the designer of color television tubes use special relativity in predicting the trajectories of these electrons?

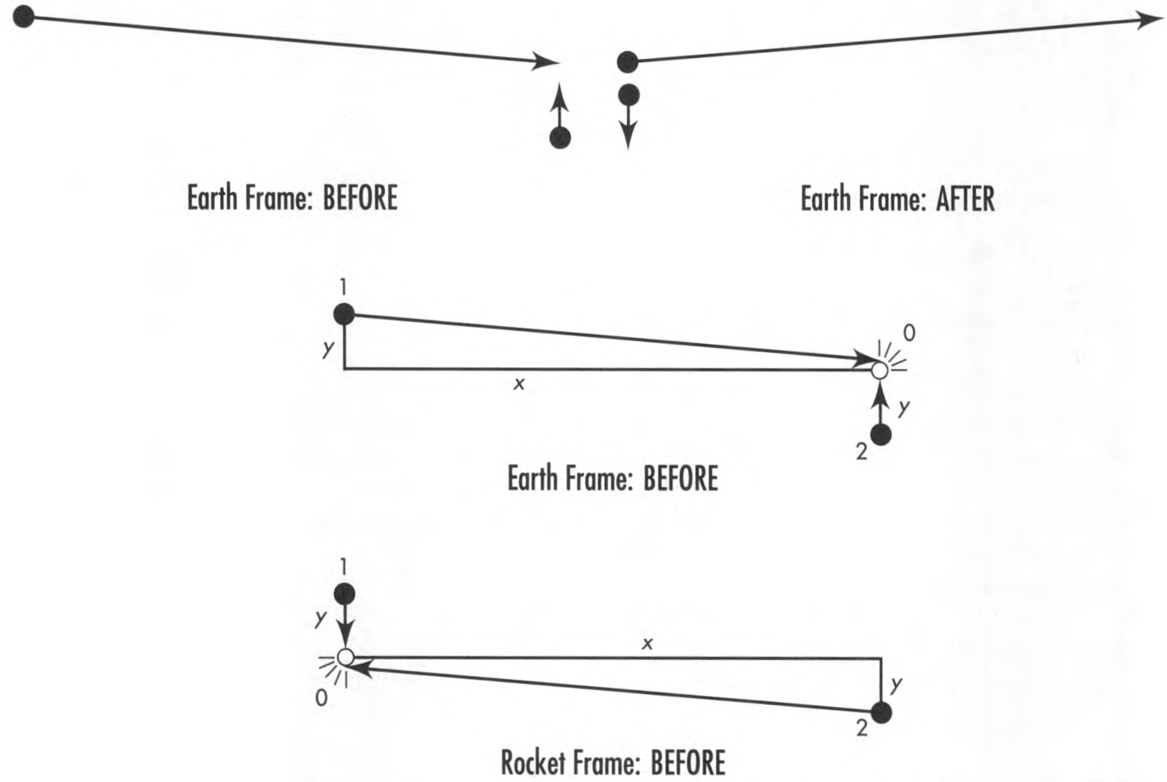
7-12 derivation of the relativistic expression for momentum—a worked example

A very fast particle interacts with a very slow particle. If the collision is a glancing one, the slow particle may move as slowly after the collision as before. Reckon the momentum of the slow-moving particle using the Newtonian expression. Now demand that momentum be conserved in the collision. From this derive the relativistic expression for momentum of the fast-moving particle.

The top figure shows such a glancing collision. After the collision each particle has the same speed as before the collision, but each particle has changed its direction of motion.

Behind this figure is a story. Ten million years ago, and in another galaxy nearly ten million light-years distant, a supernova explosion launched a proton toward Earth. The energy of this proton far exceeded anything we can give to protons in our earthbound particle accelerators. Indeed, the speed of the proton so nearly approached that of light that the proton's wristwatch read a time lapse of only one second between launch and arrival at Earth.

We on Earth pay no attention to the proton's wristwatch. For our latticework of Earth-linked observers, ages have passed since the proton was launched. Today our remote outposts warn us that the streaking proton approaches Earth. Exactly one second on our clocks before the proton is due to arrive, we launch our own proton at the slow speed one meter/second almost perpendicular to the direction of the incoming proton (BEFORE part of the top figure). Our proton saunters the one meter to the impact point. The two protons meet. So perfect is our aim and timing that after the encounter our proton simply reverses direction and returns with the same speed we gave it originally (AFTER part of the top figure). The incoming proton also does not change speed, but it is deflected upward at the same angle at which it was originally slanting downward.



EXERCISE 7-12. ***Top:*** *A symmetric elastic collision between a fast proton and a slow proton in which each proton changes direction but not speed as a result of the encounter.* ***Center:*** *Events and separations as observed in Earth frame before the collision. Here* x = *10 million light-years and* y = *1 meter, so these figures are not to scale!* ***Bottom:*** *Events and separations as observed in the rocket frame before the collision.*

How much does y-momentum of our slow-moving proton change during this encounter? Newton can tell us. At a particle speed of one meter/second, his expression for momentum, mv, is accurate. Our proton simply reverses its direction. Therefore the change in its momentum is just $2\ mv$, twice its original momentum in the y-direction.

What is the change in the y-momentum of the incoming proton, moving at extreme relativistic speed? We demand that the change in y-momentum of the fast proton be equal in magnitude and opposite in direction to the change in y-momentum of our slow proton. In brief, y-momentum is conserved. This demand, plus a symmetry argument, leads to the relativistic expression for momentum.

Key events in our story are numbered in the center figure. Event 1 is the launching of the proton from the supernova ten million years (in our frame) before the impact. Event 2 is the quiet launch of our local proton one second (in our frame) before the impact. Event 0 is the impact itself. The x-direction is chosen so that y-displacements of both protons have equal magnitude between launch and impact, namely one meter.

Now view the same events from a rocket moving along the x-axis at such a speed that events 1 and 0 are vertically above one another (bottom figure). For the rocket observer the transverse y-separations are the same as for the Earth observer (Section 3.6), so $y = 1$ meter in both frames. The order of events 1 and 2, however, is exactly reversed in time: For the rocket observer, we released our proton at high speed ten million years before impact and she releases hers one second before the collision. Otherwise the diagrams are symmetrical: To make the bottom figure look like the center one, exchange event numbers 1 and 2, then stand on your head!

Rocket observer and Earth observer do not agree on the time between events 1 and 0, but they agree on the proper time τ_{10} between them, namely one second. They also agree on the proper time τ_{20} between events 2 and 0. Moreover, because of the symmetry between the center and bottom figures, these two proper times have the same value: For the case we have chosen, the wristwatch (proper) time for each proton is one second between launch and impact.

$$\tau_{10} = \tau_{20}$$

We can use these quantities to construct expressions for the y-momenta of the two protons. Both are protons, so their masses m are the same and have the same invariant value for both observers. Because of the equality in magnitude of the y-displacements and the equality of τ_{20} and τ_{10}, we can write

$$m\frac{y}{\tau_{10}} = -m\frac{y}{\tau_{20}} \qquad \text{[both frames]}$$

The final key idea in the derivation of the relativistic expression for momentum is that the slow-moving proton travels between events 2 and 0 in an Earth-measured time that is very close in value to the proper time between these events. The vertical separation y between events 2 and 0 is quite small: one meter. In the same units, the time between them has a large value in the Earth frame: one second, or 300 million meters of light-travel time. Therefore, for such a slow-moving proton, the proper time τ_{20} between events 2 and 0 is very close to the Earth time t_{20} between these events:

$$\tau_{20} \approx t_{20} \qquad \text{[Earth frame only]}$$

Hence rewrite the both-frames equation for the Earth frame:

$$m\frac{y}{\tau_{10}} = -m\frac{y}{t_{20}} \qquad \text{[Earth frame only]}$$

The right side of this equation gives the y-momentum of the slow proton before the collision, correctly calculated using the Newtonian formula. The change in momentum of the slow proton during the collision is twice this magnitude. Now look at the left side. We claim that the expression on the left side is the y-momentum of the very fast proton. The y-momentum of the fast proton also reverses in the collision, so the change is just twice the value of the left side. In brief, this equation embodies the conservation of the y-component of total momentum in the collision. Conclusion: The left side of this equation yields the relativistic expression for y-momentum: mass times displacement divided by *proper* time for this displacement.

What would be wrong with using the Newtonian expression for momentum on the left side as well as on the right? That would mean using earth time t_{10} instead of proper time τ_{10} in the denominator of the left side. But t_{10} is the time it took the fast proton to reach Earth from the distant galaxy as recorded in the Earth frame — ten million years or 320 million million seconds! With this substitution, the equation would no longer be an equality; the left side would be 320 million million times smaller in value than the right side (smaller because t_{10} would appear in the denominator). Nothing shows more dramatically than this the radical difference between Newtonian and relativistic expressions for momentum — and the correctness of the relativistic expression that has proper time in the denominator.

This derivation of the relativistic expression for momentum deals only with its y-component. But the choice of y-direction is arbitrary. We could have interchanged y and x axes. Also the expression has been derived for particles moving with constant velocity before and after the collision. When velocity varies with time, the momentum is better expressed in terms of incremental changes in space and time. For a particle displacement dr between two events a proper time $d\tau$ apart, the expression for the magnitude of the momentum is

$$p = m\frac{dr}{d\tau}$$

One-sentence summary: In order to preserve conservation of momentum for relativistic collisions, simply replace Newton's "universal time" t in the expression for momentum with Einstein's invariant proper time τ.