CHAPTER 8

COLLIDE. CREATE. ANNIHILATE.

8.1 THE SYSTEM

an isolated island of violence

Particle physics is one of the great adventures of our time. No one can venture into the heart of it without momenergy as guide and lamp. Particles clash, yes. But however cataclysmic the encounter, it always displays one great simplicity. It takes place on a local stage, an island of violence, apart from all happenings in the outside world. In other isolated arenas of action football players form a team, actors a troupe, soldiers a platoon; but in a battle of matter and energy, the participants receive the name **system**.

What the action starts with, what particles there are, what speeds they have, what directions they take: that's the story of the system at the start of the action. We may or may not pursue in all detail every stage of every encounter, as we view the scenes of a play or watch the episodes of a game. However, nothing that claims to be an account of the clash, brief though it may be, is worthy of the name unless it reports every participant that leaves the scene with its speed and its direction. Departing, they still belong to the system. Moreover, at every step of the way from entry to departure we continue to use for the collection of participants the name *system.*

Keep score of momenergy for the system

The child keeps count of who wins and who loses in the shoot-out before he or she learns to ask questions of right and wrong, of why and wherefore. We likewise keep tabs on what goes into an encounter and what comes out only to the extent of broadcasting the participants' momenergies before and after the act of violence. We do not open up in this book the more complex story of the forces, old and new, that govern the chances for this, that, and the other outcome of a given encounter. We limit ourselves to the ground rules of momenergy conservation in an isolated system.

8.2 THREE MODEST EXPERIMENTS

elastic glass balls; inelastic wads of gum; weighing heat

A collision does not have to be violent to qualify for attention nor be exotic to make momenergy scorekeeping interesting. It is fun to begin with momenergy scorekeeping for three encounters of everyday kinds before strolling out onto the laboratory floor of high-energy particle physics.

Elastic collision: Momenergy automatically conserved

First Experiment: Elastic Collision. Suspend two identical glass marbles from the ceiling by two threads of the same length so that the marbles hang, at rest, just barely touching. Draw one back with the finger and release it (Figure 8-1). The released marble gathers speed. The speed peaks just as the first marble collides with the second. The collision is elastic: Total kinetic energy before the collision equals total kinetic energy after the collision. The elastic collision brings the first marble to a complete stop. The impact imparts to the second all the momentum the first one had. Conservation of momentum could not be clearer:

$$\begin{pmatrix} \text{total momentum} \\ \text{to the right just} \\ \textbf{before} \text{ the collision,} \\ \text{all of it resident on} \\ \text{the } \textbf{first} \text{ marble} \end{pmatrix} = \begin{pmatrix} \text{total momentum} \\ \text{to the right just} \\ \textbf{after} \text{ the collision,} \\ \text{all of it resident on} \\ \text{the } \textbf{second} \text{ marble} \end{pmatrix}$$

And energy? In the collision the two particles exchange roles. The first particle comes to a halt. The second particle moves exactly as the first one did before the collision. Hence energy too is clearly conserved.

Just before the collision and just after: How do conditions compare? Same total momentum. Same total energy. Therefore same total momenergy.

Inelastic collision: Momenergy also conserved

Second Experiment: Inelastic Collision. Replace the two glass marbles by two identical balls of putty, wax, or chewing gum (Figure 8-2). Pull them aside by equal amounts and release.

Both released balls of chewing gum gather speed, moving toward one another. The equal and opposite velocities peak just before they collide with each other. By symmetry, the momentum of the right-moving particle has the same magnitude as the momentum of the left-moving particle. However, these momenta point in opposite directions. Regarded as vectors, they sum to zero. The momentum of the *system* therefore equals zero just before the collision.

Just after the collision? The two balls have stuck together. They are both at rest; each has zero momentum. Their combined momentum is also zero. In other words,

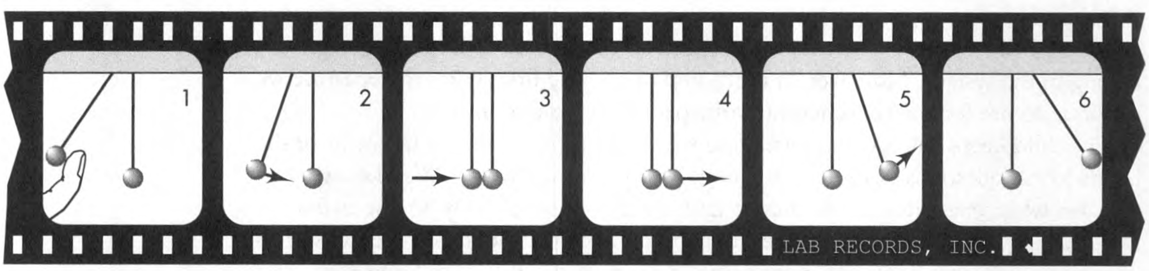


FIGURE 8-1. *One marble collides elastically with another.*

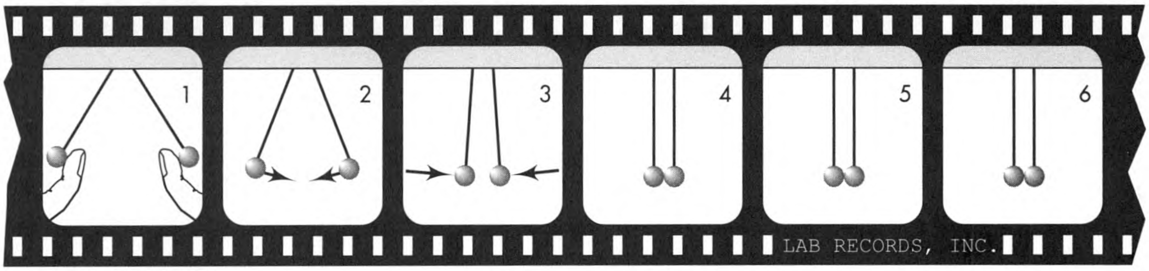


FIGURE 8-2. *One ball of chewing gum locks onto the other.*

the momentum of the *system* is zero after the collision. Zero it was also before the collision. Thus the momentum of the system is conserved.

For system energy the outcome is more perplexing. Just before the collision, each ball has an energy consisting of its mass m and its kinetic energy K. These energies add to make the total energy of the system: $E_{\text{system}} = 2m + 2K$.

Kinetic energy converted to mass

After the collision? Both balls of chewing gum are at rest, stuck together as a single blob, which now constitutes the entire system. The energy of that stationary blob must be its rest energy, equal to the mass of the system: $E_{\text{system}} = E_{\text{rest}} = M_{\text{system}}$. What is the value of that system energy? It must be the same as the energy of the system before the collision, equal to $2m + 2K$, where m is the mass of each ball before the collision. Hence, if energy is conserved, $M_{\text{system}} = 2m + 2K$. This is greater than the sum of masses of the incoming particles.

Where does this extra mass come from? The energy of relative motion of the incoming particles gets converted, during the collision, into energy of plastic deformation and heat. Each of these forms of locked-in energy yields an increment of mass. In consequence the mass of the pair of balls, stuck together as one, exceeds the sum of masses of the two balls before impact.

Third Experiment: Weighing Heat. If warmed and distorted balls of gum have more mass than cool and undistorted balls, then maybe we can measure directly the increased mass simply by heating an object and weighing it. In this case the system consists of a single large object, such as a tub of water, stationary and therefore with zero total momentum. System energy consists of the summed individual masses of all water molecules plus the summed kinetic energies of their random motions. This summed kinetic energy increases as we add heat to the water; hence its mass should increase. Can we detect the corresponding increase in weight as we heat the water in the tub?

Can we weigh heat?
Not yet!

Alas, never yet has anyone succeeded in weighing heat. In 1787 Benjamin Thompson, Count Rumford (1753–1814), tried to detect an increase in weight of barrels of water, mercury, and alcohol as their temperature rose from 29° F to 61° F (in which range ice melts). He found no effect. He concluded "that ALL ATTEMPTS TO DISCOVER ANY EFFECTS OF HEAT UPON THE APPARENT WEIGHTS OF BODIES WILL BE FRUITLESS" (capital letters his). Professor Vladimir Braginsky of the University of Moscow once described to us a new idea for weighing heat. Let a tiny quartz pellet hang on the end of a long thin near-horizontal quartz fiber, like a reeled-in fish at the end of a long supple fishing rod. A fly that settles on the fish increases its weight; the fishing rod bends a little more. Likewise heat added to the pellet will increase its mass and will bend the quartz-fiber "fishing rod" a little more. That is the idea. The sensitivity required to detect a bending so slight unfortunately surpasses the present limit of technology. Braginsky himself already has invented, published, and made available to workers all over the world a now widely applied scheme to measure very small effects. There is real hope that he—or someone else—will weigh heat and confirm what we already confidently expect.

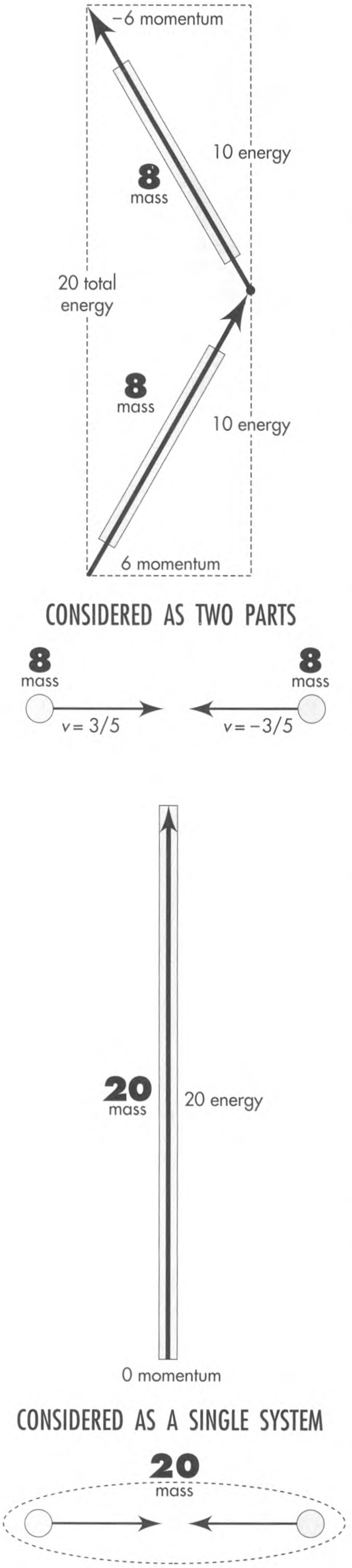


FIGURE 8-3. *Two noninteracting particles, each of mass 8, are in relative motion. Taken together, they constitute a system of mass 20. Where does the mass 20 reside? In the system!*

8.3 MASS OF A SYSTEM OF PARTICLES

energies add. momenta add. masses do not add.

No one with any detective instincts will rest content with the vague thought that heat has mass. Where within our stuck-together wads of chewing gum or Rumford's barrel of water or Braginsky's quartz pellet is that mass located? In random motions of the atoms? Nonsense. Each atom has mass, yes. But does an atom acquire additional mass by virtue of any motion? Does motion have mass? No. Absolutely not. Then where, and in what form, does the extra mass reside? Answer: Not in any part, but in the *system*.

Heat resides not in the particles individually but in the system of particles. Heat arises not from motion of one particle but from relative motions of two or more particles. Heat is a **system property.**

The mass of a system is greater when system parts move relative to each other. Of this central point, no simpler example offers itself than a system composed of a single pair of masses. Our example? Two identical objects (Figure 8-3). Each has mass 8. Relative to the laboratory frame of reference each object has momentum 6, but the two momenta are opposite in direction. The energy of each object is $E = (m^2 + p^2)^{1/2} = (8^2 + 6^2)^{1/2} = 10$.

The total momentum of the two-object system is $p_{\text{system}} = 6 - 6 = 0$. The energy of the system is $E_{\text{system}} = 10 + 10 = 20$. Therefore the mass of the system is $M_{\text{system}} = (E_{\text{system}}{}^2 - p_{\text{system}}{}^2)^{1/2} = [(20)^2 - 0^2]^{1/2} = 20$. Thus the mass of the system exceeds the sum of the masses of the two parts of the system. The mass of the system does not agree with the sum of the masses of its parts.

Energy is additive. Momentum is additive. But mass is *not* additive.

Ask where the *extra* $20 - 16 = 4$ units of mass are located? Silly question, any answer to which is also silly!

Ask where the 20 units of mass are located? Good question, with a good answer. The 20 units of mass belong to the system as a whole, not to any part individually.

Where is the life of a puppy located? Good question, with a good answer. Life is a property of the *system* of atoms we call a puppy, not a property of any part of the puppy.

Where is the *extra* ingredient added to atoms to yield a live puppy? Unacceptable question, any answer to which is also unacceptable. Life is not a property of any of the individual atoms of which the puppy is constituted. Nor is it a property of the space between the atoms. Nor is it an ingredient that has to be added to atoms. Life is a property of the puppy *system*.

Life is remarkable, but in one respect the two-object system that we are talking about is even more remarkable. Life requires organization, but the two-object system of Figure 8-3 lacks organization. Neither mass interacts with the other. Yet the total energy of the two-object system, and its total momentum, regarded from first one frame of reference, then another, then another, take on values identical in every respect to the values they would have were we dealing throughout with a single object of mass 20 units. Totally unlinked, the two objects, viewed as a system, possess the dynamic attributes — energy, momentum, and mass — of a single object.

This wider idea of mass — the mass of an isolated *system* composed of disconnected objects: what right have we to give it the name "mass"? Nature, for whatever reason, demands conservation of total momenergy in every collision. Each collision, no matter how much it changes the momenergy of each participant, leaves unchanged the sum of their momenergies, regarded as a directed arrow in spacetime — a 4-vector. Encounter or no encounter, and however complex any encounter, system momenergy does not alter. Neither in spacetime direction nor in magnitude does it ever change. But the magnitude — the length of the arrow of total momenergy, figured as we figure any spacetime interval — is system mass. Whether the system consists of a single object or of many objects, and whether these objects do or do not collide or otherwise interact with each other, this system mass never changes. That's why the concept of system mass makes sense!

Different free-float frames. Same system mass.

An example? Again, two objects of mass 8, again each moving toward a point midway between them at $v = (\text{momentum})/(\text{energy}) = (p = 6)/(E = 10) = 3/5$ the speed of light. Now, however, we analyze the two motions in a frame moving with the right-hand object (Figure 8-4). In this new frame the right-hand object is at rest: mass, $m = 8$; momentum, $p = 0$; energy, $E = [m^2 + p^2]^{1/2} = 8$. The left-hand object is approaching with a speed (addition of velocities: Section L.7 of the Special Topic following Chapter 3; also Exercise 3-11)

$$v = \frac{3/5 + 3/5}{1 + (3/5)(3/5)} = \frac{6/5}{34/25} = \frac{15}{17}$$

It has energy $E = m/[1 - v^2]^{1/2} = 8/[1 - (15/17)^2]^{1/2} = 17$ and momentum $p = vE = 15$. So much for the parts of the system! Now for the system itself. For the system the energy is $E_{\text{system}} = 8 + 17 = 25$ and the momentum is $p_{\text{system}} = 0 + 15 = 15$.

Now for the test! Does the concept of system mass make sense? In other words, does system mass turn out to have the same value in the new frame as in the original frame? It does:

$$\begin{aligned} M_{\text{system}} &= (E_{\text{system}}{}^2 - p_{\text{system}}{}^2)^{1/2} = [(25)^2 - (15)^2]^{1/2} = [625 - 225]^{1/2} \\ &= [400]^{1/2} = 20 \end{aligned}$$

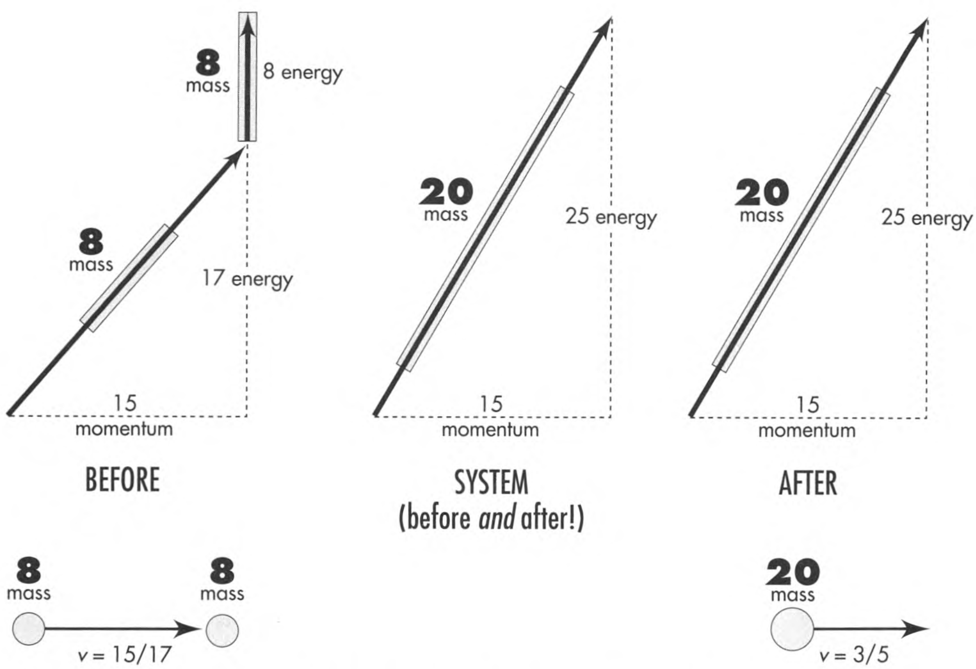


FIGURE 8-4. ***System of Figure 8-3 observed from a frame moving with the right-hand object.*** *The right-hand object is therefore initially at rest.* ***Before:*** *Arrows of momenergy for two objects before collision. Each object has a mass of eight units (shaded handles). The upper, vertical, arrow belongs to the particle originally at rest, the slanted arrow to the incoming particle.* ***System:*** *Addition of the two momenta (one of them zero!) gives the total momentum before collision. Similarly, addition of the two energies gives the total energy. Mass of the* system*—even before the two particles interact!—comes from the expression for the "hypotenuse" of a spacetimelike triangle. Result: 20 units of mass (shaded handle on center 4-vector):*

$$(\text{mass})^2 = (\text{energy})^2 - (\text{momentum})^2 = (25)^2 - (15)^2 = 625 - 225 = 400 = (20)^2$$

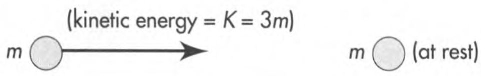
After: *The two particles now collide and amalgamate to form one particle. Arrow of total momenergy after the amalgamation is identical to arrow of total momenergy before the collision. Mass of this two-object system exceeds the mass of one object plus the mass of the other, not only after the collision but also before. Mass is not an additive quantity.*

SAMPLE PROBLEM 8-1

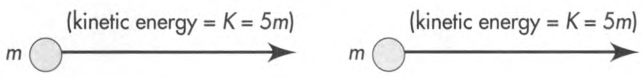
MASS OF A SYSTEM OF MATERIAL PARTICLES

Compute M_{system} for each of the following systems. The particles that make up these systems do not interact with one another. Express the system mass in terms of the unit mass m; do not use momenta or velocities in your answers. [Note: In the following diagrams, arrows represent (3-vector) momenta.]

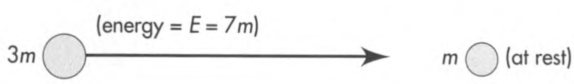
System a



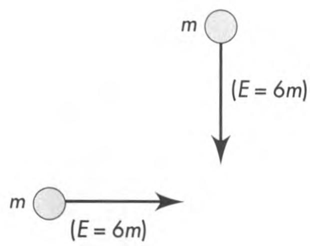
System b



System c



System d



SOLUTION

System a: System energy equals the rest energy of the two particles (the sum of their masses) plus the kinetic energy of the moving particle: $E_{\text{system}} = (m + m) + 3m = 5m$. Squared momentum of the system equals that of the moving particle: $p_{\text{system}}{}^2 = p^2 = E^2 - m^2 = (4m)^2 - m^2 = 15m^2$. Mass of the system is reckoned from the difference between the squares of energy and momentum:

$$M_{\text{system}} = [E_{\text{system}}{}^2 - p_{\text{system}}{}^2]^{1/2} = [25m^2 - 15m^2]^{1/2} = [10]^{1/2}m = 3.162\ m$$

Moreover, if the two objects collide and amalgamate, the system energy remains at the value 25, the system momentum remains at the value 15, and the system mass remains 20, as illustrated in Figure 8-4.

In summary, the mass of an isolated system has a value independent of the choice of frame of reference in which it is figured. System mass remains unchanged by encounters between the constituents of the system. And why? Because the system mass is the length (in the sense of spacetime interval) of the arrow of total momentum-energy. This momenergy total is unaffected by collisions among the parts or by any transformations, decays, or annihilations they may undergo. System mass *does* make sense!

System! System! You keep talking about "system," even when the particles do not interact, as in the system of Figure 8-3. It seems to me that you are totally arbitrary in the way you define a system. Who chooses which particles are in the system?

System b: System energy equals rest energy of the two particles plus kinetic energy of the two particles: $E_{\text{system}} = 2m + 10m = 12m$. Squared momentum of each particle is $p^2 = E^2 - m^2 = (6m)^2 - m^2 = 35m^2$ yielding $p = (35)^{1/2}m$. System momentum is twice this: $p_{\text{system}} = 2\,(35)^{1/2}\,m$. The mass of the system is

$$M_{\text{system}} = [E_{\text{system}}{}^2 - p_{\text{system}}{}^2]^{1/2} = [(12m)^2 - \{2\,(35)^{1/2}m\}^2]^{1/2}$$
$$= [144 - 140]^{1/2}m = [4]^{1/2}m = 2m$$

In this one special case the mass of the system equals the sum of masses of the objects that make up the system. We could have seen this result immediately by observing the system from a reference frame that moves along with the particles. In this frame the particles are at rest and have zero total momentum; the total energy is identical to the sum of the individual rest energies (the individual masses). So in this case the mass of the system is equal to its energy, which is equal to the sum of masses. Moreover, system mass is an invariant. Thus $2m$ is the mass of the system as reckoned in all reference frames, including the one in which System b is pictured.

System c: Total energy = system energy = $E_{\text{system}} = 7m + m = 8m$. System momentum equals the momentum of the moving particle: $p_{\text{system}}{}^2 = E^2 - m^2 = (7m)^2 - (3m)^2 = 49m^2 - 9m^2 = 40m^2$. Hence the system mass is

$$M_{\text{system}} = [64m^2 - 40m^2]^{1/2} = [24]^{1/2}m = 4.899m$$

System d: This part of the problem serves as a reminder that momentum is a Euclidean 3-vector. The squared momentum of each particle is $p^2 = E^2 - m^2 = 36m^2 - m^2 = 35m^2$. Their total momentum is *not* the algebraic sum of the momenta, because they are vectors pointing in perpendicular directions. This perpendicular orientation allows us to equate the squared system momentum to the sum of the squares of the individual momenta: $p_{\text{system}}{}^2 = 35m^2 + 35m^2 = 70m^2$. System energy is the sum of the energies (energy is a scalar and adds like a scalar!): $E_{\text{system}} = 6m + 6m = 12m$. Hence system mass is

$$M_{\text{system}} = [144m^2 - 70m^2]^{1/2} = [74]^{1/2}m = 8.602m$$

Compare this result with that of System b, which also contained two particles, each of total energy $6m$.

We do! We can draw the dashed line around any collection of objects whatever, subject to this one restriction: no object in our system may interact with any external object or experience a force from outside the system. Our system must be *isolated*. With that single limitation, the system we choose is arbitrary, has a conserved total energy, a conserved total momentum, and a system mass that is invariant—a mass that has the same value no matter in which free-float frame it is reckoned.

I can't believe the story you tell. Those two mass-8 objects, you say, may fly past each other. Then your talk about the system mass is just talk, terminology. Or they may whang into each other and amalgamate. Then your talk is all wrong, and for an obvious reason. As the objects collide they slow and come to rest relative to each other. At that instant and in that "rest frame" (the frame of Figure 8-3), each has zero momentum, and energy equal to its mass. So the total momentum of the system is zero, and its total energy is 8 + 8 = 16. That means a mass of 16. Yet you claim 20.

TABLE 8-1

CLEOPATRA'S VASE, HER BATH, AND INTERSTELLAR VACUUM: ILLUSTRATIVE FRACTIONAL CHANGES IN MASS OF SYSTEMS

System before	System after	Fractional increase in system mass (to nearest power of 10)
One-kilogram vase	Vase smashed into so many fragments that 100 centimeters^2 of glass-to-glass bonds are broken	10^{-18}
Bath water at 15° C	Bath water at 40° C	10^{-12}
Water (H_2O)	Atomic hydrogen (H) and oxygen (O)	10^{-9}
Earth	All molecules of Earth lifted against the pull of their mutual gravity to infinite separation from one another	10^{-9}
Hydrogen atom in lowest energy state	Electron withdrawn to infinite separation from nucleus	10^{-8}
Deuteron	Deuteron separated into proton and neutron	10^{-3}
Neutron star	Widely separated iron atoms at rest with respect to each other	10^{-1}
A vacuum, before it is zapped by converging photons	Electron–positron pair bound as a positronium atom	Infinite fractional increase

Slow and come to rest? Yes. But that means force: "elastic," gravitational, electromagnetic, or nuclear force. That's the new and valuable point you make here. And those particles, pushing against that force, store up energy. This energy, too, has to be put into the bookkeeping. When amalgamating particles come to rest relative to one another, the energy of interaction "balances the books"—it so happens—and leads to a final mass of 20, greater than the sum of masses of the original objects. For the figuring of system mass, however, we really don't have to get into this detail. It is enough for us to know that total momentum is conserved, $p_{\text{system}} = 0$ in Figure 8-3, and total energy—in whatever way it is apportioned between the objects and the fields of force that act between them—is also conserved, $E_{\text{system}} = 20$. The length, in the sense of interval, of the 4-vector of momenergy for the system remains unchanged: $M_{\text{system}} = 20$.

System energy increase? System mass can increase.

What about a system that is *not* isolated? A system that has—and keeps—zero momentum, but receives an increment of energy? Then its mass rises by an amount exactly equal to that input of energy. The increase in mass is the same whether that energy goes into altering the relative motion of the parts of the system or increasing the energy of interaction between them or some combination of motion and interaction. Supply energy to a system by heating it or setting it into internal vibration or fracturing the bonds between its parts? Each is a guaranteed way to increase the mass of the system (Table 8-1)!

8.4 ENERGY WITHOUT MASS: PHOTON

light moves with zero aging.
photons move with zero mass.

A striking example of the primacy of momenergy over mass is furnished by a **quantum of light** colliding with an electron.

Quantum? A quantum of luminous energy of a given color or, in more technical terms, light of a given wavelength or frequency of vibration. Max Planck discovered in 1900 that light of a given color comes only in quanta—"hunks"—of energy of a standard amount, an amount completely determined by the color (Table 8-2). We can have one quantum, one hunk, one **photon,** of green light, or two, or fifteen, but never two and a half.

Nothing did more to raise the light quantum, the hunk of luminous energy, the photon, to the status of a particle than experiments carried out by 28-year-old Arthur Holly Compton at Washington University, St. Louis, in 1920. Shining X-rays of known wavelength (and hence of known frequency and known quantum energy) on a variety of different substances, he measured the wavelength (and hence the quantum energy) of the emergent "scattered" X-rays. He got identical changes in wavelength at identical angles of observation from many kinds of materials. There was no way he could explain this result except to say that the scattering object was in every case the same, an electron, whatever the atom in which the electron happened to reside.

Compton demonstrates quantum of radiation—photon!

But why did the change of wavelength have a unique value, the same for all materials at a given angle of scattering? Every idea of classical physics failed to fit, Compton found. "Compton arrived at his revolutionary quantum theory for the scattering process rather suddenly in late 1922," a biographer tells us. "He now treated the interaction as a simple collision between [an X-ray quantum] and a free electron . . . [He] found that [this hypothesis gave results] which agreed perfectly with his data . . . When Compton reported his discovery at meetings of the American Physical Society, it aroused great interest and strong opposition . . ." By 1927, however, his finding was generally accepted and in that year won him the Nobel Prize.

What does it mean to treat a photon on the same footing as a particle? It means this: attribution to the photon of an energy and a momentum, in other words momenergy.

TABLE 8-2

MOMENTUM AND ENERGY CARRIED BY ONE PHOTON, ONE QUANTUM, ONE HUNK OF LUMINOUS ENERGY OF VARIOUS "COLORS"

(Unit of energy used in this table: electron-volt or eV, the amount of energy given to an electron by accelerating it through an electrical potential difference of one volt)

Source of electromagnetic radiation	Momentum (and energy) of a single quantum	Frequency in vibrations per second	Wavelength in meters
KDKA, Pittsburgh: world's first radio broadcast station	4.22×10^{-9} eV	1.02×10^{6}	294
A sample infrared beam	1.24×10^{-2} eV	3×10^{12}	10^{-4}
Yellow radiation from a sodium arc lamp	2.11 eV	5.09×10^{14}	5.90×10^{-7}
Ultraviolet light from a mercury arc lamp	4.89 eV	1.18×10^{15}	2.54×10^{-7}
Ultraviolet star radiation of just barely sufficient quantum energy to strip a hydrogen atom of its electron	13.6 eV	3.29×10^{15}	0.91×10^{-7}
Each of two gamma rays given off in the mutual annihilation of a slow positron and a slow electron	5.11×10^{5} eV	1.23×10^{20}	2.43×10^{-12}
Each of two gamma rays given out when a neutral pi meson, at rest, decays	6.75×10^{7} eV	1.63×10^{22}	1.84×10^{-14}
Each of two gamma rays given off in the mutual annihilation of a slow proton and a slow antiproton	0.938×10^{9} eV	2.27×10^{23}	1.32×10^{-15}

Photon momenergy points in lightlike direction

In what direction in spacetime does the photon's arrow of momenergy point? In a lightlike direction, because the photon—a quantum of light—travels with light speed!

When we turn from spacetime to a particular free-float frame of reference and observe a pulse of light at one event along its worldline and then observe it at a second event (Figure 8-5), we know in advance something important about the interval between the two events: It equals zero.

$$
\begin{aligned}
(\text{interval})^2 &= (\text{distance between two events})^2 - (\text{time between two events})^2 \\
&= (\text{difference between two quantities of identical magnitude}) \\
&= 0
\end{aligned}
$$

Photon momenergy: magnitude zero (photon mass = 0)

A photon in a pulse of light has a momenergy arrow with a tip and a tail, like the momenergy vector for any other particle. Between the tip and tail there is a magnitude. The magnitude for the photon, however, has the value zero—zero because this arrow points in the same direction in spacetime as the worldline of the light pulse (Figure 8-5). For that reason its space component (momentum) and its time component (energy) are equal. And, of course, we express the square of this magnitude as we express the square of any interval, as a *difference* between the squared timelike and spacelike separations between the two ends of the arrow:

$$
\begin{aligned}
&(\text{magnitude of momenergy arrow of photon})^2 \\
&\qquad = (\text{photon energy})^2 - (\text{photon momentum})^2 \\
&\qquad = (\text{photon mass})^2 = 0
\end{aligned}
$$

In brief, the lightlike character of the arrow of photon momenergy tells us that (1) photon mass equals zero and (2) the magnitude of momentum, or punch-delivering power, of the photon is identical in value with the energy of the photon:

$$(\text{photon energy}) = (\text{magnitude of photon momentum})$$

and

$$(\text{photon mass}) = 0$$

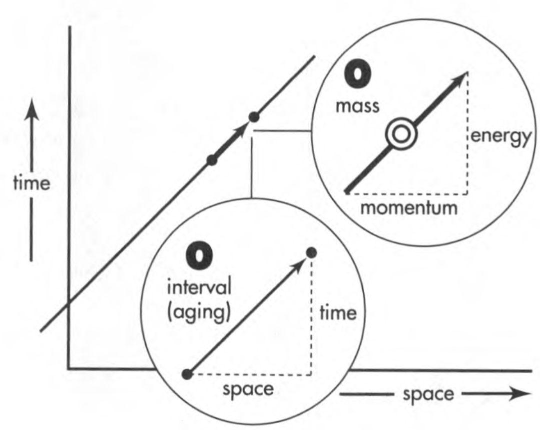


FIGURE 8-5. ***Worldline of a photon.*** *Note its "unit slope in spacetime."* ***Insets:*** *Unit slope of worldline means equal space and time separations between events on this worldline, hence zero interval between them—and zero aging for the photon. Momenergy of the same photon, also with unit slope, symbolizing three properties of the photon: it has zero mass (hence the big zero as an invariant "handle"), it travels with light speed, and it has a momentum identical in magnitude with its energy.*

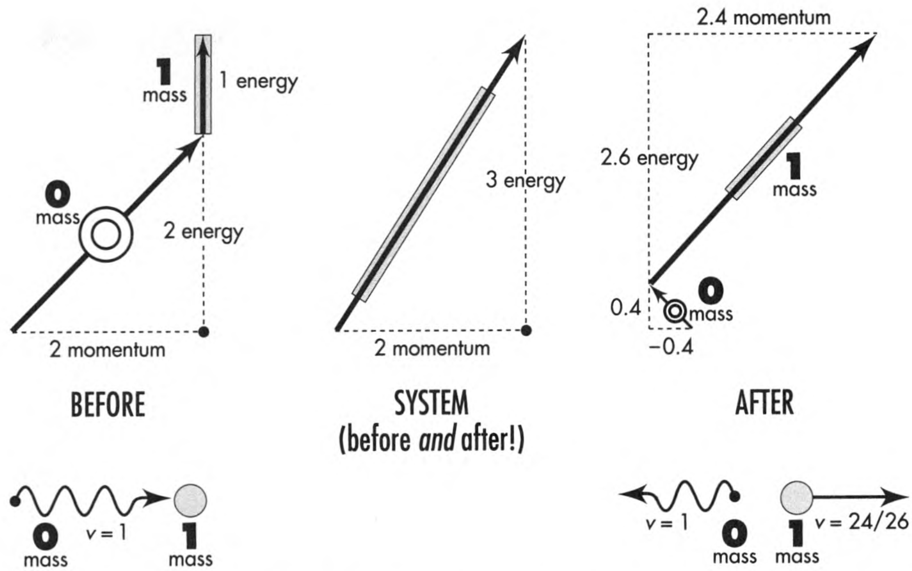


FIGURE 8-6. ***Backscattering of a photon by a free electron.*** *The wiggly arrow symbol represents a photon. Energy, momentum, and mass of all particles are expressed in units of electron mass.* ***Before:*** *The electron at rest has an energy equal to its mass (vertical arrow); the photon has an energy (and a momentum) of 2 electron masses (angled arrow).* ***System:*** *Arrow of total momenergy. (What is the mass of the* system?) ***After:*** *Arrows of momenergy of knocked-on electron (labeled 1) and backscattered photon (labeled 0) after the encounter. Arrow of total momenergy of the system remains the same (is conserved!) during this process.*

Figure 8-5 summarizes these features of the elementary quanta of visible light and other electromagnetic radiations. For a "handle" on the momenergy 4-vector of a photon — representative of its magnitude — we choose a stylized zero, 0.

Compton collision analyzed

Nothing shows these revolutionary features of light to better advantage than the very collision process studied by Arthur Compton: the encounter between a single photon and a single electron. We take the electron, loosely bound though it may be in one or another outer orbit of an atom, as essentially free and essentially at rest — at rest compared to the swift motion in which it finds itself after the high-energy photon hits it (Figure 8-6).

To simplify all numbers, we pick for the photon energy a value typical of gamma rays, considerably greater than that of the X-rays with which Compton worked but easily available today from various sources of radioactivity: 1.022 MeV (million electron-volts). We pick this number because we want to express all energies in units of electron mass, 9.11×10^{-31} kilograms or 0.511 MeV. Our choice of photon energy equals exactly two electron masses. Convenient!

Incoming photons of this energy, encountering an electron, are scattered by the electron sometimes in one direction, sometimes in another, and sometimes straight backward. In that most extreme of encounters — backward scattering — an interchange of momentum takes place that nevertheless preserves total momentum and also total energy, as illustrated in Figure 8-6. The electron is kicked forward with a momentum of $2.4 = 12/5$ times the electron mass, and the photon bounces backward with a momentum (and energy) of $0.4 = 2/5$ times the electron mass, much less than the two-electron masses of momentum (and energy) with which it approached.

SAMPLE PROBLEM 8-2

MASS OF A SYSTEM THAT INCLUDES PHOTONS

A photon has no rest energy — that is, no mass of its own. However, a photon can contribute energy and momentum to a system of objects. Hence the presence of one or more photons in a system can increase the mass of that system. More: A system consisting entirely of zero-mass photons can itself have nonzero mass!

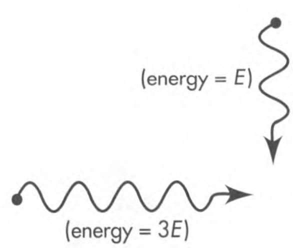
Find system mass M_{system} for each of the following systems. The particles that make up these systems do not interact with one another. Express the system mass in terms of the unit mass m (or the unit energy E in the photons-only systems). Use only energy and mass in your answers: no momenta or velocities.

System a

System b

System c

System d



SOLUTION

System a: System energy equals the rest energy m of the material particle plus the energy $E = 3m$ of the photon: $E_{system} = m + 3m = 4m$. The momentum of the system is equal to the momentum of the photon, which is equal to its energy: $p_{system} = 3m$. The mass of the system is reckoned from the difference of the square of energy and momentum:

$$M_{system} = [E_{system}{}^2 - p_{system}{}^2]^{1/2} = [(4m)^2 - (3m)^2]^{1/2} = [16m^2 - 9m^2]^{1/2}$$
$$= [7]^{1/2}m = 2.646m$$

System b: System energy equals the sum of the energies of the two photons: $E_{system} = 3E + E = 4E$. System momentum equals sum of momenta of the two photons — which in this case also equals the sum of the energies of the two photons: $p_{system} = 3E + E = 4E$. Therefore system mass equals zero:

$$M_{system} = [E_{system}{}^2 - p_{system}{}^2]^{1/2} = [(4E)^2 - (4E)^2]^{1/2} = 0$$

We could have predicted this result immediately. Two photons moving along in step are, as regards momentum and energy, completely equivalent to a single photon of energy equal to the sum of energies of the separate photons. And a single photon has, of course, zero mass.

System c: Total energy = system energy = $E_{system} = 3E + E = 4E$. System momentum equals the difference between the rightward momentum of the first particle and the leftward momentum of the second particle: $p_{system} = 3E - E = 2E$. Hence the system mass is

$$M_{system} = [16E^2 - 4E^2]^{1/2} = [12]^{1/2}E = 3.464E$$

Why can't we simply make a single photon by adding the energies of the two photons, as in system b? Because energies add as scalars, and momenta add as 3-vectors. In this case the total energy is $4E$ and the total momentum is $2E$. No way to make a single photon out of this; for a photon, energy and momentum must have equal magnitudes!

System d: This part serves as an additional reminder that momentum is a 3-vector. The system energy equals $E_{system} = E + 3E = 4E$. The squared momentum of the system equals the sum of squares of the momenta of the separate particles, since they move in perpendicular directions in this frame: $p_{system}{}^2 = E^2 + (3E)^2 = 10E^2$.

Hence system mass is:

$$M_{system} = [16E^2 - 10E^2]^{1/2} = [6]^{1/2}E = 2.449E$$

8.5 PHOTON USED TO CREATE MASS

photon hits electron, creates electron–positron pair

It should not be surprising that a photon can deliver energy without having any mass of its own. After all, an electron does have mass of its own; yet an electron traveling sufficiently close to light speed can impart to its target an amount of energy ten, a hundred, or a thousand times as great as its own mass. Not mass but momentum governs the size of punch that either photon or electron can deliver.

Matter is born

Incredibly, however, a photon in the presence of an electron can create matter out of empty space. To bring about this process, double the energy of the quantum of radiant energy shown in Figure 8-6. When a photon with energy equal to four electron masses hits an electron at rest, the photon most often recoils; in other words, it suffers backward scattering, an instance of the Compton process. Occasionally, however, the impacting photon produces out of empty space, near the struck electron, a new pair of electrons, one with a negative electric charge like all everyday electrons, the other with an identical amount of positive charge. The electron with positive charge has the name **positron** (Box 8-1).

This process goes on all the time high in Earth's atmosphere, where cosmic rays pour in from outer space. There, however, energies of cosmic-ray photons often far exceed four electron masses. In consequence, the struck electron and the two newly created electrons go off in slightly different directions and at different speeds. However, when the energy of the incoming photon is sufficiently finely tuned, in the immediate vicinity of an energy of four electron masses, the three particles can stick together as a super-light molecule, a **polyelectron**, a system analogous to what chemists call the hydrogen molecule ion (Figure 8-7).

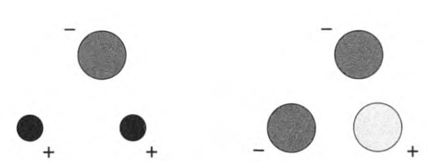


FIGURE 8-7. *Comparison and contrast.* ***Left:*** *Two protons and an electron forming the hydrogen molecule ion of chemistry. (A proton is much more massive than an electron but can be envisioned as occupying less volume.)* ***Right:*** *Two electrons and a positron, forming a* ***polyelectron*** *created by impact of a properly tuned photon (about 2 MeV of energy) on an electron at rest.*

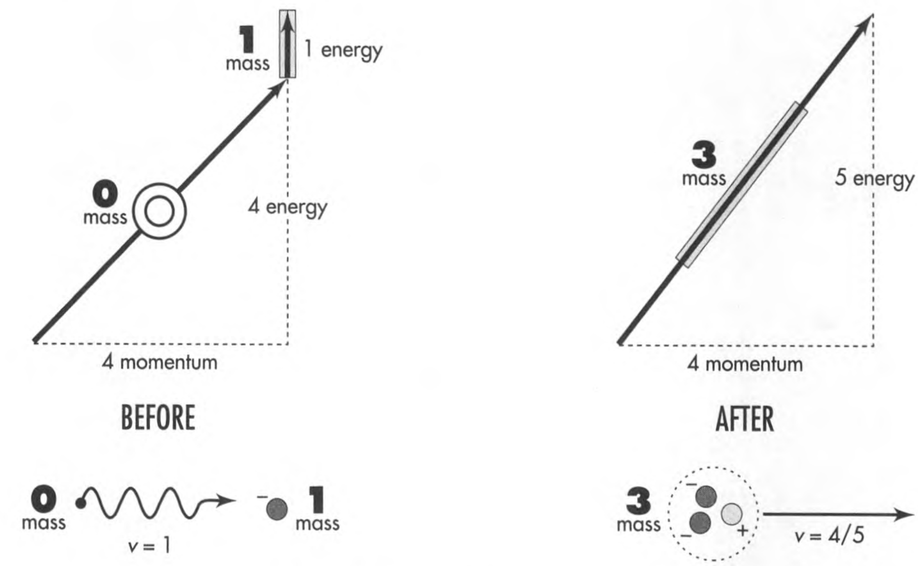


FIGURE 8-8. ***Conservation of energy and momentum in the process of creating a pair (a positive and a negative electron) in the field of an electron. Before:*** *A photon that has energy (and momentum) equal to four electron masses (sloping arrow) strikes an electron essentially at rest (vertical arrow).* ***After:*** *The photon has ceased to exist, and the two newly created particles have gone off in company with the original electron at 80 percent of light speed—a combined "particle" of three electron masses.*

Why does it take a light quantum with an energy of *four* electron masses to create (Figure 8-8) a polyelectron, a super-light hydrogen molecule ion, an object with a mass of three electron masses (in truth, a tiny bit less than three electron masses because of the negative binding energy among the three particles)? The question becomes all the more insistent when we recall that the electron that got hit already brought to the consummation of the deal a rest energy equal to one electron mass.

System momentum means not all system energy available to create particles

In brief, why do we have to put in five electron masses of energy to get out a three-electron-mass product? Simply asking this question points out where the explanation lies. The incident photon brings in a great momentum, and the electron with which it reacts has no momentum. So all that momentum has to go into the output product, the polyelectron. Since the polyelectron must have momentum, it must also have kinetic energy—energy not available for creating additional mass. In consequence, that object has so much energy of motion that only a much diminished part of the energy of the incident photon is available for the creation process itself.

8.6 MATERIAL PARTICLE USED TO CREATE MASS

proton hits proton, creates proton–antiproton pair

Any energetic particle can create other particles

Particles other than the photon can also create particles. A particle of any type can carry enough energy to create particles similar to or different from itself. Each such creation must not only follow momenergy conservation laws of special relativity, but it is also subject to the law of conservation of total electric charge and other conservation laws, as described in elementary particle physics.

BOX 8-1

BACKYARD ZOO OF PARTICLES

This is not a textbook of particle physics, but our examples include interactions between common particles. Here are brief descriptions of some of them.

Electron

Electrons form the outer structure of every atom and rattle around in approximately 99.99999999999 percent of its volume. The mass of the electrons of an atom, however, accounts for only about one two-thousandth of its mass or less. The electron carries a negative "elementary" electrical charge. Every accepted theory of particle physics treats the electron itself as an elementary particle — it is not made up of anything more fundamental. The **positron** is the antiparticle of the electron, with the same mass but a positive elementary charge. When positron and electron meet, sooner or later they mutually annihilate, yielding two or more high-energy photons (gamma rays). This will be the fate of the positron and one of the electrons in the polyelectron discussed in Section 8.5 soon after they begin to orbit one another.

Proton

The proton (Greek for "the first one") is, with the neutron, the most massive constituent of atomic nuclei. The simplest atom, hydrogen, in its most abundant form has a single proton as nucleus. The proton has a positive charge equal in magnitude to that of the electron, but a mass almost two thousand times as great as that of the electron. As far as we know the proton is stable; experiments have shown its lifetime to be greater than 10^{31} years — *very* much longer than the current age of the universe (about 10^{10} years). Particle physicists postulate that protons (and neutrons) are composed of still-more-elementary particles called quarks. The **antiproton,** antiparticle of the proton, has mass equal to that of the proton but negative unit charge. When it encounters a proton, the two particles annihilate, sometimes creating gamma rays but more often other particles not listed in this box.

Neutron

The neutron (from Latin *neuter* — "neither"; neither positively nor negatively charged) is similar to the proton but has no charge and has slightly greater mass. It is a constituent of all nuclei except for the most abundant form of elementary hydrogen. When not in a nucleus, the neutron decays into a proton, electron, and neutrino with half-life of about 10 minutes.

Photon

The photon, the quantum of light, has zero mass. Its properties are described in Section 8.4.

Neutrino

There are several kinds of neutrinos, all of which appear to have zero mass and to move at light speed. The neutrino (Italian for "little neutral one") has no charge and interacts only weakly with ordinary matter: Neutrinos of certain energies can pass through a block of lead one light-year thick with only a 50–50 chance of being absorbed! An immense flux of neutrinos passes continually through our bodies without injuring us. "Ten million trillion [10^{19}] neutrinos will speed harmlessly through your brain and body in the time it takes to read this sentence. By the time you have read this sentence, they will be farther away than the moon."

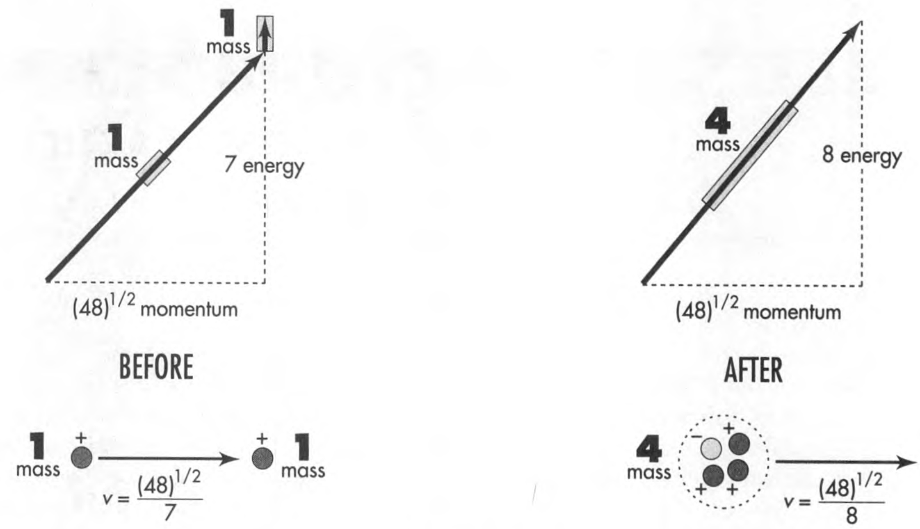


FIGURE 8-9. ***Conservation of energy and momentum in the process of creation of a proton–antiproton pair by the impact of a proton on another proton. Before:*** *The incoming proton (sloping arrow) moves with a speed $(48)^{1/2}/7 = 99$ percent that of light. The target proton initially stands at rest (vertical arrow).* ***After:*** *The resulting three protons and one antiproton are kicked to the right at $(48)^{1/2}/8$ $= 87$ percent of light speed.*

Figure 8-9 shows "the creation of a proton–antiproton pair by a proton in the presence of another proton." The antiproton has mass equal to that of the proton but carries a negative unit charge (Box 8-1). The interaction shown leaves all four resulting particles moving along together. The resulting particles stay together when the incoming particle has the lowest energy that can create the additional pair. This minimum energy is called the **threshold energy.** We don't want the four particles to move apart after the creation. If they did, we would have to supply the incoming particle with additional kinetic energy. It would have to carry an energy greater than the threshold energy. We discuss here the threshold energy of the incoming proton.

Threshold energy defined

Magnitudes of the momenergy vectors displayed in Figure 8-9 are expressed in "natural units" for the proton, namely the mass of the proton itself, 1.67×10^{-27} kilograms or 938.27 MeV. This time the numbers are not all integers: the momentum of the system has a value equal to the square root of 48, or 6.928 proton masses.

The creation of a proton–antiproton pair by a PROTON requires a total of eight proton units of energy to create two proton units of mass. In contrast the creation of an electron–antielectron pair by a PHOTON requires a total of only four electron units of energy to create two electron units of mass. Why is the photon process so much more efficient (in units of mass of the struck particle) than the proton process? Answer: The photon is annihilated in the creation process. In contrast, the incoming proton is not annihilated; the bookkeeper must keep the incoming proton on the payroll, providing momenergy after the collision to keep the proton in step with the other three particles. This after-collision momenergy of the proton is not available to be applied to other products of the collision. Therefore a proton of given total energy can create less mass than a photon of the same energy when each strikes a stationary target.

"Efficiency" of particle production

8.7 CONVERTING MASS TO USABLE ENERGY: FISSION, FUSION, ANNIHILATION

fission and fusion both slide down the energy hill toward the minimum, iron. electron and positron annihilate to yield two energetic photons.

For a final perspective on the evanescence of mass and the preservation of momenergy, turn from processes where mass is created to three processes in which mass is destroyed: fission, fusion, and annihilation.

Anyone who first hears about the splitting of a nucleus (fission) as a source of energy, and the joining of two nuclei (fusion) also as a source of energy might gain the mistaken impression that a perpetual motion machine has been invented. Could we split and join the same nucleus over and over again, each time releasing energy? No. Here's why. Fission occurs in the splitting of uranium, for instance when a neutron strikes a uranium nucleus:

$$ {}^{1}_{0}\mathrm{n} + {}^{235}_{92}\mathrm{U} \longrightarrow {}^{236}_{92}\mathrm{U} \longrightarrow {}^{95}_{37}\mathrm{Rb} + {}^{141}_{55}\mathrm{Cs} $$

In this equation the lower-left subscript tells the number of protons in the given nucleus and the upper-left superscript shows number of protons plus neutrons in the nucleus. The process described by this equation rearranges the 236 nucleons, that is, 92 protons plus 144 neutrons, into a configuration that comes a bit closer to that most stable of all available nuclear configurations, the iron nucleus:

Fission and fusion: Both go from looser to tighter binding

$$ {}^{56}_{26}\mathrm{Fe} $$

But fusion too, for example the process of uniting two rather light nuclei such as "heavy hydrogen" or deuterons to form a helium nucleus,

$$ {}^{2}_{1}\mathrm{D} + {}^{2}_{1}\mathrm{D} \longrightarrow {}^{4}_{2}\mathrm{He} $$

can also be regarded as one step along the way toward rearranging nucleons (protons and neutrons) to achieve the iron configuration or something like it.

In brief, we can get energy out of nucleon rearrangement processes that move from looser binding of both heavier and lighter nuclei toward tighter binding of the (intermediate-mass) iron nucleus (Figure 8-10). In neither fission nor fusion, however, is the fraction of mass converted into energy as great as one percent. (For an example of fusion reaction in Sun, see Sample Problem 8-5, especially **c**.)

Annihilation is interesting because it can convert 100 percent of matter into radiation. Annihilation is interesting, too, because it has been demonstrated on the microscopic scale. A slow positive electron, a positron, joining up by chance to orbit with an everyday negative electron, eventually unites with it to annihilate them both and produce sometimes two, sometimes three light quanta (photons—called **gamma rays** in the case of these high energies):

Annihilation converts 100% of matter into radiation

$$ e^{+} + e^{-} \rightarrow 2 \text{ or } 3 \text{ photons} $$

Figure 8-11 displays the balance of energy and momentum in the two-quantum annihilation process.

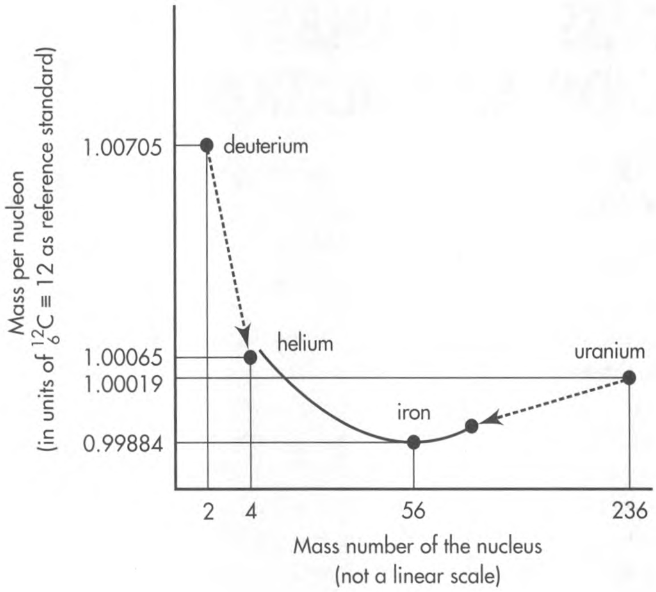


FIGURE 8-10. *Both the conversion of deuterium to the more massive helium in fusion and the conversion of uranium to lighter nuclei in fission decrease the mass per nucleon, both toward the most stable of nuclei, iron.*

Why 2 or 3 photons? Why can't just a single photon be emitted in this process?

Brief answer: Conservation of momentum. Fuller answer: Look at Figure 8-11. Before annihilation, the system has zero total momentum. A single photon remaining after the annihilation could not have zero momentum, no matter in which direction it moved! The presence of a single photon after the collision could not satisfy conservation of momentum. So annihilation never does and never can end up giving only a single photon.

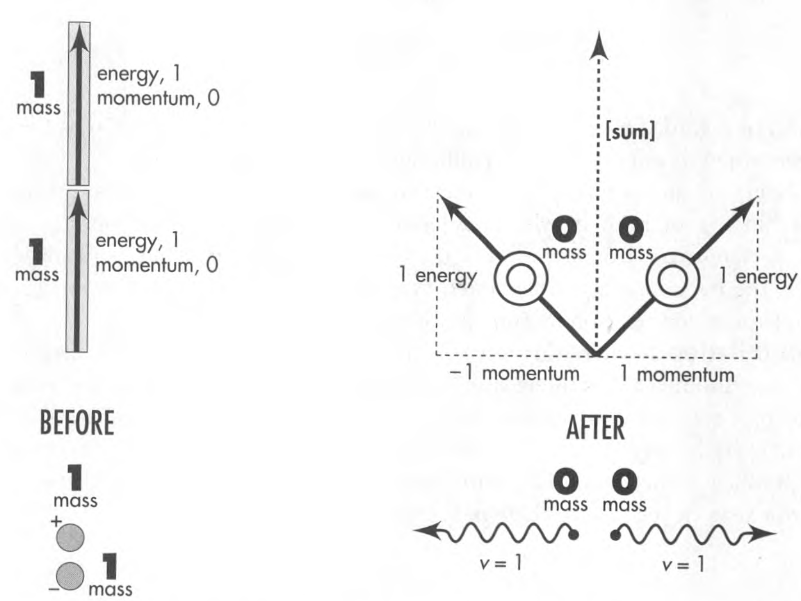


FIGURE 8-11. ***Momenergy conservation in the two-photon electron–positron annihilation process. Before:*** *Before annihilation each oppositely charged particle has rest energy and no momentum.* ***After:*** *The two particles have annihilated, creating two high-energy photons (gamma rays). The two photons fly apart in opposite directions; total momentum remains zero.*

BOX 8-2

ANALYZING A PARTICLE ENCOUNTER

Conservation of total momenergy! In any given free-float frame that means conservation of total energy and conservation of each of three components of total momentum. In no way does the power and scope of this principle make itself felt more memorably than the analysis of simple encounters of this, that, and the other kind in an isolated system of particles. "Analyzing an encounter" means using conservation laws and other relations to find unknown masses, energies, and momenta of particles in terms of known quantities. Sometimes a complete analysis is not possible; the information provided may be insufficient. Here are suggested steps in analyzing an encounter. Sample Problems 8-3 and 8-4 illustrate these methods.

1. **Draw a diagram** of particles *before* and particles *after* the interaction. Label particles entering with numbers or letters and particles leaving with different numbers or letters (even if they are the same particles). Use arrows to show particle directions of motion and label with symbols their masses, energies, and momenta, whether initially known or unknown.

2. Write down algebraically the **conservation of total energy.** Do not forget to include the rest energy — the mass m — of any particle not moving in the chosen free-float frame.

3. Write down algebraically the **conservation of total momentum.** Do not forget that momentum is a vector. In general this means demanding conservation of each of three components of total momentum.

4. Try to **solve** for unknowns in terms of knowns, still using symbols.

   a. Make liberal use of the relation $m^2 = E^2 - p^2$, where $p^2 = p_x{}^2 + p_y{}^2 + p_z{}^2$. For a photon or neutrino, mass equals zero and $E = p$ (in magnitude: Pay attention to the direction of the momentum vector $\mathbf{p}$ — or its sign if motion is in one space dimension).

   b. Do NOT use speed v of a particle unless forced to by requirements of the problem. Relativistic particles typically move with speeds very close to light speed, so speed proves to be a poor measure of significance. Increase by one percent the speed of a particle moving at $v = 0.99$ and you increase its energy by a factor of almost 10.

   c. Substitute numerical values into resulting equations as late as possible. Before substituting numerical values, check that all values are expressed in concordant units.

5. **Check** your result. Check units of the solution. Is the order of magnitude of numerical results reasonable? Substitute limiting values, for example letting energy of an incoming particle become very large (and very small). Is the limiting-case result reasonable?

Is there any **general conclusion** you can draw from your specific solution? Does this exercise illustrate a deep principle or lead to an even more interesting application of conservation laws?

SAMPLE PROBLEM 8-3

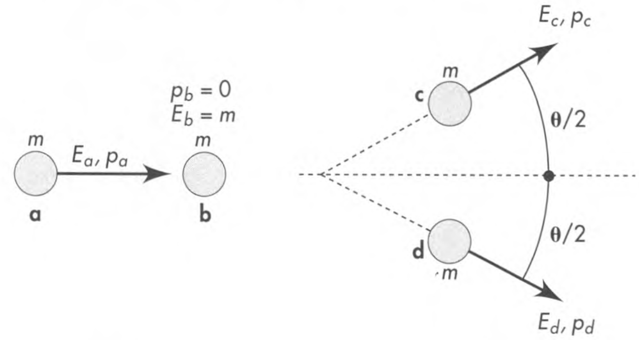
SYMMETRIC ELASTIC COLLISION

A proton of mass m and kinetic energy K in the laboratory frame strikes a proton initially at rest in that frame. The two protons undergo a *symmetric elastic* collision: the outgoing protons move in directions that make equal and opposite angles $\theta/2$ with the line of motion of the original incoming particle. Find energy and momentum of each outgoing particle and angle θ between their outgoing directions of motion for this symmetric case.

Historical note: When impact speed is small compared to the speed of light, this separation of directions, θ, is 90 degrees, according to Newtonian mechanics. Early cloud-chamber tracks sometimes showed symmetric collisions with angles of separation substantially less than 90 degrees, thereby giving evidence for relativistic mechanics and providing the first reliable measurements of impact energy.

SOLUTION, following steps in Box 8-2

1. **Draw a diagram** and label all four particles with letters:



BEFORE AFTER

Symmetry of this diagram implies that the two outgoing particles have equal energy and equal magnitude of momentum; that is, $E_c = E_d$ and (in magnitude) $p_c = p_d$.

2. **Conservation of energy:** Energy of each particle equals mass plus kinetic energy. And the masses don't change in this reaction. Therefore total kinetic energy after the encounter (divided equally between the two particles) equals the (known) total kinetic energy before the encounter, all localized on one particle. In brief: $K_c = K_d = K_a/2 = K/2$. Simple answer to one of the three questions we were asked!

3. **Conservation of momentum:** By symmetry, the vertical components of momenta of the outgoing particles cancel. Horizontal components add, leading to the relation

$$p_{\text{tot}} = p_a = p_c \cos(\theta/2) + p_d \cos(\theta/2) = 2p_d \cos(\theta/2)$$

or, in brief,

$$p_a = 2p_d \cos(\theta/2) \qquad \text{[conservation of momentum]}$$

4. **Solve for the unknown angle θ:** Along the way find the other requested quantity, the magnitude $p_c = p_d$ of the momenta after the collision. To that end, first find the momentum p_a before the collision, using the general formula for the momentum of an individual particle:

$$p = [E^2 - m^2]^{1/2} = [(K + m)^2 - m^2]^{1/2} = (K^2 + 2mK + m^2 - m^2)^{1/2}$$
$$= (K^2 + 2mK)^{1/2}$$

Therefore

$$p_a = (K^2 + 2mK)^{1/2}$$

From conservation of energy, $K_c = K_d = K/2$. Therefore

$$p_d = [(K/2)^2 + 2m(K/2)]^{1/2}$$

Substitute these expressions for p_a and p_d into the equation for conservation of momentum:

$$(K^2 + 2mK)^{1/2} = 2[(K/2)^2 + 2m(K/2)]^{1/2}\cos(\theta/2)$$

Square both sides and solve for $\cos^2(\theta/2)$ to obtain

$$\cos^2(\theta/2) = \frac{K + 2m}{K + 4m}$$

Now apply to this result the trigonometric identity

$$\cos^2(\theta/2) \equiv \frac{(\cos\theta + 1)}{2}$$

After some manipulation, obtain the desired result:

$$\cos\theta = \frac{(K/m)}{(K/m) + 4}$$

Here K is the kinetic energy of the incoming particle, m the mass of either particle, and θ the angle between outgoing particles. This result assumes (1) an elastic collision (kinetic energy conserved), (2) one particle initially at rest, (3) equal masses of the two particles, and (4) the symmetry of outgoing paths shown in the diagram.

5a. Limiting case: Low energy. In the case of low energy (Newtonian limit), the incoming particle has a kinetic energy K very much less than its rest energy m, so the ratio K/m approaches zero. In the limit, $\cos\theta$ becomes zero and $\theta = 90$ degrees. This is the accepted Newtonian result for low velocities (except for an exactly head-on collision, in which case the incoming particle stops dead and the struck particle moves forward with the same speed and direction as the original incoming particle).

5b. Limiting case: High energy. For extremely high-energy elastic collisions, the incident particle has a kinetic energy very much greater than its rest energy, so the ratio K/m increases without limit. In this case the quantity 4 in the denominator becomes negligible compared with K/m, so numerator and denominator both approach the value K/m, with the result $\cos\theta \longrightarrow 1$ and $\theta \longrightarrow 0$. This means that in the special symmetric case discussed here both resulting particles go forward in the same direction as the incoming particle, sharing equally the kinetic energy of the incoming particle.

For an incoming particle of very high energy, the elastic collision described here is only one of several possible outcomes. Alternative processes include creation of new particles.

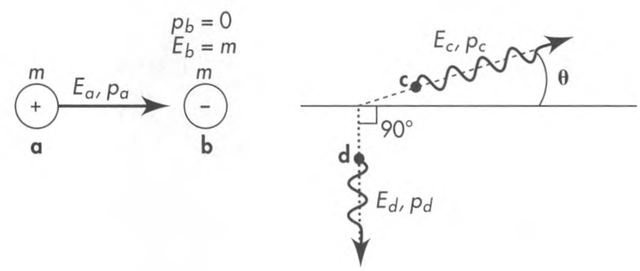
SAMPLE PROBLEM 8-4

ANNIHILATION

A positron of mass m and kinetic energy equal to its mass strikes an electron at rest. They annihilate, creating two high-energy photons. One photon enters a detector placed at an angle of 90 degrees with respect to the direction of the incident positron. What are the energies of both photons (in units of mass of the electron) and direction of motion of the second photon?

SOLUTION, following steps in Box 8-2

1. **Draw a diagram** and label the particles with letters.



BEFORE — AFTER

2. **Conservation of energy** expressed in the symbols of the diagram, and including the rest energy of the initial stationary particle:

$$E_{\text{tot}} = E_a + m = E_c + E_d$$

3. **Conservation of each component of total momentum:**

$$p_{x\,\text{tot}} = p_a = p_c \cos\theta \qquad \text{[horizontal momentum]}$$
$$p_{y\,\text{tot}} = 0 = p_c \sin\theta - p_d \qquad \text{[vertical momentum]}$$

4. **Solve:** First of all, the problem states that the kinetic energy K of the incoming positron equals its rest energy m. Therefore its total energy $E_a = m + K = m + m = 2m$. Second, the outgoing particles are photons, for which $p_c = E_c$ and $p_d = E_d$ in magnitude, respectively. With these substitutions, the three conservation equations become

$$E_a + m = 2m + m = 3m = E_c + E_d \qquad \text{[conservation of energy]}$$
$$p_a = E_c \cos\theta \qquad \text{[conservation of horizontal momentum]}$$
$$E_d = E_c \sin\theta \qquad \text{[conservation of vertical momentum]}$$

SAMPLE PROBLEM 8-5

CONVERSION OF MASS TO ENERGY IN SUN

Luminous energy from Sun pours down on the outer atmosphere of Earth at a rate of 1372 watts per square meter of area that lies perpendicular to the direction of this radiation. The figure 1372 watts per square meter has the name **solar constant.** The radius of Earth equals approximately 6.4×10^6 meters and the Earth-Sun distance equals 1.5×10^{11} meters. The mass of Sun is approximately 2.0×10^{30} kilograms.

These are three equations in three unknowns E_c and E_d and θ. Square both sides of the second and third equations, add them, and use a trigonometric identity to get rid of the angle θ:

$$p_a^2 + E_d^2 = E_c^2(\cos^2\theta + \sin^2\theta) = E_c^2$$

Substitute $p_a^2 = E_a^2 - m^2$ on the left side of this equation and again use $E_a = 2m$ to obtain a first expression for E_c^2:

$$E_c^2 = E_a^2 - m^2 + E_d^2 = 4m^2 - m^2 + E_d^2 = 3m^2 + E_d^2$$

Now solve the equation of conservation of energy for E_c and square it to obtain a second expression for E_c^2:

$$E_c^2 = (3m - E_d)^2 = 9m^2 - 6mE_d + E_d^2$$

Equate these two expressions for E_c^2 and subtract E_d^2 from both sides to obtain

$$3m^2 = 9m^2 - 6mE_d$$

Solve for unknown E_d:

$$E_d = \frac{9m^2 - 3m^2}{6m} = \frac{6m^2}{6m} = m$$

This yields our first unknown. Use this result and conservation of energy to find an expression for E_c:

$$E_c = 3m - E_d = 3m - m = 2m$$

Finally, angle θ comes from conservation of vertical momentum. For a photon $p = E$, so

$$\sin\theta = \frac{p_d}{p_c} = \frac{E_d}{E_c} = \frac{m}{2m} = \frac{1}{2}$$

from which $\theta = 30$ degrees. We have now solved for all unknowns: $E_c = 2m$, $E_d = m$, and $\theta = 30$ degrees.

5. **Limiting cases:** There is no limiting case here, since the energy of the incoming positron is specified fully in terms of the mass m common to electron and positron.

---

a. How much mass is converted to energy every second in Sun to supply the luminous energy that falls on Earth?

b. What *total* mass is converted to energy every second in Sun to supply luminous energy?

c. Most of Sun's energy comes from burning hydrogen nuclei (mostly protons) into helium nuclei (mostly a two-proton–two-neutron combination). Mass of the proton equals 1.67262×10^{-27} kilogram, while the mass of a helium nucleus of this kind equals 6.64648×10^{-27} kilogram. How many metric tons of hydrogen must Sun convert to helium every second to supply its luminous output? (One metric ton is equal to 1000 kilograms, or 2200 pounds.)

d. Estimate how long Sun will continue to warm Earth, neglecting all other processes in Sun and emissions from Sun.

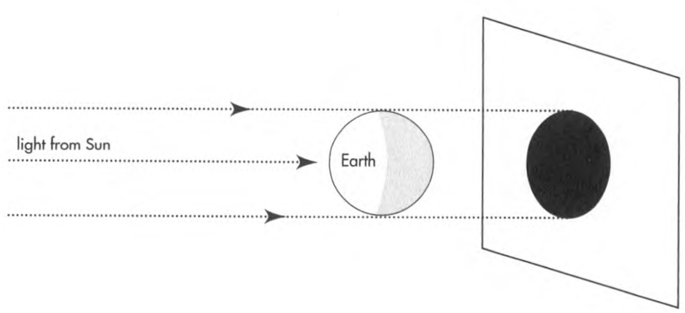
SOLUTION

a. One watt equals one joule per second = one kilogram meter²/second³. We want to measure energy in units of mass—in kilograms. Do this by dividing the number of joules by the square of the speed of light (Section 7.5 and Table 7-1):

$$\frac{1372 \text{ joules}}{c^2} = \frac{1.372 \times 10^3 \text{ kilogram meters}^2/\text{second}^2}{9.00 \times 10^{16} \text{ meters}^2/\text{second}^2}$$
$$= 1.524 \times 10^{-14} \text{ kilograms}$$

Thus every second 1.524×10^{-14} kilogram of luminous energy falls on each square meter perpendicular to Sun's rays. The following calculations are based on a simplified model of Sun (see last paragraph of this solution). Therefore we use the approximate value 1.5×10^{-14} kilogram per second and two-digit accuracy.

What total luminous energy falls on Earth per second? It equals the solar constant (in kilograms per square meter per second) times some area (in square meters). But what area? Think of a huge movie screen lying behind Earth and perpendicular to Sun's rays (see the figure). The shadow of Earth on this screen forms a circle of radius equal to the radius of Earth. This shadow represents the zone of radiation removed from that flowing outward from Sun. Call the area of this circle the cross-sectional area A of Earth. Earth's radius $r = 6.4 \times 10^6$ meters, so the cross-sectional area A seen by incoming Sunlight equals $A = \pi r^2 = 1.3 \times 10^{14}$ meters². Hence a total luminous energy equal to $(1.5 \times 10^{-14}$ kilograms/meter²$) \times (1.3 \times 10^{14}$ meters²$) = 2.0$ kilograms fall on Earth every second. This equals the mass converted every second in Sun to supply the light incident on Earth.



8.8 SUMMARY

mass: the magnitude of the 4-vector called momenergy

"Mass can be converted into energy and energy can be converted into mass"—this is a loose and sometimes misleading way to summarize some consequences of the two

b. Assume that Sun delivers sunlight at the same "solar-constant rate" to every part of a sphere surrounding Sun of radius equal to the Earth–Sun distance. The area of this large sphere has the value $4\pi R^2$ where $R = 1.5 \times 10^{11}$ meters, the average distance of Earth from Sun. This area equals 2.8×10^{23} meters^2. Therefore Sun converts a total of 2.8×10^{23} meters$^2 \times 1.5 \times 10^{-14}$ kilograms/meter^2 (from **a**) $= 4.2 \times 10^9$ kilograms of mass into luminous energy every second, or about 4 million metric tons per second.

c. Through a series of nuclear processes not described here, four protons transform into a helium nucleus consisting of two protons and two neutrons. The four original protons have a mass $4 \times 1.67262 \times 10^{-27} = 6.69048 \times 10^{-27}$ kilogram. The helium nucleus has a mass 6.64648×10^{-27} kilogram. The difference, 0.04400×10^{-27} kilogram, comes out mostly as light. (We cannot use two-digit accuracy here, because the important result is a difference between nearly equal numbers.)

The ratio of hydrogen burned to mass converted equals $6.69048/0.04400 = 150$ (back to two-digit accuracy!). So for each kilogram of mass converted to electromagnetic radiation, 150 kilograms of hydrogen burn to helium. In other words, about 0.7 percent of the rest energy (mass) of the original hydrogen is converted into radiation. Hence in order to convert 4.2×10^9 kilograms per second into radiation, Sun burns $150 \times 4.2 \text{ x } 10^9$ kilograms per second $= 6.3 \times 10^{11}$ kilograms of hydrogen into helium per second—about 630 million metric tons each second.

d. We can reckon Sun's mass by figuring how much Sun gravity it takes to guide our planet around in an orbit of 8 light-minute radius and one year time of circuit. Result: about 2.0×10^{30} kilograms. If Sun were all hydrogen, then the process of burning to helium at the present rate of 6.3×10^{11} kilograms every second would take $(2.0 \times 10^{30}$ kilograms$)/(6.3 \times 10^{11}$ kilograms/second$) = 3.2 \times 10^{18}$ seconds. At 32 million seconds per year, this would last about 10^{11} years, or 100 billion years.

Of course the evolution of a star is more complicated than the simple conversion of hydrogen into helium-plus-radiation. Other nuclear reactions fuse helium into more massive nuclei on the way to the most stable nucleus, iron-56 (Section 8.7). These other reactions occur at higher temperatures and typically proceed at faster rates than the hydrogen-to-helium process. Sun emits a flood of neutrinos (invisible; detected with elaborate apparatus; amount presently uncertain by a factor of 2, carry away less than 1 percent of Sun's output). Sun also loses mass as particles blown away from the surface, called the **solar wind.** And stars do not convert all their hydrogen to helium and other nuclei—or live for 100 billion years. According to current theory, the lifetime of a star like Sun equals approximately 10 billion years (10^{10} years). We believe Sun to be 4 to 5 billion years old. The remaining 6 billion years (6×10^9 years) or so should be sufficient time for our descendants to place themselves in the warmth of nearby stars.

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principles that are basic and really accurate: (1) The total momenergy of an isolated system of particles remains unchanged in a reaction; (2) The invariant magnitude of the momenergy of any given particle equals the mass of that particle.

How much sound information about physics can be extracted from these basic principles? What troubles sometimes arise from accepting a too loose formulation of the "principle of equivalence of mass and energy"? Some answers to these questions appear in the dialog that follows, which serves also as a summary of this chapter.

DIALOG: USE AND ABUSE OF THE CONCEPT OF MASS

Does an isolated system have the same mass as observed in every inertial (free-float) reference frame?

Yes. Given in terms of energy E and momentum p by $m^2 = E^2 - p^2$ in one frame, by $m^2 = (E')^2 - (p')^2$ in another frame. Mass of an isolated system is thus an *invariant*.

Does its *energy* have the same value in every inertial frame?

No. Energy is given by $E = (m^2 + p^2)^{1/2}$ or

$$E = m/(1 - v^2)^{1/2}$$

or

$$E = (\text{mass}) + (\text{kinetic energy}) = m + K$$

Value depends on the frame of reference from which the particle (or isolated system of particles) is observed. Value is lowest in the frame of reference in which the particle (or system) has zero momentum (zero *total* momentum in the case of an isolated system of particles). In that frame, and in that frame only, energy equals mass.

Does energy equal zero for an object of zero mass, such as a photon or neutrino or graviton?

No. Energy has value $E = (0^2 + p^2)^{1/2} = p$ (or in conventional units $E_{conv} = cp_{conv}$). Alternatively one can say—formally—that the entire energy resides in the form of *kinetic* energy ($K = p$ in this special case of *zero* mass), none at all in the form of rest energy. Thus,

$$E = (\text{mass}) + (\text{kinetic energy}) = 0 + K = K = p$$

(case of zero mass only!).

Can a photon—that has no mass—give mass to an absorber?

Yes. Light with energy E transfers mass $m = E$ ($= E_{conv}/c^2$) to a heavy absorber (Exercise 8.5).

Invariance of mass: Is that feature of nature the same as the principle that all electrons in the universe have the same mass?

No. It is true that all elementary particles of the same kind have the same mass. However, that is a fact totally distinct from the principle that the mass of an isolated system has identical value in whatever free-float frame it is figured (invariance of system mass).

Invariance of mass: Is that the same idea as the conservation of the momenergy of an isolated system?

No. Conservation of momenergy—the principle valid for an isolated system—says that the momenergy 4-vector figured *before* the constituents of a system have interacted is identical to the momenergy 4-vector figured *after* the constituents have interacted. In contrast, invariance of mass—the magnitude of the momenergy 4-vector—says that that mass is the same in *whatever* free-float frame it is figured.

Momenergy: Is that a richer concept than mass?

Yes. Momenergy 4-vector reveals mass and more: the motion of object or system with the mass

Conservation of the momenergy of an isolated system: Does this imply that collisions and interactions within an isolated system cannot change the system's mass?

Yes. Mass of an isolated system, being the magnitude of its momenergy 4-vector, can never change (as long as the system remains isolated).

Conservation of the momenergy of an isolated system: Does this say that the constituents that enter a collision are necessarily the same in individual mass and in number as the constituents that leave that collision?

No! The constituents often change in a high-speed encounter.
Example 1: Collision of two balls of putty that stick together—after collision hotter and therefore very slightly more massive than before.
Example 2: Collision of two electrons (e^-) with sufficient violence to create additional mass, a pair consisting of one ordinary electron and one positive electron (positron: e^+):

$$e^- \text{ (fast)} + e^- \text{ (at rest)} \longrightarrow e^+ + 3e^-.$$

Example 3: Collision that radiates one or more photons:

$$e^- \text{ (fast)} + e^- \text{ (at rest)} \longrightarrow$$

$$2\begin{pmatrix}\text{electrons of}\\ \text{intermediate}\\ \text{speed}\end{pmatrix} + \begin{pmatrix}\text{electromagnetic}\\ \text{energy (photons)}\\ \text{emitted in the}\\ \text{collision process}\end{pmatrix}$$

In all three examples the *system* momenergy and *system* mass are each the same before as after.

Can I figure the mass of an isolated *system* composed of a number, n, of freely-moving objects by simply adding the masses of the individual objects? **Example:** Collection of fast-moving molecules.

Ordinarily NO, but yes in one very special case: Two noninteracting objects move freely and in step, side by side. Then the mass of the system *does* equal the sum of the two individual masses. In the general case, where the system parts are moving relative to each other, the relation between system mass and mass of parts is not additive. The length, in the sense of interval, of the 4-vector of total momenergy is not equal to the sum of the lengths of the individual momenergy 4-vectors, and for a simple reason: In the general case those vectors do not point in the same spacetime direction. *Energy* however, does add and *momentum* does add:

$$E_{\text{system}} = \sum_{i=1}^{n} E_i \qquad \text{and} \qquad p_{x,\text{system}} = \sum_{i=1}^{n} p_{x,i}$$

From these sums the mass of the system can be evaluated:

$$M^2_{\text{system}} = E^2_{\text{system}} - p^2_{x,\text{system}} - p^2_{y,\text{system}} - p^2_{z,\text{system}}$$

Can we simplify this expression for the mass of an isolated system composed of freely moving objects when we observe it from a free-float frame so chosen as to make the total momentum be zero?

Yes. In this case the mass of the system has a value given by the sum of energies of individual particles:

$$M_{\text{system}} = E_{\text{system}} = \sum_{i=1}^{n} E_i \qquad \text{[in zero-total momentum frame]}$$

Moreover, the energy of each particle can always be expressed as sum of rest energy m plus kinetic energy K:

$$E_i = m_i + K_i \qquad (i = 1, 2, 3, \ldots, n)$$

So the mass of the system exceeds the sum of the masses of its individual particles by an amount equal to the total kinetic energy of all particles (but only as observed in the frame in which *total* momentum equals zero):

$$M_{\text{system}} = \sum_{i=1}^{n} m_i + \sum_{i=1}^{n} K_i \qquad \text{[in zero-total momentum frame]}$$

For slow particles (Newtonian low-velocity limit) the kinetic energy term is negligible compared to the mass term. So it is natural that for years many thought that the mass of a system is the sum of the masses of its parts. However, such a belief leads to incorrect results at high velocities and is wrong as a matter of principle at all velocities.

What's the meaning of mass for a system in which the particles interact as well as move?

The energies of interaction have to be taken into account. They therefore contribute to the total energy, E_{system}, that gives the mass

$$M_{\text{system}} = (E^2_{\text{system}} - p^2_{\text{system}})^{1/2}$$

How do we find out the mass of a system of particles (Table 8-1) that are held—or stick—together?

Weigh it! Weigh it by conventional means if we are here on Earth and the system is small enough, otherwise by determining its gravitational pull on a satellite in free-float orbit about it.

Does mass measure "amount of matter"?

Nature does not offer us any such concept as "amount of matter." History has struck down every proposal to define such a term. Even if we could count number of atoms or by any other counting method try to evaluate amount of matter, that number would not equal mass. First, mass of the specimen changes with its temperature. Second, atoms tightly bonded in a solid weigh less—are less massive—than the same atoms free. Third, many of nature's atoms undergo radioactive decay, with still greater changes of mass. Moreover, around us occasionally, and continually in stars, the number of atoms and number of particles themselves undergo change. How then speak honestly? Mass, yes; "amount of matter," no.

Does the explosion in space of a 20-megaton hydrogen bomb convert 0.93 kilogram of mass into energy (fusion, Section 8.7)? [$\Delta m = \Delta E_{\text{rest, conv}}/c^2 =$ $(20 \times 10^6$ tons TNT$) \times (10^6$ grams/ton$) \times (10^3$ calories/gram of "TNT equivalent"$) \times (4.18$ joules/calorie$)/c^2 = (8.36 \times 10^{16}$ joules$)/(9 \times 10^{16}$ meters^2/second$^2) = 0.93$ kilogram]

Yes *and* no! The question needs to be stated more carefully. Mass of the system of expanding gases, fragments, and radiation has the *same* value immediately after explosion as before; mass M of the system has not changed. However, hydrogen has been transmuted to helium and other nuclear transformations have taken place. In consequence the *makeup* of mass of the system

$$M_{\text{system}} = \sum_{i=1}^{n} m_i + \sum_{i=1}^{n} K_i \qquad \text{[in zero-total momentum frame]}$$

has changed. The first term on the right — sum of masses of individual constituents — has *decreased* by 0.93 kilogram:

$$\left(\sum_{i=1}^{n} m_i\right)_{\text{after}} = \left(\sum_{i=1}^{n} m_i\right)_{\text{before}} - 0.93 \text{ kilogram}$$

The second term — sum of kinetic energies, including "kinetic energy" of photons and neutrinos produced — has *increased* by the same amount:

$$\left(\sum_{i=1}^{n} K_i\right)_{\text{after}} = \left(\sum_{i=1}^{n} K_i\right)_{\text{before}} + 0.93 \text{ kilogram}$$

The first term on the right side of this equation — the original heat content of the bomb — is practically zero by comparison with 0.93 kilogram. Thus part of the mass of *constituents* has been converted into energy; but the mass of the *system* has not changed.

The mass of the products of a nuclear *fission* explosion (Section 8.7: fragments of split nuclei of uranium, for example) — contained in an underground cavity, allowed to cool, collected, and weighed — is this mass less than the mass of the original nuclear device?

Yes! The key point is the waiting period, which allows heat and radiation to flow away until transmuted materials have practically the same heat content as that of original bomb. In the expression for the mass of the system

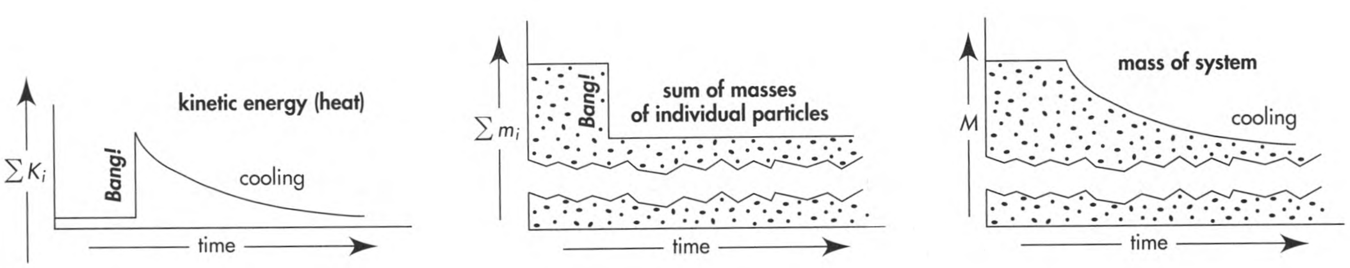
$$M_{\text{system}} = \sum_{i=1}^{n} m_i + \sum_{i=1}^{n} K_i \qquad \text{[in zero-total momentum frame]}$$

the second term on the right, the kinetic energy of thermal agitation — whose value rose suddenly at the time of explosion but dropped during the cooling period — has undergone no net alteration as a consequence of the explosion followed by cooling.

In contrast, the sum of masses

$$\sum m_i$$

has undergone a permanent decrease, and with it the mass M of *what one weighs* (after the cooling period) has dropped (see the figure).



Does Einstein's statement that mass and energy are equivalent mean that energy is the *same* as mass?

No. Value of energy depends on the free-float frame of reference from which the particle (or isolated system of particles) is regarded. In contrast, value of mass is independent of inertial frame. Energy is only the *time* component of a momenergy 4-vector, whereas mass measures *entire magnitude* of that 4-vector. The time component gives the magnitude of the momenergy 4-vector only in the special case in which that 4-vector has no space component; that is, in a frame in which the momentum of the particle (or the total momentum of an isolated system of particles) equals zero. Only as measured in this special **zero-momentum** frame does energy have the same value as mass.

Then what *is* the meaning of Einstein's statement that mass and energy are equivalent?

Einstein's statement refers to the reference frame in which the particle is at rest, so that it has zero momentum p and zero kinetic energy K. Then $E = m + K \rightarrow m + 0$. In that case the energy is called the **rest energy** of the particle:

$$E_{\text{rest}} = m$$

In this expression, recall, the energy is measured in units of mass, for example kilograms. Multiply by the conversion factor c^2 to express energy in conventional units, for example joules (Table 7-1). The result is Einstein's famous equation:

$$E_{\text{rest, conv}} = mc^2$$

Many treatments of relativity fail to use the subscript "rest"—needed to remind us that this equivalence of mass and energy refers only to the *rest* energy of the particle (for a system, the total energy in the zero-total-momentum frame).

Without delving into all fine points of legalistic phraseology, how significant is the conversion factor c^2 in the equation $E_{\text{rest, conv}} = mc^2$?

The conversion factor c^2, like the factor of conversion from seconds to meters or miles to feet (Box 3-2), today counts as a detail of convention, rather than as a deep new principle.

If the factor c^2 is not the central feature of the relationship between mass and energy, what *is* central?

The distinction between mass and energy is this: Mass is the magnitude of the momenergy 4-vector and energy is the time component of the *same* 4-vector. Any feature of any discussion that emphasizes this contrast is an aid to understanding. Any slurring of terminology that obscures this distinction is a potential source of error or confusion.

Is the mass of a moving object greater than the mass of the same object at rest?

No. It is the same whether the object is at rest or in motion; the same in all frames.

Really? Isn't the mass, M, of a system of freely moving particles given, not by the sum of the masses m_i of the individual constituents, but by the sum of

Ouch! The concept of "relativistic mass" is subject to misunderstanding. That's why we don't use it. First, it applies the name mass—belonging to the energies E_i *(but only in a frame in which total momentum of the system equals zero)?* Then why not give E_i a new name and call it "relativistic mass" of the individual particle? Why not adopt the notation

$$m_{i,\,\mathrm{rel}} = E_i = m_i + K_i \quad ?$$

With this notation, can't one then write

$$M = \sum_{i=1}^{n} m_{i,\,\mathrm{rel}} \quad ? \qquad \text{[in zero-total momentum frame]}$$

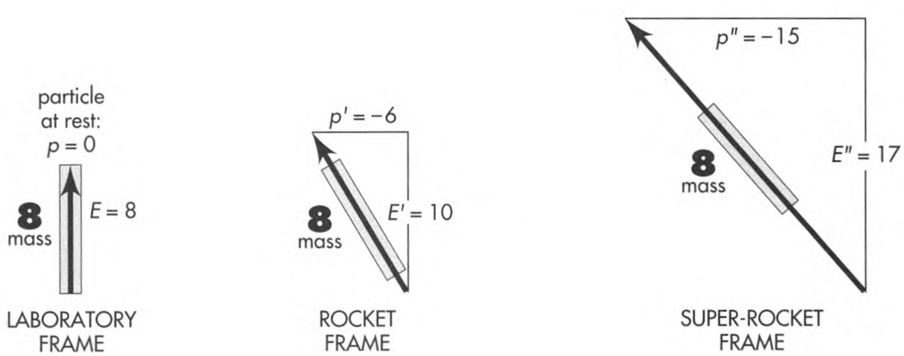
magnitude of a 4-vector—to a very different concept, the time component of a 4-vector. Second, it makes increase of energy of an object with velocity or momentum appear to be connected with some change in internal structure of the object. In reality, the increase of energy with velocity originates not in the object but in the geometric properties of spacetime itself.

In order to make this point clear, should we call invariant mass of a particle its "rest mass"?

That is what we called it in the first edition of this book. But a thoughtful student pointed out that the phrase "rest mass" is also subject to misunderstanding: What happens to the "rest mass" of a particle when the particle moves? In reality mass is mass is mass. Mass has the same value in all frames, is invariant, no matter how the particle moves. [Galileo: "In questions of science the authority of a thousand is not worth the humble reasoning of a single individual."]

Can any simple diagram illustrate this contrast between mass and energy?

Yes. The figure shows the momentum-energy 4-vector of the same particle as measured in three different frames. Energy differs from frame to frame. Momentum differs from frame to frame. Mass (magnitude of 4-vector, represented by the length of handles on the arrows) has the same value, $m = 8$, in all frames.



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Final quotation in Box 8-1: Timothy Ferris, *Coming of Age in the Milky Way* (Anchor Books, Doubleday, New York, 1988), page 344.

Sample Problem 8-5 was suggested by Chet Raymo's science column in the *Boston Globe,* May 2, 1988, page 35.

Galileo quote in final dialog: Galileo Galilei, *Dialogo dei due massimi sistemi del mundo,* Landini, Florence. Translation by S. Drake, *Galileo Galilei — Dialogue Concerning the Two Chief World Systems — Ptolemaic and Copernican,* University of California Press, Berkeley and Los Angeles, 1953.

ACKNOWLEDGMENTS

We thank colleagues old and young for the comments that helped us clarify, formulate, and describe the concept of mass in this chapter and in the final dialog, and very specially Academician Lev B. Okun, Institute of Theoretical and Experimental Physics, Moscow, for correspondence and personal discussions. We believe that our approach agrees with that in two of his articles, both entitled "The Concept of Mass," which appeared in *Physics Today,* June 1989, pages 31–36, and *Soviet Physics-Uspekhi,* Volume 32, pages 629–638 (July 1989).

CHAPTER 8 EXERCISES

You now have at your disposal the power of special relativity to provide physical insight and accurate predictions about an immense range of phenomena, from nucleus to galaxy. The following exercises give only a hint of this range. Even so, there are too many to carry out as a single assignment or even several assignments. For this reason—and to anchor your understanding of relativity—we recommend that you continue to enjoy these exercises as your study moves on to other subjects. The following table of contents is intended to help organize this ongoing attention.

Reminder: In these exercises the symbol v (in other texts sometimes called β) stands for speed as a fraction of the speed of light c. Let v_{conv} be the speed in conventional units; then $v \equiv v_{\text{conv}}/c$.

CONTENTS

MASS AND ENERGY

8-1 examples of conversion

a How much mass does a 100-watt bulb dissipate (in heat and light) in one year?

b The total electrical energy generated on Earth during the year 1990 was probably between 1 and 2×10^{13} kilowatt-hours. To how much mass is this energy equivalent? In the actual production of this electrical energy is this much mass converted to energy? Less mass? More mass? Explain your answer.

c Eric Berman, pedaling a bicycle at full throttle, produces one-half horsepower of *useful* power (1 horsepower = 746 watts). The human body is about 25 percent efficient; that is, 75 percent of the food burned is converted to heat and only 25 percent is converted to useful work. How long a time will Eric have to ride to lose one kilogram by the conversion of mass to energy? How can reducing gymnasiums stay in business?

8-2 relativistic chemistry

One kilogram of hydrogen combines chemically with 8 kilograms of oxygen to form water; about 10^8 joules of energy is released.

a Ten metric tons (10^4 kilograms) of hydrogen combines with oxygen to produce water. Does the resulting water have a greater or less mass than the original hydrogen and oxygen? What is the magnitude of this difference in mass?

b A smaller amount of hydrogen and oxygen is weighed, then combined to form water, which is weighed again. A very good chemical balance is able to detect a fractional change in mass of 1 part in 10^8. By what factor is this sensitivity more than enough — or insufficient — to detect the fractional change in mass in this reaction?

PHOTONS

8-3 pressure of light

a Shine a one-watt flashlight beam on the palm of your hand. Can you feel it? Calculate the total force this beam exerts on your palm. *Should* you be able to feel it? A particle of what mass exerts the same force when you hold it at Earth's surface?

b From the solar constant (1.372 kilowatts/square meter, Sample Problem 8-5) calculate the pressure of sunlight on an Earth satellite. Consider both reflecting and absorbing surfaces, and also "real" surfaces (partially absorbing). Why does the color of the light make no difference?

c A spherical Earth satellite has radius $r = 1$ meter and mass $m = 1000$ kilograms. Assume that the satellite absorbs all the sunlight that falls on it. What is the acceleration of the satellite due to the force of sunlight, in units of g, the gravitational acceleration at Earth's surface? For a way to reduce this "disturbing" acceleration, see Figure 9-2.

d It may be that particles smaller than a certain size are swept out of the solar system by the pressure of sunlight. This certain size is determined by the equality of the outward force of sunlight and the inward gravitational attraction of Sun. Estimate this critical particle size, making any assumptions necessary for your estimate. List the assumptions with your answer. Does your estimated size depend on the particle's distance from Sun?

Reference: For pressure of light measurement in an elementary laboratory, see Robert Pollock, *American Journal of Physics,* Volume 31, pages 901–904 (1963). Pollock's method of determining the pressure of light makes use of resonance to amplify a small effect to an easily measured magnitude. Dr. Pollock developed this experiment in collaboration with the same group of first-year students at Princeton University with whom the authors had the privilege to work out the presentation of relativity in the first edition of this book.

8-4 measurement of photon energy

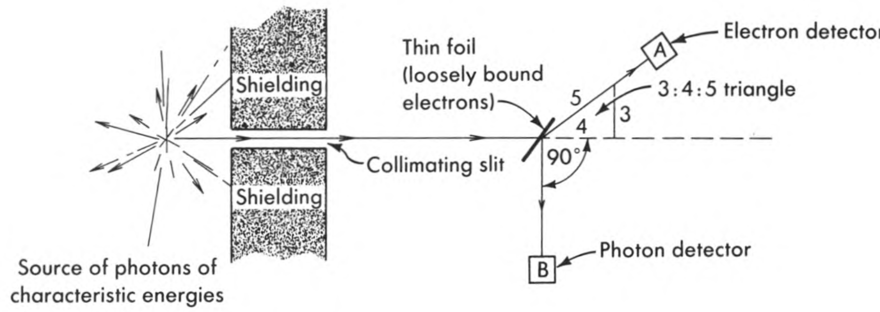
A given radioactive source emits energetic photons (X-rays) or very energetic photons (gamma rays) with energies characteristic of the particular radioactive nucleus in question. Thus a precise energy measurement can often be used to determine the composition of even a tiny specimen. In the apparatus diagrammed in the figure on page 255, only those events are detected in which a count on detector A (knocked-on electron) is accompanied by a count on detector B (scattered photon). What is the energy of the incoming photons that are detected in this way, in units of the rest energy of the electron?

8-5 Einstein's derivation: equivalence of energy and mass — a worked example

Problem

From the fact that light exerts pressure and carries energy, show that this energy is equivalent to mass and hence — by extension — show the equivalence of all energy to mass.

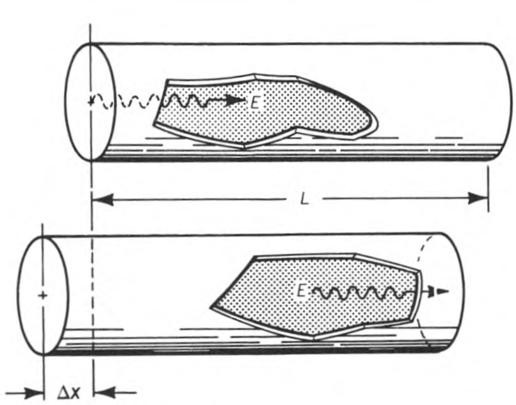
Commentary: The equivalence of energy and mass is such an important consequence that Einstein very early, after his relativistic derivation of this result, sought and found an alternative elementary physical line of reasoning that leads to the same conclusion. He envisaged a closed box of mass M initially at rest, as shown in the first figure. A directed burst of electro-



EXERCISE 8-4. *Measurement of photon energy.*

magnetic energy is emitted from the left wall. It travels down the length L of the box and is absorbed at the other end. The radiation carries an energy E. But it also carries momentum. This one sees from the following reasoning. The radiation exerts a pressure on the left wall during the emission. In consequence of this pressure the box receives a push to the left, and a momentum, p. But the momentum of the system as a whole was zero initially. Therefore the radiation carries a momentum p opposite to the momentum of the box. How can one use knowledge of the transport of energy and momentum by the radiation to deduce the mass equivalent of the radiation? Einstein got his answer from the argument that the center of mass of the system was not moving before the transport process and therefore cannot be in motion during the transport process. But the box obviously carries mass to the left. Therefore the radiation must carry mass to the right. So much for Einstein's reasoning in broad outline. Now for the details.

From relativity Einstein knew that the momentum p of a directed beam of radiation is equal to the energy E of that beam (Section 8.4; both p and E measured in units of mass). However, this was known before Einstein's relativity theory, both from Maxwell's theory of electromagnetic radiation and from direct observation of the pressure exerted by light on a mirror suspended in a vacuum. This measurement had first successfully been carried out by E. F. Nichols and G. F. Hull between 1901 and 1903. (By now the experiment has been so simplified and increased in sensitivity that it can be carried out in an elementary laboratory. See the reference for Exercise 8-3.)



EXERCISE 8-5, first figure. *Transfer of mass by radiation.*

Thus the radiation carries momentum and energy to the right while the box carries momentum and mass to the left. But the center of mass of the system, box plus radiation, cannot move. So the radiation must carry to the right not merely energy but mass. How much mass? To discover the answer is the object of these questions.

a What is the velocity of the box during the time of transit of the radiation?

b After the radiation is absorbed in the other end of the box, the system is once again at rest. How far has the box moved during the transit of the radiation?

c Now demand that the center of mass of the system be at the same location both before and after the flight of the radiation. From this argument, what is the mass equivalent of the energy that has been transported from one end of the box to the other?

Solution

a During the transit of the radiation the momentum of the box must be equal in magnitude and opposite in direction to the momentum p of the radiation. The box moves with a very low velocity v. Therefore the Newtonian formula Mv suffices to calculate its momentum:

$$Mv = -p = -E$$

From this relation we deduce the velocity of the box,

$$v = -E/M$$

b The transit time of the photon is very nearly

$t = L$ meters of light-travel time. In this time the box moves a distance

$$\Delta x = vt = -EL/M$$

c If the radiation transported no mass from one end of the box to the other, and if the box were the sole object endowed with mass, then this displacement Δx would result in a net motion of the center of mass of the system to the left. But, Einstein reasoned, an isolated system with its center of mass originally at rest can never set itself into motion nor experience any shift in its center of mass. Therefore, he argued, there must be some countervailing displacement of a part of the mass of the system. This transport of mass to the right can be understood only as a new feature of the radiation itself. Consequently, during the time the box is moving to the left, the radiation must transport to the right some mass m, as yet of unknown magnitude, but such as to ensure that the center of mass of the system has not moved. The distance of transport is the full length L of the box diminished by the distance Δx through which the box has moved to the left in the meantime. But Δx is smaller than L in the ratio E/M. This ratio can be made as small as one pleases for any given transport of radiant energy E by making the mass M of the box sufficiently great. Therefore it is legitimate to take the distance moved by the radiation as equal to L itself. Thus, with arbitrarily high precision, the condition that the center of mass shall not move becomes

$$M\Delta x + mL = 0$$

Calculate the mass m and find, using Δx from part **b**,

$$m = -\Delta x M/L = -(-EL/M)(M/L)$$

or, finally,

$$m = E$$

In conventional units, we have the famous equation

$$E_{\text{conv}} = mc^2$$

We conclude that the process of emission, transport, and reabsorption of radiation of energy E is equivalent to the transport of a mass $m = E$ from one end of the box to the other end. The simplicity of this derivation and the importance of the result makes this analysis one of the most interesting in all of physics.

Discussion: The mass equivalence of radiant energy implies the mass equivalence of thermal energy and—by extension—of other forms of energy, according to the following reasoning. The energy that emerges from the left wall of the box may reside there originally as heat energy. This thermal energy excites a typical atom of the surface from its lowest energy state to a higher energy state. The atom returns from this higher state to a lower state and in the course of this change sends out the surplus energy in the form of radiation. This radiant energy traverses the box, is absorbed, and is ultimately converted back into thermal energy. Whatever the details of the mechanisms by which light is emitted and absorbed, the net effect is the transfer of heat energy from one end of the box to the other. To say that mass has to pass down the length of the box when radiation goes from one wall to the other therefore implies that mass moves when thermal energy changes location. The thermal energy in turn is derived from chemical energy or the energy of a nuclear transformation or from electrical energy. Moreover, thermal energy deposited at the far end of the tube can be converted back into one or another of these forms of energy. Therefore these forms of energy—and likewise all other forms of energy—are equivalent in their transport to the transport of mass in the amount $m = E$.

How can one possibly uphold the idea that a pulse of radiation transports mass? One already knows that a photon has zero mass, by virtue of the relation (Section 8.4)

$$(\text{mass})^2 = (\text{energy})^2 - (\text{momentum})^2 = 0$$

Moreover, what is true of the individual photon is true of the pulse of radiation made up of many such photons: The energy and momentum are equal in magnitude, so that the mass of the radiation necessarily vanishes. Is there not a fundamental inconsistency in saying in the same breath that the mass of the pulse is zero and that radiation of energy E transports the mass $m = E$ from one place to another?

The source of our difficulty is some confusion between two quite different concepts: (1) energy, the time component of the momentum–energy 4-vector, and (2) mass, the magnitude of this 4-vector. When the system divides itself into two parts (radiation going to the right and box recoiling to the left) the components of the 4-vectors of the radiation and of the recoiling box add up to identity with the components of the original 4-vector of the system before emission, as shown in the second figure. However, the magnitudes of the 4-vectors (magnitude = mass) are not additive. No one dealing with Euclidean geometry would expect the length of one side of a triangle to be equal to the sum of the lengths of the other two sides. Similarly in Lorentz geometry. The mass of the system (M) is not to be considered as equal to the sum of the mass of the radiation (zero) and the mass of the recoiling box (less than M). But components of 4-vectors *are* additive; for example,

$$\begin{pmatrix}\text{energy of}\\ \text{system}\end{pmatrix} = \begin{pmatrix}\text{energy of}\\ \text{radiation}\end{pmatrix} + \begin{pmatrix}\text{energy of}\\ \text{recoiling box}\end{pmatrix}$$

Thus we see that the energy of the recoiling box is $M - E$. Not only is the energy of the box reduced by the emission of radiation from the wall; also its mass is reduced (see shortened length of 4-vector in diagram). Thus the radiation takes away mass from the wall of the box even though this radiation has zero mass. The inequality

$$\begin{pmatrix}\text{mass of}\\ \text{system}\end{pmatrix} \neq \begin{pmatrix}\text{mass of}\\ \text{radiation[zero]}\end{pmatrix} + \begin{pmatrix}\text{mass of}\\ \text{recoiling box}\end{pmatrix}$$

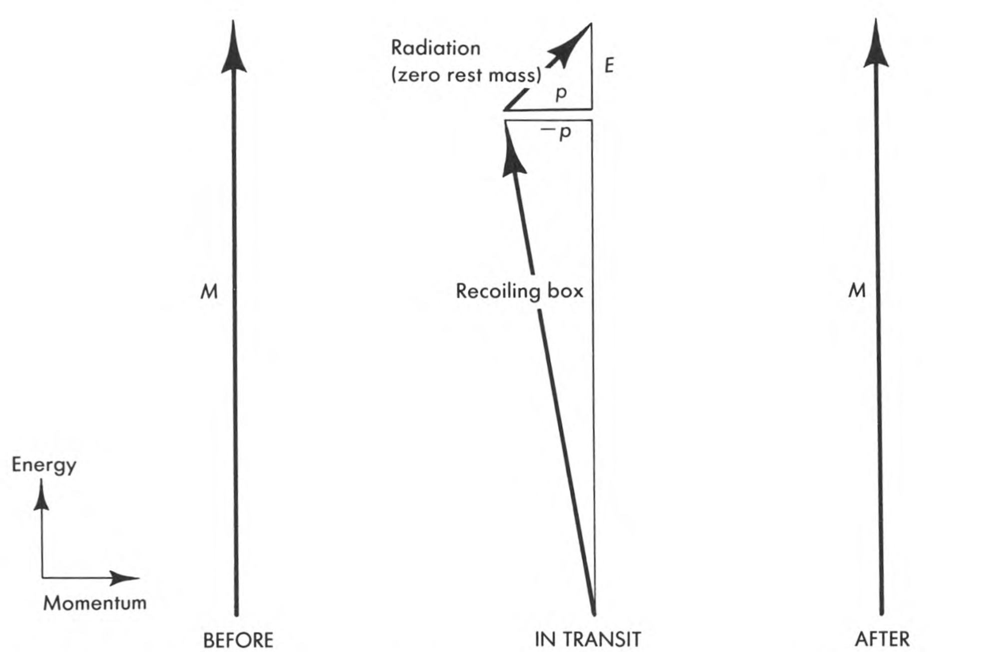
is as natural in spacetime geometry as is the inequality $5 \neq 3 + 4$ for a 3-4-5 triangle in Euclidean geometry.

What about the gravitational attraction exerted by the system on a test object? Of course the redistribution of mass as the radiation moves from left to right makes some difference in the attraction. But let the test object be at a distance r so great that any such redistribution has a negligible effect on the attraction. In other words, all that counts for the pull on a unit test object is the total mass M as it appears in Newton's formula for gravitational force:

$$\begin{pmatrix}\text{force per}\\ \text{unit mass}\end{pmatrix} = \frac{GM}{r^2}$$

Even so, will not the distant detector momentarily experience a less-than-normal pull while the radiation is in transit down the box? Is not the mass of the radiation zero, and is not the mass of the recoiling box reduced below the original mass M of the system? So is not the total attracting mass less than normal during the process of transport? No! The mass of the system—one has to say again—is not equal to the sum of the masses of its several parts. It is instead equal to the magnitude of the total momentum–energy 4-vector of the system. And at no time does either the total momentum (in our case zero!) or the total energy of the system change—it is an isolated system. Therefore neither is there any change in the magnitude M of the total momentum–energy 4-vectors shown in the second figure. So, finally, there is never any change in the gravitational attraction.

There is one minor swindle in the way this problem has been presented: The box cannot in fact move as a rigid body. If it could, then information about the emission of the radiation from one end could be obtained from the motion of the other end before the arrival of the radiation itself—this information would be transmitted at a speed greater than that of light! Instead, the recoil from the emission of the radiation travels along the sides of the box as a vibrational wave, that is, with the speed of sound, so that this wave arrives at the other end long after the radiation does. In the meantime the absorption of the radiation at the second end causes a second vibrational wave which travels back along the sides of the box. The addition of the vibration of the box to the prob-



EXERCISE 8-5, second figure. *Radiation transfers mass from place to place even though the mass of the radiation is zero!*

lem requires a more complicated analysis but does not change in any essential way the results of the exercise.

References: A. Einstein, *Annalen der Physik,* Volume 20, pages 627–633 (1906). For a more careful treatment of the box, see A. P. French, *Special Relativity* (W. W. Norton, New York, 1968), pages 16–18 and 27–28.

8-6 gravitational red shift

Note: Exercises 8-6 and 8-7 assume an acquaintance with the following elementary facts of gravitation.

(1) A very small object—or a spherically symmetric object of any radius—with mass M attracts an object of mass m—also small or spherically symmetric—with a force

$$F = \frac{GMm}{r^2}$$

Here r is the distance between the centers of the two objects and G is the Newtonian constant of gravitation, $G = 6.67 \times 10^{-11}$ (meter)^3/(kilogram-second^2).

(2) The work required to move a test particle of unit mass from r to $r + dr$ against the gravitational pull of a fixed mass M is $GM(dr/r^2)$. Translated from conventional units of energy to units of mass this work is

$$dW_{\text{conv}} = \frac{GM}{c^2}\frac{dr}{r^2} = M^* \frac{dr}{r^2}$$

per unit of mass contained in the test particle.

(3) The symbol $M^* = GM/c^2$ in this formula has a simple meaning. It is the mass of the center of attraction translated from units of kilograms to units of meters. For example, the mass of Earth ($M_{\text{Earth}} = 5.974 \times 10^{24}$ kilograms) expressed in length units is $M^*_{\text{Earth}} = 4.44 \times 10^{-3}$ meters, and the mass of Sun ($M_{\text{Sun}} = 1.989 \times 10^{30}$ kg) is $M^*_{\text{Sun}} = 1.48 \times 10^3$ meters.

(4) Start the test particle at a distance r from the center of attraction of mass M and carry it to an infinite distance. The work required is $W = M^*/r$ in units of mass per unit of mass contained in the test particle.

So much for the minitutorial. Now to business.

a What fraction of your rest energy is converted to potential energy when you climb the Eiffel Tower (300 meters high) in Paris? Let g^* be the acceleration of gravity in meters/meter^2 at the surface of Earth:

$$g^* = \frac{GM_{\text{Earth}}}{c^2}\frac{1}{r^2_{\text{Earth}}} = \frac{M^*_{\text{Earth}}}{r^2_{\text{Earth}}} = \frac{g}{c^2}$$

b What fraction of one's rest energy is converted to potential energy when one climbs a very high ladder that reaches higher than the gravitational influence of Earth? Assume that Earth does not rotate and is alone in space. Does the fraction of the energy that is lost in either part **a** or part **b** depend on your original mass?

c Apply the result of part **a** to deduce the fractional energy change of a photon that rises vertically to a height z in a uniform gravitational field g^*. Photons have zero mass; one can say formally that they have only kinetic energy $E = K$. Thus photons have only one purse—the kinetic energy purse—from which to pay the potential energy tax as they rise in the gravitational field. Light of frequency f is composed of photons of energy $E = hf/c^2$ (see Exercise 8-31). Show that the fractional energy loss for photons rising in a gravitational field corresponds to the following fractional change in frequency:

$$\frac{\Delta f}{f} = -g^* z \qquad \text{[uniform gravitational field]}$$

Note: We use f for frequency instead of the usual Greek nu, ν, to avoid confusion with v for speed.

d Apply the result of part **b** to deduce the fractional energy loss of a photon escaping to infinity. (To apply **b** for this purpose is an approximation good to one percent when this fractional energy loss itself is less than two percent.) Specifically, let the photon start from a point on the surface of an astronomical object of mass M (kilograms) or M^* (meters) $= GM/c^2$ and radius r. From the fractional energy loss, show that the fractional change of frequency is given by the expression

$$\frac{\Delta f}{f} = -\frac{M^*}{r} \qquad \text{[escape field of spherical object]}$$

This decrease in frequency is called the **gravitational red shift** because, for visible light, the shift is toward the lower-frequency (red) end of the visible spectrum.

e Calculate the fractional gravitational red shifts for light escaping from the surface of Earth and for light escaping from the surface of Sun.

Discussion: The results obtained in this exercise are approximately correct for light moving near Earth, Sun, and white dwarf (Exercise 8-7). Only general relativity correctly describes the motion of light very close to neutron star or black hole (Box 9-2).

8-7 density of the companion of Sirius

Note: This exercise uses a result of Exercise 8-6.

Sirius (the Dog Star) is the brightest star in the heavens. Sirius and a small companion revolve about one another. By analyzing this revolution using Newtonian mechanics, astronomers have determined that the mass of the companion of Sirius is roughly equal to the mass of our Sun (M is about 2×10^{30} kilograms; M^* is about 1.5×10^{3} meters). Light from the companion of Sirius is analyzed in a spectrometer. A spectral line from a certain element, identified from the pattern of lines, is shifted in frequency by a fraction 7×10^{-4} compared to the frequency of the same spectral line from the same element in the laboratory. (These figures are experimentally accurate to only one significant figure.) Assuming that this is a gravitational red shift (Exercise 8-6), estimate the average density of the companion of Sirius in grams/centimeter^3. This type of star is called a **white dwarf** (Box 9-2).

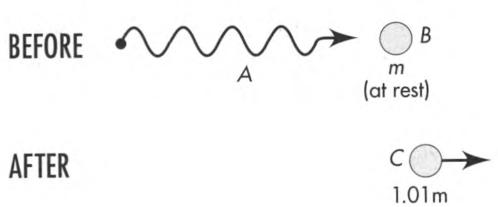
CREATIONS, TRANSFORMATIONS, ANNIHILATIONS

8-8 nuclear excitation

A nucleus of mass m initially at rest absorbs a gamma ray (photon) and is excited to a higher energy state such that its mass is now $1.01\ m$.

a Find the energy of the incoming photon needed to carry out this excitation.

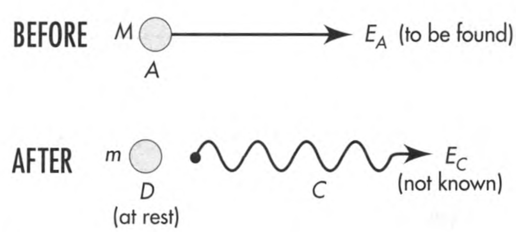
b Explain why the required energy of the incoming photon is greater than the change of mass of the nucleus.



EXERCISE 8-8. *Excitation of a nucleus by a gamma ray.*

8-9 photon braking

A moving radioactive nucleus of known mass M emits a gamma ray (photon) in the forward direction and drops to its stable nonradioactive state of known mass m. Find the energy E_A of the incoming nucleus (BEFORE diagram in the figure) such that the resulting mass m nucleus is at rest (AFTER diagram). The unknown energy E_C of the outgoing gamma ray should not appear in your answer.



EXERCISE 8-9. *Stopping a nucleus by emission of a gamma ray.*

8-10 photon integrity

Show that an isolated photon cannot split into two photons going in directions other than the original direction. (Hint: Apply the laws of conservation of momentum and energy and the fact that the third side of a triangle is shorter than the sum of the other two sides. What triangle?)

8-11 pair production by a lonely photon?

A gamma ray (high-energy photon, zero mass) can carry an energy greater than the rest energy of an electron–positron pair. (Remember that a positron has the same mass as the electron but opposite charge.) Nevertheless the process

$$\text{(energetic gamma ray)} \longrightarrow \text{(electron)} + \text{(positron)}$$

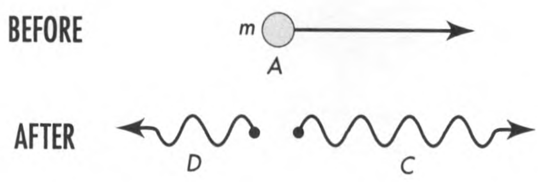
cannot occur in the absence of other matter or radiation.

a Prove that this process is incompatible with the laws of conservation of momentum and energy as employed in the laboratory frame of reference. Analyze the alleged creation in the frame in which electron and positron go off at equal but opposite angles $\pm\phi$ with the extended path of the incoming gamma ray.

b Repeat the demonstration—which then becomes much more impressive—in the center-of-momentum frame of the alleged pair, the frame of reference in which the total momentum of the two resulting particles is zero.

8-12 photoproduction of a pair by two photons

Two gamma rays of different energies collide in a vacuum and disappear, bringing into being an electron–positron pair. For what ranges of energies of the two gamma rays, and for what range of angles between their initial directions of propagation, can this reaction occur? (Hint: Start with an analysis of the reaction at threshold; at threshold the electron and positron are relatively at rest.)



EXERCISE 8-13. *Decay of positronium in flight.*

8-13 decay of positronium

A moving "atom" called positronium (an electron and positron orbiting one another) of mass m and initial energy E decays into two gamma rays (high-energy photons) that move in opposite directions along the line of motion of the initial atom. Find the energy of each gamma ray, E_C and E_D, in terms of the mass m and energy E_A of the initial particle. Check that $E_C = E_D$ in the case that the initial particle is at rest.

8-14 positron–electron annihilation I

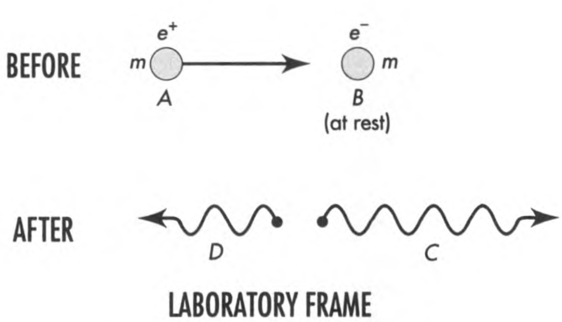
A positron e^+ of mass m and kinetic energy K is annihilated on a target containing electrons e^- (same mass m) practically at rest in the laboratory frame:

$$e^+(\text{fast}) + e^-(\text{at rest}) \longrightarrow \text{radiation}$$

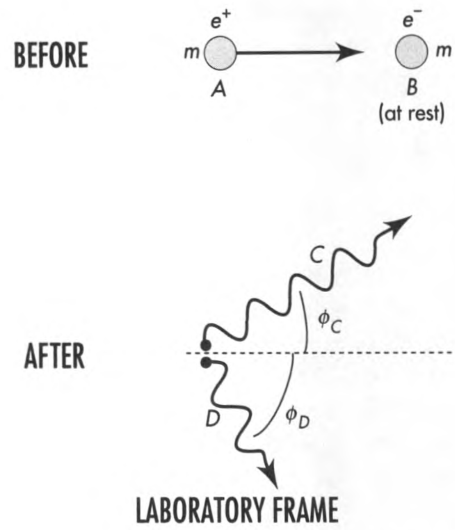
a By considering the collision in the center-of-momentum frame (the frame of reference in which the total momentum of the initial particles is equal to zero), show that it is necessary for at least two gamma rays (rather than one) to result from the annihilation.

b Return to the laboratory frame, shown in the figure. The outgoing photons move on the line along which the positron approaches. Find an expression for the energy of each outgoing photon. Let your derivation be free of any reference to velocity.

c Using simple approximations, evaluate the answer to part **b** in the limiting cases (1) very small K and (2) very large K. (Very small and very large compared with what?)



EXERCISE 8-14. *Positron–electron annihilation.*



EXERCISE 8-15, first figure. *Positron–electron annihilation.*

8-15 positron–electron annihilation II

A positron e^+ of mass m and kinetic energy K is annihilated on a target containing electrons e^- (same mass m) practically at rest in the laboratory frame:

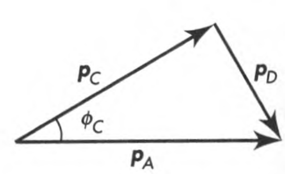
$$e^+(\text{fast}) + e^-(\text{at rest}) \longrightarrow \text{radiation}$$

The resulting gamma rays go off at different angles with respect to the direction of the incoming positron, as shown in the first figure.

a Derive an expression for the energy of one of the gamma rays in the laboratory frame as a function of the angle between the direction of emergence of that gamma ray and the direction of travel of the positron before its annihilation. The gamma ray energy should be a function of only the energy and mass of the incoming positron and the angle of the outgoing gamma ray. (Hint: Use the law of cosines, as applied to the second figure.)

$$p_D{}^2 = p_A{}^2 + p_C{}^2 - 2p_A p_C \cos\phi_C$$

b Show that for outgoing gamma rays moving along the positive and negative x-direction, the results of this exercise reduce to the results of Exercise 8-14.



EXERCISE 8-15, second figure. *Conservation of vector momentum means that the momentum triangle is closed.*

8-16 creation of proton–antiproton pair by an electron

What is the threshold kinetic energy K_{th} of the incident electron for the following process?

electron (fast) + proton (at rest) $\longrightarrow$
electron + antiproton + two protons

8-17 colliders

How much more violent is a collision of two protons that are moving toward one another from opposite directions than a collision of a moving proton with one at rest?

Discussion: When a moving particle strikes a stationary one, the energy available for the creation of new particles, for heating, and for other interactions — or, in brief, the available interaction energy — is less than the initial energy (the sum of the rest and kinetic energies of the initial two particles). Reason: The particles that are left over after the reaction have a net forward motion (law of conservation of momentum), the kinetic energy of which is available neither for giving these particles velocity relative to each other nor for producing more particles. For this reason much of the particle energy produced in accelerators is not available for studying interactions because it is carried away in the kinetic energy of the products of the collision.

However, in the center-of-momentum frame, the frame in which the total momentum of the system is equal to zero, no momentum need be carried away from the interaction. Therefore the energy available for interaction is equal to the total energy of the incoming particles.

Is there some way that the laboratory frame can be made also the center-of-momentum frame? One way is to build two particle accelerators and have the two beams collide head on. If the energy and masses of the particles in each beam are respectively the same, then the laboratory frame is the center-of-momentum frame and all the energy in each collision is available interaction energy. It is easier and cheaper to achieve the same efficiency by arranging to have particles moving in opposite directions in the same accelerator. A magnetic field keeps the particles in a circular path, "storing" them at their maximum energy for repeated tries at interaction. Such a facility is called a **collider.** The figure on page 262 gives some details of a particular collider.

a What is the total available interaction energy for each encounter in the laboratory frame of the Tevatron shown on page 262?

b Now transform to a frame in which one of the incoming particles is at rest (transformation given in Exercise 7-5). This would be the situation if we tried to build an accelerator in which moving antiprotons hit a stationary target of, say, liquid hydrogen (made of protons and electrons). [Simplify: At 0.9 TeV = 9×10^{11} eV what is the effective speed v of the proton? What is its momentum compared with its energy? What is the value of the time stretch factor $\gamma = E/m$?] If the target protons were at rest, what energy, in TeV, would the incoming antiproton need to have in order to yield the same interaction energy as that achieved in the Tevatron?

Wait a minute! You keep telling us that energy and momentum have different values when measured with respect to different reference frames. Yet here you assume the "interaction energy" is the same in the Tevatron laboratory frame as it is in the rest frame of a proton that moves with nearly the speed of light in the Tevatron frame. Is the energy of a system different in different frames, or is it the same?

There is an important distinction between the total energy of a system and the "available interaction energy," just as there is an important distinction between your money in the bank and "ready cash" in the bank that you can spend. If some of your money in the bank has been put in escrow for payment on a house you are buying, then you cannot spend that part of your bank money to buy a new car. Similarly, the total energy of the proton–antiproton system is much smaller in the Tevatron laboratory frame than in the frame in which the proton is initially at rest, but all of the Tevatron laboratory-frame energy can be spent — used to create new particles, for example. In contrast, only a minute fraction of the energy in the frame in which the proton is initially at rest can be spent to create new particles, since total momentum must be conserved; most of the total energy is kept "in escrow" for this purpose. The number and kinds of new particles created must be the same for all observers! Therefore the "available interaction energy" must be the same for all observers. The central point here is that the Tevatron collider design makes all of the energy in the proton–antiproton system "available" for use in the laboratory.

EXERCISE 8-17. ***Top:*** *Aerial view of the Tevatron ring at Fermi National Accelerator Laboratory in Batavia, Illinois. The ring is 6.3 kilometers in circumference.* ***Bottom:*** *View along the tunnel of the Tevatron. Protons (positive charge) and antiprotons (antiparticle of the proton: same mass, negative charge) circulate in separate beams in opposite directions in the same vacuum chamber in the lower ring of superconducting magnets shown in the photo. The upper ring of regular magnets accelerates protons from 8 GeV to 150 GeV. Some of these protons are injected into the lower set of magnets directly, rotating clockwise. Other protons strike a copper target and create antiprotons at a lower energy that are accumulated over approximately 15 hours in a separate ring (not shown) and then reaccelerated to 150 GeV and inserted into the lower ring, circulating counterclockwise. (Opposite charge, opposite motion yields same magnetic force toward the center, hence counterrotation around the same circle.) Then particles in both beams in the lower ring of magnets are accelerated at the same time from 150 GeV to a final energy of 0.9 TeV per particle. (1 teraelectron-volt = 10^{12} electron-volts, or approximately 1000 times the rest energy of the proton or antiproton.) After acceleration, the beams are switched magnetically so that they cross each other at multiple intersection points around the ring, allowing protons and antiprotons to collide in the laboratory center-of-momentum frame. Detectors at the points of intersection monitor products of the collisions. Protons and antiprotons that do not interact at one intersection are not wasted; they may interact at another intersection point or on subsequent trips around the ring. The particles are allowed to coast around and around at full energy for as long as 24 hours as they interact. Question: Approximately how many revolutions around the ring does a given proton or antiproton make in 24 hours?* Photographs courtesy of Fermi Laboratory.

DOPPLER SHIFT

8-18 Doppler shift along the x-direction

Note: Recall Exercise L-5 in the Special Topic on Lorentz Transformation, following Chapter 3.

Apply the momenergy transformation equations (Exercise 7-5) to light moving in the positive x-direction for which $p_x = p = E$.

a Show that the relation between photon energy E' in the rocket frame and photon energy E in the laboratory frame is given by the equation

$$E = \gamma\,(1 + v)\,E' = \frac{(1 + v)\,E'}{(1 - v^2)^{1/2}}$$
$$= \frac{(1 + v)\,E'}{[(1 - v)(1 + v)]^{1/2}} = \left[\frac{1 + v}{1 - v}\right]^{1/2} E'$$

[photon moves along positive x-direction]

b Use the Einstein relation between photon energy E and classical wave frequency f, namely $E_{\text{conv}} = hf$ or $E = hf/c^2$ and $E' = hf'/c^2$, to derive the transformation for frequency

$$f = \left[\frac{1 + v}{1 - v}\right]^{1/2} f'$$

[wave motion along positive x-direction]

This is the Doppler shift equation for light waves moving along the positive x-direction.

Note: We use f for frequency instead of the usual Greek nu, ν, to avoid confusion with v for speed.

c Show that for a wave moving along the negative x-direction, the equation becomes

$$f = \left[\frac{1 - v}{1 + v}\right]^{1/2} f'$$

[wave motion along negative x-direction]

d Derive the corresponding equations that convert laboratory-measured frequency f to rocket-measured frequency f' for waves moving along both positive and negative x-directions.

8-19 Doppler equations

A photon moves in the xy laboratory plane in a direction that makes an angle ϕ with the x-axis, so that its components of momentum are $p_x = p\cos\phi$ and $p_y = p\sin\phi$ and $p_z = 0$.

a Use the Lorentz transformation equations for the momentum–energy 4-vector (Exercise 7-5) and the relation $E^2 - p^2 = 0$ for a photon to show that in the rocket frame, moving with speed v_{rel} along the laboratory x-direction, the photon has an energy E' given by the equation

$$E' = E\gamma\,(1 - v_{\text{rel}}\cos\phi)$$

and moves in a direction that makes an angle ϕ' with the x'-axis given by the equation

$$\cos\phi' = \frac{\cos\phi - v_{\text{rel}}}{1 - v_{\text{rel}}\cos\phi}$$

b Derive the inverse equations for E and $\cos\phi$ as functions of E', $\cos\phi'$, and v_{rel}. Show that the results are

$$E = E'\gamma\,(1 + v_{\text{rel}}\cos\phi')$$
$$\cos\phi = \frac{\cos\phi' + v_{\text{rel}}}{1 + v_{\text{rel}}\cos\phi'}$$

c If the frequency of the light in the laboratory frame is f, what is the frequency f' of the light in the rocket frame? Use the Einstein relation between photon energy E and classical wave frequency f, namely $E_{\text{conv}} = hf$ or $E = hf/c^2$ and $E' = hf'/c^2$, to derive the transformations for frequency

$$f' = f\gamma(1 - v_{\text{rel}}\cos\phi)$$
$$f = f'\gamma(1 + v_{\text{rel}}\cos\phi')$$

This difference in frequency due to relative motion is called the **Doppler shift.**

Note: We use f for frequency instead of the usual Greek nu, ν, to avoid confusion with v for speed.

d For wave motion along the positive and negative x-direction, show that the results of this exercise reduce to the results of Exercise 8-18.

e Discussion question: Do the Doppler equations enable one to determine the rest frame of the source that emits the photons?

8-20 the physicist and the traffic light

A physicist is arrested for going through a red light. In court he pleads that he approached the intersection at such a speed that the red light looked green to him. The judge, a graduate of a physics class, changes the charge to speeding and fines the defendant one dollar for every kilometer/hour he exceeded the local speed limit of 30 kilometers/hour. What is the fine? Take the wavelength of green light to be 530 nanometers $= 530 \times 10^{-9}$ meter) and the wavelength of red light to be 650 nanometers. The relation between wavelength λ and frequency f for light is $f\lambda = c$. Notice that the light propagates in the negative x-direction ($\phi = \phi' = \pi$).

8-21 speeding light bulb

A bulb that emits spectrally pure red light uniformly in all directions in its rest frame approaches the observer from a very great distance moving with nearly the speed of light along a straight-line path whose perpendicular distance from the observer is b. Both the color and the number of photons that reach the observer per second from the light bulb vary with time. Describe these changes qualitatively at several stages as the light bulb passes the observer. Consider both the Doppler shift and the headlight effect (Exercises 8-19 and L-9).

8-22 Doppler shift at the limb of Sun

Sun rotates once in about 25.4 days. The radius of Sun is about 7.0×10^8 meters. Calculate the Doppler shift that we should observe for light of wavelength 500 nanometers $= 500 \times 10^{-9}$ meter) from the edge of Sun's disk (the **limb**) near the equator. Is this shift toward the red end or toward the blue end of the visible spectrum? Compare the magnitude of this Doppler shift with that of the gravitational red shift of light from Sun (Exercise 8-6).

8-23 the expanding universe

Note: Recall Exercise 3-10.

a Light from a distant galaxy is analyzed by a spectrometer. A spectral line of wavelength 730 nanometers $= 730 \times 10^{-9}$ meters is identified (from the pattern of other lines) to be one of the lines of hydrogen that, for hydrogen in the laboratory, has the wavelength 487 nanometers. If the shift in wavelength is a Doppler shift, how fast is the observed galaxy moving relative to Earth? Notice that the light propagates in a direction opposite to the direction of motion of the galaxy ($\phi = \phi' = \pi$).

b There is independent evidence that the observed galaxy is 5×10^9 light years away. Estimate the time when that galaxy parted company from our own galaxy—the Milky Way—using the simplifying assumption that the speed of recession was the same throughout the past (that is, not slowed down by the gravitational attractions between one galaxy and another). The astronomer Edwin Hubble discovered in 1929 that this time—whose reciprocal is called the Hubble constant, and which may itself therefore appropriately be called the Hubble time—has about the same value for all galaxies whose distances and speeds can be measured. Hence the concept of the expanding universe.

c Will allowance for the past effect of gravitation in slowing the expansion increase or decrease the estimated time back to the start of this expansion?

Reference: E. Hubble, *Proceedings of the U. S. National Academy of Sciences,* Volume 15, pages 168–173 (1929).

8-24 twin paradox using the Doppler shift

The Twin Paradox (Chapter 4 and Exercises 4-1 and 5-8) can be resolved elegantly using the Doppler shift as follows. Paul remains on Earth. His twin sister Penny travels at a high speed, v, to a distant star and returns to Earth at the same speed. Both Penny and Paul observe a distant variable star whose light gets alternately dimmer and then brighter with a frequency f in the Earth frame (f' in the rocket frame). This variable star is very much farther away than the length of Penny's path and is in a direction perpendicular to this path in the Earth frame. Both observers will count the same total number of pulsations of the variable star during Penny's round trip. Use this fact and the expression for the Doppler shift at the 90-degree laboratory angle of observation (Exercise 8-19) to verify that at the end of the trip described in Chapter 4, Penny will be only 20 years older while Paul will have aged 202 years.

Reference: E. Feenberg, *American Journal of Physics,* Volume 27, page 190 (1959).

8-25 Doppler line broadening

The average kinetic energy of a molecule in a gas at temperature T degrees Kelvin is $(3/2)kT$. (The constant k is called the Boltzmann constant and has the value 1.38×10^{-23} joules/degree Kelvin). Molecules of gas move in random directions. Calculate the average speed from the low-velocity approximation of Newtonian mechanics. Estimate the fractional change in frequency due to the Doppler shift that will be observed in light emitted from a molecule in a gas at temperature T. Will this shift increase or decrease the observed frequency of the emitted light? This effect, called Doppler broadening of spectral lines, is one reason why a given spectral line from a gas excited in an electric discharge contains a range of frequencies around a central frequency.

8-26 $E_{\text{rest conv}} = mc^2$ from the Doppler shift

Einstein's famous equation in conventional units, $E_{\text{rest conv}} = mc^2$, and the relativistic expression for energy can be derived from (1) the relativistic expression for momentum (derived separately, for example in Exercise 7-12), (2) the conservation laws, and (3) the

Doppler shift (Exercise 8-18). In conventional units, a photon has energy $E_{conv} = hf$, where h is Planck's constant and f is the frequency of the corresponding classical wave. (We use f for frequency instead of the usual Greek nu, ν, to avoid confusion with v for speed.) Divide by c^2 to convert to units of mass: $E = hf/c^2$. Expressed in units of mass, a photon has equal energy and momentum. Therefore the momentum of a photon is also given by the equation $p = hf/c^2$. Momentum does differ from energy, however, in that it is a 3-vector. In one dimensional motion, the *sign* of the momentum (positive for motion to the right, negative for motion to the left) is important, as in the analysis below.

A particle of mass m_{before} emits two photons in opposite directions while remaining at rest in the laboratory frame. Conservation of momentum requires these two photons to have equal and opposite momenta and therefore to correspond to the same classical frequency f. In consequence, they also have the same energy.

a First result: Energy released $=\Delta m$. Now view this process from a rocket frame moving at speed $v = v_{conv}/c$ along the direction of flight of the two photons. The particle moves in this frame, but does not change velocity on emitting the photons. The photon emitted in the same direction as the rocket motion will be upshifted in energy (and in corresponding classical frequency) as compared with the energy observed in the laboratory; the other backward-moving photon will be downshifted. We can calculate this frequency shift using the Doppler formulas (Exercise 8-18). Use the expression $m\gamma v$ for momentum of a particle, equation (7-8), to state the conservation of momentum (notice the minus sign before the second photon term, representing the photon moving to the left):

$$m_{before} v\gamma = m_{after} v\gamma + \frac{hf}{c^2}\left[\frac{1+v}{1-v}\right]^{1/2} - \frac{hf}{c^2}\left[\frac{1-v}{1+v}\right]^{1/2}$$

Simplify this expression to

$$m_{before} = m_{after} + 2hf/c^2 \qquad (1)$$

or

$$m_{before} - m_{after} = \Delta m = 2hf/c^2 = \text{energy released}$$

Conservation of momentum in both frames implies a change in particle mass equal to the total energy of the emitted photons. Multiply the mass-units result by c^2 to convert to conventional units and the equation in the well-known form

$$\text{energy released (conventional units)} = (\Delta m)c^2$$

b Second result: $E_{rest} = m$. Now add the condition that energy is conserved in the laboratory frame:

$$E_{before} = E_{after} + 2hf/c^2 \qquad (2)$$

Compare equations (1) and (2). These two equations both describe a particle at rest. Show that they are consistent if $E_{before} = m_{before}$ and $E_{after} = m_{after}$ and that therefore in general

$$E_{rest} = m$$

or, in conventional units,

$$E_{rest\ conv} = mc^2$$

c Third result: At any speed, $E = m\gamma$. Next add the condition that energy be conserved in the rocket frame. Place primes on expressions for rocket-measured energy of the particle and use the Doppler equations to transform the classical frequency back to the laboratory value f. Show that the result is

$$E'_{before} = E'_{after} + (2hf/c^2)\gamma \qquad (3)$$

The salient difference between equations (2) and (3) is that in the rocket frame the particle is in motion. Deduce that the general expression for energy of a particle includes the stretch factor gamma:

$$E = m\gamma$$

or, in conventional units,

$$E_{conv} = m\gamma c^2$$

Reference: Fritz Rohrlich, *American Journal of Physics,* Volume 58, pages 348–349 (April 1990).

8-27 everything goes forward

"Everything goes forward" is a good rule of thumb for interactions between highly relativistic particles and stationary targets. In the laboratory frame, many particles and gamma rays resulting from collisions continue in essentially the same direction as the incoming particles.

The first figure (top) shows schematically the collision of two protons in the center-of-momentum frame, the frame in which the system has zero total momentum. A great many different particles are created in the collision, including a gamma ray (the fastest possible particle) that by chance moves perpendicular to the line of motion of the incoming particles: $\phi' = \pi/2$ radians.

The first figure (bottom) shows the same interaction in the laboratory frame, in which one proton is initially at rest. At what angle ϕ does the product gamma ray move in this frame?

a From the Doppler equations (Exercise 8-19), show that the outgoing angle ϕ for the gamma ray in the laboratory frame is given by the expression

$$\cos\phi = v_{\text{rel}} \tag{1}$$

b What is the speed v_{proton} of the rightward-moving proton in the laboratory frame? We define the laboratory frame by riding at speed v_{rel} on the leftward-moving proton in the center-of-momentum frame. Therefore the rightward-moving proton also moves with speed v_{rel} in the center-of-momentum frame. Use the law of addition of velocities to find the speed of the rightward-moving proton in the laboratory frame (Section L.7 and Exercise 3.11).

$$v_{\text{proton}} = \frac{2v_{\text{rel}}}{1 + v_{\text{rel}}^2} \tag{2}$$

c In order to solve equation (1) for ϕ, we need to know the value of v_{rel}. Equation (2) is a quadratic in v_{rel}. Show that the solution is

$$v_{\text{rel}} = \frac{1}{v_{\text{proton}}}\left[1 - \frac{1}{\gamma_{\text{proton}}}\right] \tag{3}$$

Here γ_{proton} is the stretch factor γ using the proton velocity v_{proton}.

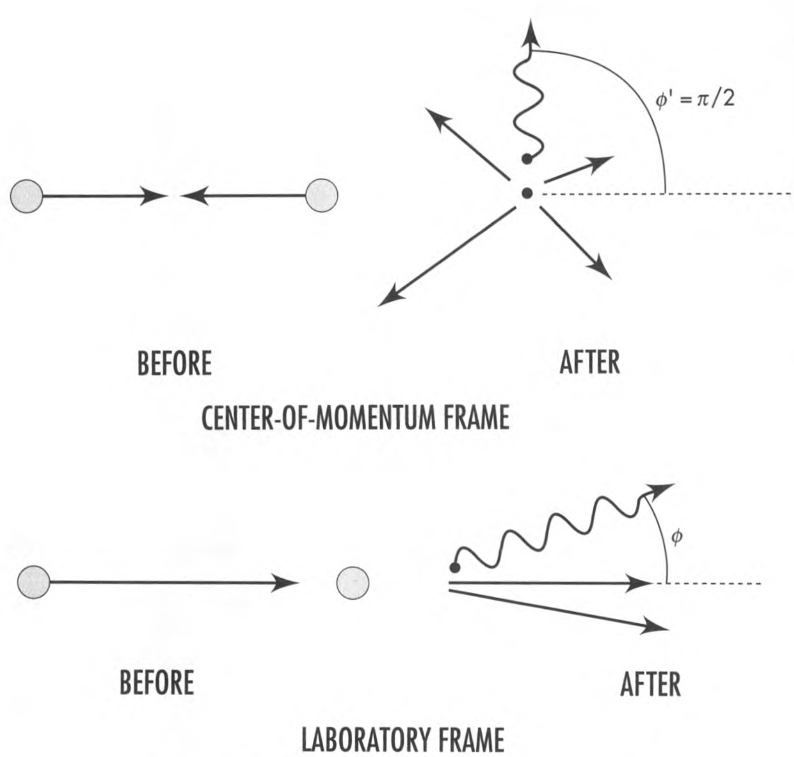
d We are interested in finding the angle ϕ when the incoming proton is highly relativistic. In this case $v_{\text{proton}} \approx 1$. From the approximation for small angles (ϕ expressed in radians)

$$\cos\phi \approx 1 - \phi^2/2 \qquad |\phi| << 1$$

show that the angle ϕ is given approximately by the expression

$$\phi \approx \left[\frac{2}{\gamma_{\text{proton}}}\right]^{1/2} \tag{4}$$

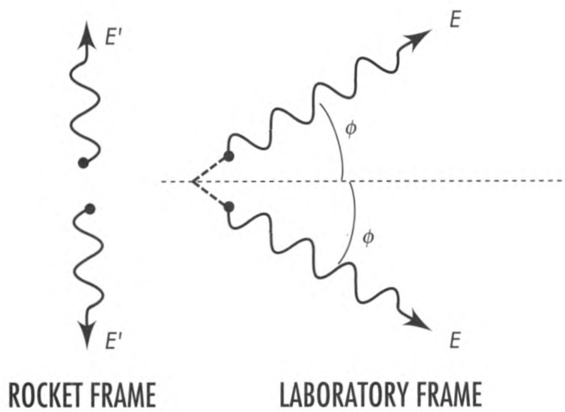
e What is the value of ϕ in radians and in degrees for incident protons of energy $E = 200$ GeV? For incident protons of energy 2×10^4 GeV? (1 GeV $= 10^9$ electron-volts. Mass of the proton is approximately 1 GeV.)



EXERCISE 8-27, first figure. *In the* **center-of-momentum frame** *two incoming protons collide, creating many particles, among them a gamma ray that moves perpendicular to the original line of motion. In the* **laboratory frame,** *in which one proton is initially at rest, in what direction does the gamma ray move?*

8-28 decay of π^o-meson

A π^o meson (neutral pi-meson) moving in the x-direction with a kinetic energy in the laboratory frame equal to its mass m decays into two photons. In the rocket frame in which the meson is at rest these photons are emitted in the positive and negative y'-directions, as shown in the figure. Find the energies of the two photons in the rocket frame (in units of the mass of the meson) and the energies and directions of propagation of the two photons in the laboratory frame.

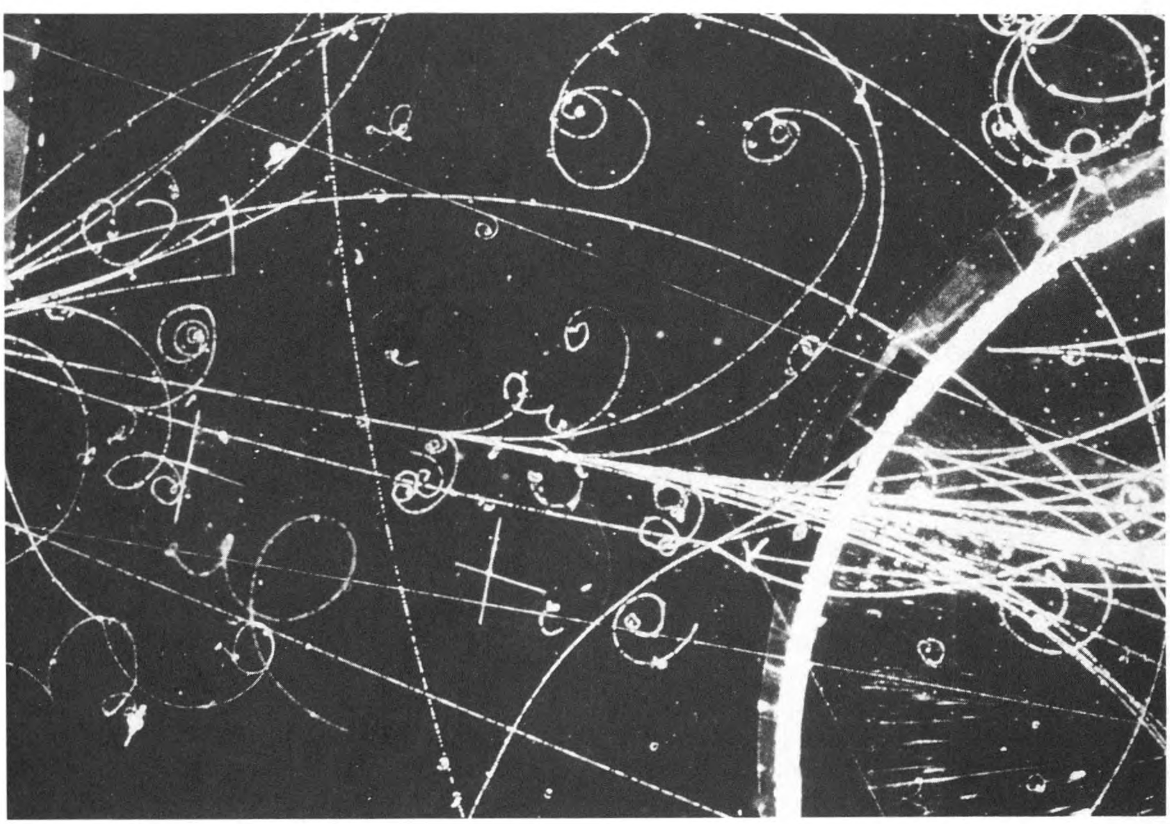


EXERCISE 8-28. *Two photons resulting from the decay of a π°-meson, as observed in rocket and laboratory frames.*

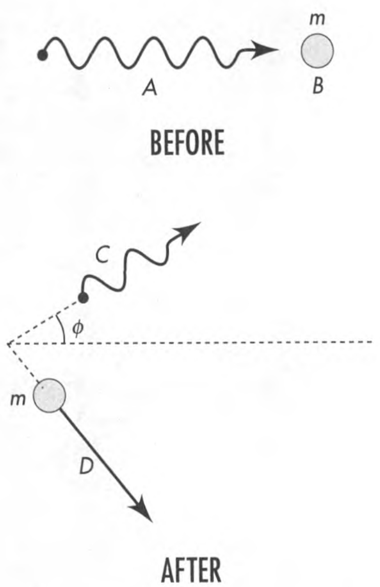
COMPTON SCATTERING

8-29 Compton scattering

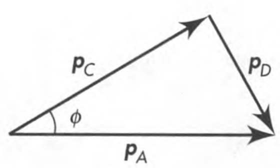
Analyze Compton scattering of an incident photon that collides with and recoils from an electron that is initially at rest. Compton scattering in one dimension was discussed in Section 8.4. Here we analyze Compton scattering in two dimensions. The goal is to determine the reduced energy of the photon that has been scattered with a change of direction measured by the



EXERCISE 8-27, second figure. *Forward spray of particles created in collisions near the middle of the picture. An incident particle, probably a charged π-meson, enters from the left with energy approximately 100 to 200 times its rest energy and strikes a nucleus of neon or hydrogen. Curving paths in the imposed magnetic field are probably knock-on electrons. These and the cascade of other particles move initially in the same direction as the incoming π-meson: "Everything goes forward!"* Photograph courtesy of Fermi Laboratory.



EXERCISE 8-29, first figure. *Compton scattering of a photon from an electron initially at rest. The angle ϕ is called the scattering angle.*



EXERCISE 8-29, second figure. *Conservation of vector momentum means that the momentum triangle is closed.*

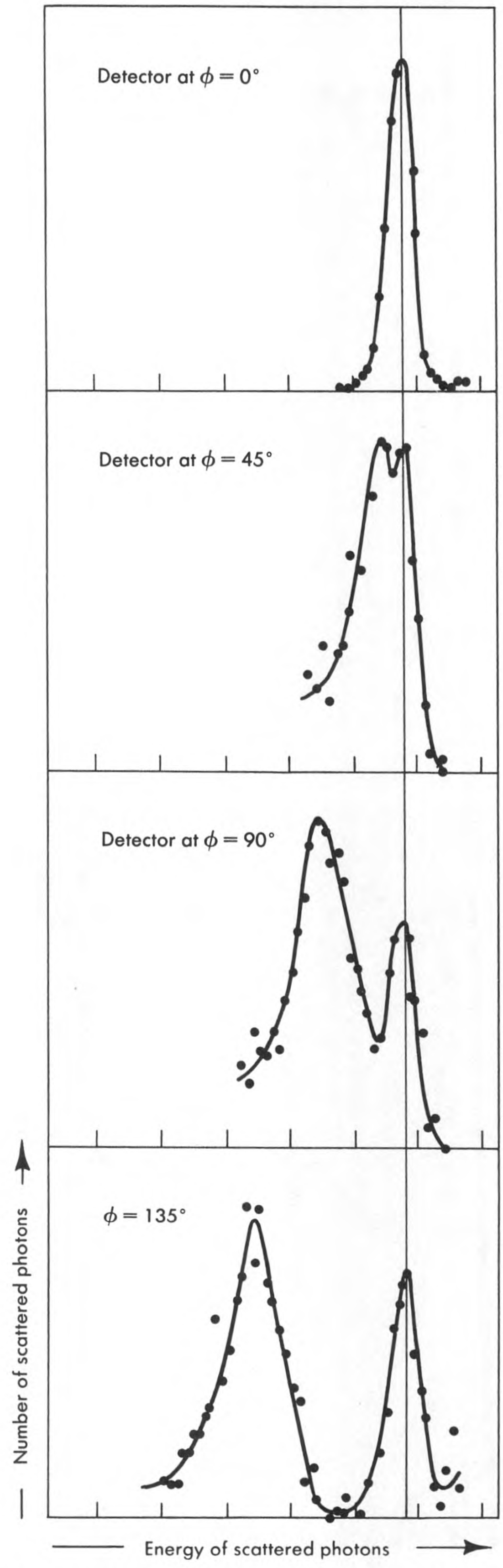
angle ϕ. The angle ϕ is called the **scattering angle.** Use the notation in the first figure. Do not use frequency or wavelength or Planck's constant or speed in your analysis — only the laws of conservation of momentum and energy plus equations:

$$E^2 - p^2 = m^2 \qquad \text{[for an electron]}$$
$$E^2 - p^2 = 0 \qquad \text{[for a photon]}$$

Discussion: The conservation of momentum is a vector conservation law. This means that the vector sum of the momenta after the collision equals the momentum of the photon before the collision. In other words, the vectors form a triangle, as shown in the second figure. Apply the law of cosines to this figure:

$$p_D{}^2 = p_A{}^2 + p_C^2 - 2\, p_A\, p_C \cos \phi$$

a Now replace all momenta with energies (easy for photons, more awkward for the electron), com-



EXERCISE 8-29, third figure. *Results of the Compton experiment in which photons were scattered from the electrons in a graphite target. At each angle of the detector except $\phi = 0$ there are some photons scattered with loss of energy (electron recoils by itself) and other photons scattered with little or no loss of energy (electron and atom recoil as a unit).*

bine with the conservation of energy, and derive the Compton scattering formula:

$$E_{\text{scattered}} = \frac{E_{\text{incident}}}{1 + \dfrac{E_{\text{incident}}}{m}(1 - \cos\phi)}$$

Exercise 8-30 gives some examples of this result.

b Compton's original experiments showed that some photons were scattered without a measurable change of energy. These photons were scattered by electrons that did not leave the atom in which they were bound, so that the entire atom recoiled as a unit. Assume that the energy of the incoming photon is at most a few times the rest energy of the electron. In this case, show that the energy change is negligible for photons scattered by electrons tightly bound to an atom of average mass (say $10 \times 2000 \times$ mass of an electron). See the third figure.

Reference: A. H. Compton, *Physical Review,* Volume 22, pages 409–413 (1923).

8-30 compton scattering examples

a A gamma ray photon of energy equal to twice the mass of the electron scatters from an electron initially at rest. Provide the following answers in units of MeV. (Mass of the electron is 0.511 MeV.) From the Compton scattering formula find the energy of the scattered photon for scattering angles 0, 90, and 180 degrees. If you have access to a computer, calculate this energy at 10-degree increments between zero and 180 degrees and plot the resulting curve of energy vs. angle.

b In a new set of experiments, the incident gamma ray has energy equal to five times the rest energy of the electron. Repeat the calculations of part **a** for this case.

8-31 energy of a photon and frequency of light

Planck found himself forced in 1900 to recognize that light of frequency f (vibrations/second) is composed of quanta (Planck's word) or photons (Einstein's later word), each endowed with an energy $E = hf/c^2$ (energy in units of mass) where h is a universal constant of proportionality called Planck's constant. How can Planck's formula possibly make sense when—as we now know—not only E but also f depend upon the frame of reference in which the light is observed? (We use f for frequency instead of the usual Greek nu, ν, to avoid confusion with v for speed.)

a A photon moves along the positive x-axis. Results of Exercise 8-18 show the relation between the energy of this photon measured in the rocket frame and its energy measured in the laboratory frame. A classical electromagnetic wave moves along the positive x-axis. Results of Exercise L-5 (at the end of the Special Topic following Chapter 3) show the relation between the frequency of this wave measured in the rocket frame and its energy measured in the laboratory frame. Compare these two results to show that if we associate photons with a light wave in one coordinate system, this association will hold in all coordinate systems.

b The theory of relativity does not tell us the value of Planck's constant h in the formula $E = (h/c^2)f$ that relates photon energy (in units of mass) to classical wave frequency. Experiment shows the constant h to have the value 6.63×10^{-34} joule-second. Show that if energy is measured in conventional units, the relation between energy and frequency has the form

$$E_{\text{conv}} = hf \qquad \text{[energy in conventional units]}$$

c Show that the formula for Compton scattering (Exercise 8-29) becomes

$$f_{\text{scattered}} = \frac{f_{\text{incident}}}{1 + \dfrac{hf_{\text{incident}}}{mc^2}(1 - \cos\phi)}$$

In the 1920s there was great resistance to the idea that when the electron is "shaken" by the electric field of wave at one frequency it should scatter (reemit) this radiation at a lower frequency.

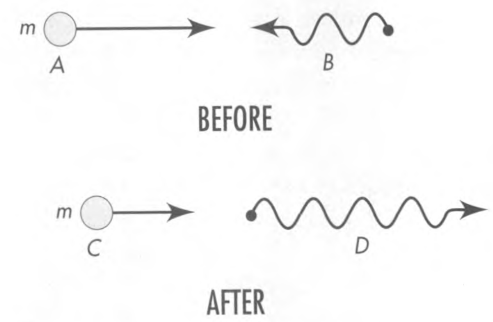
8-32 inverse Compton scattering

In Compton's original experiment an X-ray photon scattered with reduced energy from an electron initially at rest. In contrast, a photon scattered from a moving electron can increase the energy of the photon. Such an interaction is called **inverse Compton scattering.** The figure (page 270) shows an example.

When a high-energy electron collides head on with a low-energy photon, what is the energy of the outgoing photon? Answer this question using parts **a–e** or by some other method.

a Write down equations of conservation of energy and momentum, using subscripts A through D from the figure.

b Recall that the energy of a photon is equal to the magnitude of its momentum. Use this to simplify



EXERCISE 8-32. *Inverse Compton scattering. A low-energy photon is scattered by a high-energy electron.*

the conservation equations, taking leftward momentum to be negative.

c We are not interested in the energy or the momentum of outgoing electron C. Therefore solve the energy equation for E_C and the momentum equation for p_C, square and subtract the two sides, and use $E_C^2 - p_C^2 = m$. What happens to $E_A{}^2$ and $p_A{}^2$ on the other side of the resulting equation? For now keep terms in the first power of p_A without substituting the awkward equivalent $p_A = (E_A{}^2 + m^2)^{1/2}$.

d Solve the resulting equation for the energy of the outgoing photon.

e Now consider an important special case in which the incoming electron is extremely energetic, with an energy of, say, thousands of times its rest energy as measured in the laboratory. Show that this case the incoming electron behaves in essential respects as a photon: $p_A \approx +E_A$. Simplify your equation of part **d** to show that under these circumstances the outgoing photon has the energy of the incoming electron *no matter what the energy of the incoming photon.*

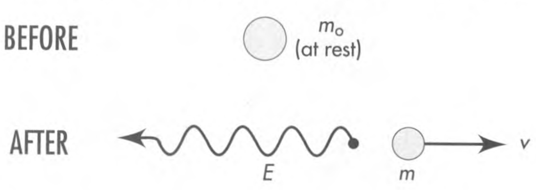
TESTS OF RELATIVITY

Note: Exercises 8-33 through 8-39 form a connected tutorial on tests of relativity. Some of these exercises depend on each other and on earlier exercises, especially Exercise 8-6.

8-33 photon energy shift due to recoil of emitter

Note: This exercise uses the results of Exercise 8-25.

A free particle of initial mass m_o and initially at rest emits a photon of energy E. The particle (now of mass m) recoils with velocity v, as shown in the figure.



EXERCISE 8-33. *Recoil of a particle that emits a photon.*

a Write down the conservation laws in a form that makes no reference to velocity. Consider the case in which the fractional change in mass in the emission process is very small compared to unity. Show that for this special case the photon has an energy $E_o = m_o - m$. For the general case show that

$$E = E_o\left(1 - \frac{E_o}{2m_o}\right)$$

or

$$\frac{E - E_o}{E_o} = \frac{\Delta E}{E_o} = -\frac{E}{2m_o}$$

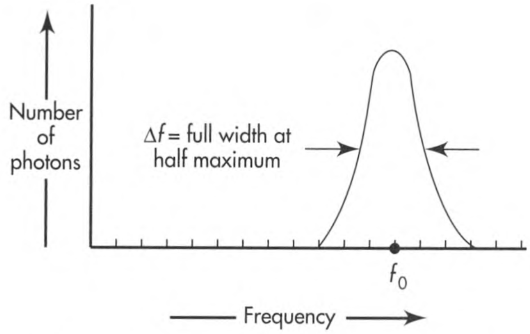
b Show that this shift in energy for visible light ($E_{o\,\text{conv}} \sim 3$ eV) emitted from atoms ($mc^2 \sim 10 \times 10^9$ eV) in a gas is very much less than the Doppler shift due to thermal motion (Exercise 8-25) even for temperatures as low as room temperature ($kT \sim 1/40$ eV).

8-34 recoilless processes

a A free atom of iron ^{57}Fe—formed in a so-called "excited state" by the radioactive decay of cobalt ^{57}Co—emits from its nucleus a gamma ray (high-energy photon) of energy 14.4 keV and transforms to a "normal" ^{57}Fe atom. By what fraction is the energy of the emitted ray shifted because of the recoil of the atom? The mass of the ^{57}Fe atom is about equal to that of 57 protons.

b That not all emitted gamma rays experience this kind of frequency shift was the important discovery made in 1958 by R. L. Mössbauer at the age of 29. He showed that when radioactive nuclei embedded in a solid emit gamma rays, some significant fraction of these atoms fail to recoil as free atoms. Instead they behave as if locked rigidly to the rest of the solid. The recoil in these cases is communicated to the solid as a whole. The solid being heavier than one atom by many powers of 10, these events are called **recoilless processes.** For gamma rays emitted in recoilless processes, the m_o in Exercise 8-33 is the mass of the entire chunk in which the iron atoms are embedded. When this chunk has a mass of one gram, by what fraction is the frequency of the emitted ray shifted in this "recoilless" process?

c The gamma rays emitted from excited ^{57}Fe atoms do not have a precisely defined energy but are spread over a narrow energy range—or frequency range—or natural line width, shown as a bell-shaped curve in the figure. (The physical basis for this curve is explained by quantum physics.) The full width of this curve at half maximum is denoted by $\Delta\nu$. R. V. Pound and G. A. Rebka selected ^{57}Fe for experiments



EXERCISE 8-34. *Natural line width of photons emitted from ^{57}Fe.*

with recoilless processes because the fractional ratio $\Delta f/f_0$ has the very small value 6×10^{-13} for the 14.4 keV gamma ray from ^{57}Fe. How much is the natural line width, Δf, of ^{57}Fe expressed in cycles/second? Compare the fractional natural line width with the fractional shift due to recoil of a free iron atom. And compare it with the fractional shift of a gamma ray from a recoilless process.

Reference: For a more detailed account of Mössbauer's discovery—for which the German scientist was awarded the Nobel prize in 1961—see S. DeBenedetti, "The Mössbauer Effect," *Scientific American,* Volume 202, pages 72–80 (April 1960). For the selection of ^{57}Fe, see R. V. Pound and G. A. Rebka, Jr., *Physical Review Letters,* Volume 3, pages 439–441 (1959).

Pound and Rebka's application of recoilless processes thus put into one's hands a resonance phenomenon sharp in frequency to the fantastic precision of 6 parts in 10^{13}. Exercise 8-35 deals with detection of this radiation. Exercise 8-36 uses motion (Doppler shift) as a means for producing controlled changes of a few parts in 10^{13}—or much larger changes—in the effective frequency of source or detector or both. To what uses can radiation of precisely defined frequency be put? There are many uses. For instance, the effect is the basis of important techniques in solid-state physics, molecular physics, and biophysics. One can detect the change in the natural frequency of radiation from ^{57}Fe atoms caused by other atoms in the neighborhood—and by external magnetic fields—and in this way analyze the interaction between the iron atom its surroundings. Here we aim at detection of various effects predicted by relativity.

8-35 resonant scattering

The nucleus of normal ^{57}Fe absorbs gamma rays at the resonant energy of 14.4 keV much more strongly than it absorbs gamma rays of any nearby energy. The energy absorbed in this way is converted to internal energy of the nucleus and transmutes the ^{57}Fe to the "excited state." After a time this excited nucleus drops back to the "normal state," emitting the excess energy in various forms in all directions. Therefore the number of gamma rays transmitted through a thin sheet containing ^{57}Fe will be less at the 14.4 keV resonance energy than at any nearby energy. This process is called **resonant scattering.**

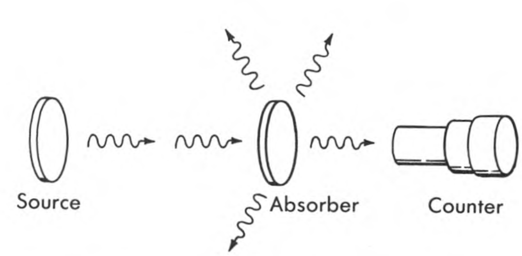
a Show that when a gamma ray of the resonant energy E_0 is incident on a free iron atom initially at rest then the free nucleus cannot absorb the gamma ray at its resonant energy, because the process cannot satisfy both the law of conservation of momentum and the law of conservation of energy.

b Show that both conservation laws are satisfied when an iron atom embedded in a one-gram crystal absorbs such a gamma ray by a recoilless process, in which the entire crystal absorbs the momentum of the incident gamma ray. ("Satisfied"? For momentum, yes; for energy, no. However, the fractional discrepancy in energy—equivalent to the fractional discrepancy in frequency—is less than 6 parts in 10^{13} and therefore small enough so that the iron nucleus is "unable to notice" the discrepancy and therefore absorbs the gamma ray.)

8-36 measurement of Doppler shift by resonant scattering

In the experimental arrangement shown in the figure, a source containing excited ^{57}Fe nuclei emits (among other radiations) gamma rays of energy E_0 by a recoilless process. An absorber containing ^{57}Fe nuclei in the normal state absorbs some of these gamma rays by another recoilless process and reemits this energy in various forms in all directions. Thus the counting rate on a gamma ray counter placed as shown is less for an absorber containing normal ^{57}Fe than for an equivalent absorber without normal ^{57}Fe. Now the source is moved toward the absorber with speed v.

a What must be the velocity of the source if the gamma rays are to arrive at the absorber shifted in frequency by 6 parts in 10^{13}? Express your answer in centimeters/second.



EXERCISE 8-36. *Resonant scattering of photons.*

b Will the counting rate of the counter increase or decrease under these circumstances?

c What will happen to this counting rate if the source is moved away from the absorber with the same speed?

d Make a rough plot of counting rate of the counter as a function of the source velocity toward the absorber (positive velocity) and away from the absorber (negative velocity).

e **Discussion question:** Does this method allow one to measure the "absolute velocity" of the source, in violation of the Principle of Relativity (Chapter 3)?

8-37 test of the gravitational red shift I

A 14.4-keV gamma ray emitted from ^{57}Fe without recoil travels vertically upward in a uniform gravitational field. By what fraction will the energy of this photon be reduced in rising to a height z (Exercise 8-6)? An absorber located at this height must move with what speed and in what direction in order to absorb such gamma rays by recoilless processes? Calculate this velocity when the height is 22.5 meters. Plot the counting rate as a function of absorber velocity expected if (a) the gravitational red shift exists, and (b) there is no gravitational red shift. A frequency shift of $\Delta f/f_o = (2.56 \pm 0.03) \times 10^{-15}$ was determined in an experiment conducted by R. V. Pound and J. L. Snider. You will notice that this shift is very much smaller than the natural line width $\Delta f/f_o = 6 \times 10^{-13}$ (see the figure for Exercise 8-34). Therefore the result depended on a careful exploration of the shape of this line and was derived statistically from a large number of photon counts.

References: Original experiment: R. V. Pound and G. A. Rebka, Jr., *Physical Review Letters,* Volume 4, pages 337–341 (1960). Improved experiment: R. V. Pound and J. L. Snider, *Physical Review,* Volume 140, pages B788–B803 (1965).

8-38 test of the gravitational red shift II

On June 18, 1976, a Scout D rocket was launched from Wallops Island, Virginia, carrying an atomic hydrogen-maser clock as the payload. It achieved a maximum altitude of 10^7 meters. By means of microwave signals, its clock was compared with an identical clock at the surface of Earth. The experiment used continuous comparison of these two clocks as the payload rose and fell. Simplifying (and somewhat misrepresenting) the experiment, we report their result as a fractional frequency red shift at the top of the trajectory due to gravitational effects of $\Delta f/f = 0.945 \times 10^{-10} \pm 6.6 \times 10^{-15}$.

Modify the analysis of Exercise 8-6 to make a prediction about this experiment and compare your prediction with the results of the Scout D rocket experiment.

References: Description of experiment and preliminary results: R. F. C. Vessot and M. W. Levine, *General Relativity and Gravitation,* Volume 10, Number 3, pages 181–204 (1979). Final results: R. F. C. Vessot, M. W. Levine, and others, *Physical Review Letters,* Volume 45, pages 2081–2084 (1980). Popular explanation: Clifford M. Will, *Was Einstein Right?* (Basic Books, New York, 1986), pages 42–64.

8-39 test of the twin paradox

For Penny to leave her twin brother Paul behind in the laboratory, go away at high speed, return, and find herself younger than stay-at-home Paul is so contrary to everyday experience that it is astonishing to find that the experiment has already been done and the prediction upheld! Chalmers Sherwin pointed out that the twins can be identical iron atoms just as well as living beings. Let one iron atom remain at rest. Let the other make one forth-and-back trip. Or many round trips. The percentage difference in aging of the twin atoms is the same after a million round trips as after one round trip—and it is easier to measure. How does one get the second atom to make many round trips? By embedding it in a hot piece of iron, so that it vibrates back and forth about a position of equilibrium (thermal agitation!). How does one measure the difference in aging? In the case of Penny and Paul the number of birthday firecrackers that each sets off during their separation are counted. In the experiment with iron atoms one compares not the number of flashes of firecrackers up to the time of meeting but the frequency of the photons emitted by recoilless processes, and thus—in effect—the number of ticks from two identical nuclear clocks in the course of one laboratory second. In other words, one compares the effective frequency of INTERNAL nuclear vibrations (not to be confused with the back-and-forth vibration of the iron atom as a whole!) as observed in the laboratory for (a) an iron nucleus at rest and (b) an iron nucleus in a hot specimen.

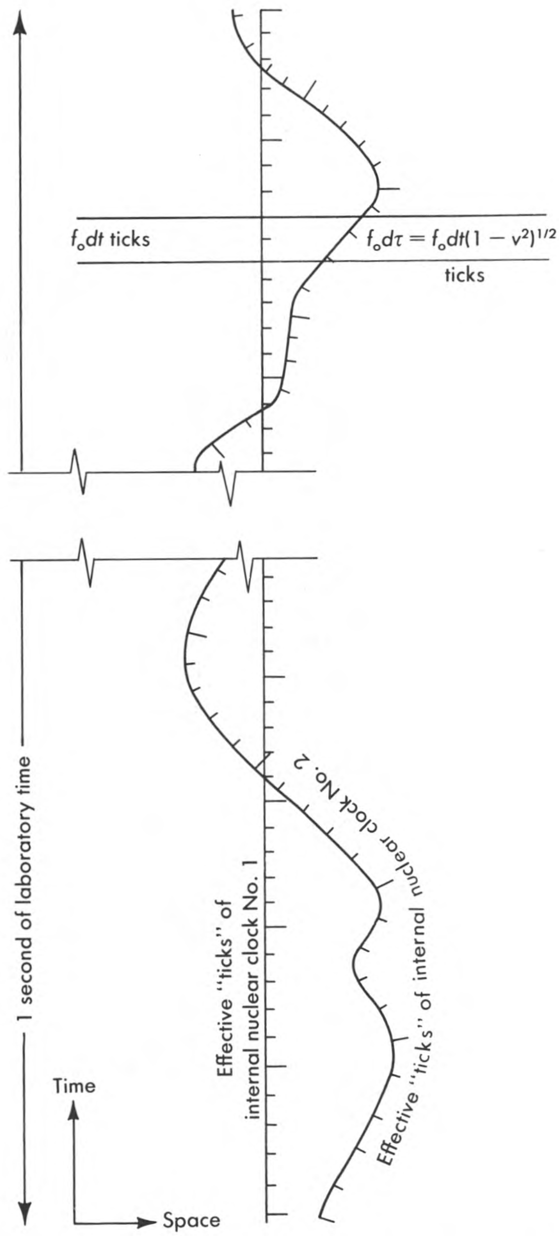
It is difficult to obtain an iron nucleus at rest. Therefore the actual experiment compared the effective internal nuclear frequency for two crystals of iron with a difference of temperature ΔT. R. V. Pound and G. A. Rebka, Jr., measured that a sample warmed up by the amount $\Delta T = 1$ degree Kelvin underwent a fractional change in effective frequency of $\Delta f/f_o = (-2.09 \pm 0.24) \times 10^{-15}$ (fewer vibrations; fewer clock ticks; fewer birthdays; more youthful!). (We use f for frequency instead of the usual Greek nu, ν, to avoid confusion with v for speed.)

To simplify thinking about the experiment, go back to the idea that one iron atom is at rest and the other is in thermal agitation at temperature T; predict the fractional lowering in number of internal vibrations in the hot sample per laboratory second; and compare with experiment.

Discussion: The figure compares the effective "ticks" of the two "internal nuclear clocks" in the laboratory time dt. Note that the speed of thermal agitation is about 10^{-5} the speed of light. What algebraic approximation suggests itself for the discrepancy factor $1 - (1 - v^2)^{1/2}$? How much is the deficit in number of "ticks" (for hot atom versus atom at rest) in the lapse of laboratory time dt? Show that the cumulative deficit in number of "ticks" from the hot atom in one second is $f_o(v^2/2)_{avg}(1 \text{ second})$ where $(v^2)_{avg}$ means "the time average value of the square of the atomic speed" (relative to the speed of light). Note that the mean kinetic energy of thermal agitation of a hot iron atom (mass $m_{Fe} = 57\ m_{proton}$) is given by the classical kinetic theory of gases:

$$(1/2)\ m_{Fe}(v^2)_{avg}c^2 = (3/2)\ kT$$

Here k is Boltzmann's factor of conversion between two units of energy, degrees and joules (or degrees and ergs); $k = 1.38 \times 10^{-23}$ joule/degree Kelvin ($k = 1.38 \times 10^{-16}$ erg/degree Kelvin). How does the experimental result of Pound and Rebka compare with the result of your calculation?



EXERCISE 8-39. *Comparison of nuclear clock at rest with nuclear clock in thermal motion.*

References: Chalmers W. Sherwin, *Physical Review,* Volume 120, pages 17–21 (1960). R. V. Pound and G. A. Rebka, Jr., *Physical Review Letters,* Volume 4, pages 274–275 (1960).

FREE-FOR-ALL!

8-40 momentum without mass?

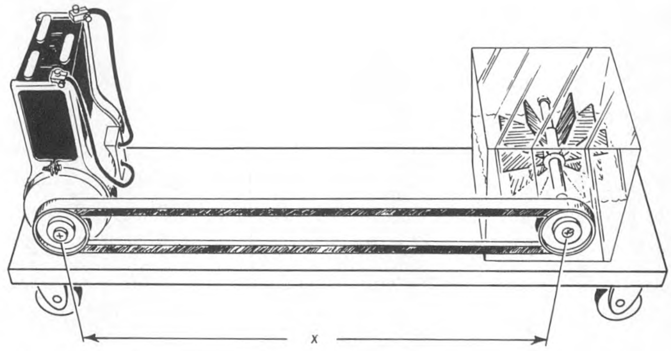
A small motor mounted on a board is powered by a battery mounted on top of it, as shown in the figure on page 274. By means of a belt the motor drives a paddlewheel that stirs a puddle of water. The paddlewheel mechanism is mounted on the same board as the motor but a distance x away. The motor performs work at a rate dE/dt.

a How much mass is being transferred per second from the motor end of the board to the paddlewheel end of the board?

b Mass is being transferred over a distance x at a rate given by your answer to part **a**. What is the momentum associated with this transfer of mass? Since this momentum is small, Newtonian momentum concepts are adequate.

c Let the mounting board be initially at rest and supported by frictionless rollers on a horizontal table. The board will move! In which direction? What happens to this motion when the battery runs down? How far will the board have moved in this time?

d Show that an observer on the board sees the energy being transferred by the belt; an observer on the table sees the energy being transferred partly by the belt and partly by the board; an observer riding one way on the belt sees the energy being transferred partly by the belt moving in the other direction and partly by the board. Evidently it is not always possible



EXERCISE 8-40. *Transfer of mass without net transfer of particles or radiation.*

to make a statement satisfactory to all observers about the path by which energy travels from one place to another or about the speed at which this energy moves from one place to another!

8-41 the photon rocket and interstellar travel

The "perfect" rocket engine combines matter and antimatter in a controlled way to yield photons (high-energy gamma rays), all of which are directed out the rear of the rocket. Suppose we start with a spaceship of initial mass M_o, initially at rest. At burnout the remaining spaceship moves with speed v and has a mass equal to the fraction f of the original mass. For a given fraction f, we want to know the final rocket speed v or, better yet, the time stretch factor $\gamma = 1/(1 - v^2)^{1/2}$. (**Note:** Here, f is *not* frequency.)

a What is the total energy of the system initially? Let E_{rad} stand for the total energy of radiation after burnout. Find an expression for the total energy of the system after burnout and set up the conservation of energy equation.

b Similarly, set up the conservation of momentum equation. What is the total momentum of the system initially? The momentum of the radiation at burnout? The momentum of the spaceship at burnout?

c Eliminate E_{rad} between the two conservation equations. Show that the result can be written

$$\gamma f + \gamma v f = 1$$

d From the definition of γ, show that $\gamma v = (\gamma^2 - 1)^{1/2}$ and hence that the equation of part **c** can be written in the form

$$f^2 - 2\gamma f + 1 = 0$$

e What is the value of the fraction $f =$ (final spaceship mass)/(initial spaceship mass) for a time stretch factor $\gamma = 10$? In your opinion, is it possible to construct a spaceship whose shell and payload is this small a fraction of takeoff mass?

f Substitute the result of part **e** into the conservation of energy equation in part **a**. Show that the total energy of emitted radiation is less than the mass of fuel consumed. Why?

g Does your analysis apply to takeoff from Earth's surface? From Earth orbit? From somewhere else? What safety precautions apply to the backward blast of gamma rays?

h You are the astronaut assigned to this spaceship. Do you want to stop at your distant destination star or fly past at high speed? Do you want to return to Earth? Do you want to stop at Earth on your return or merely wave in passing? Must all fuel for the entire trip be on board at takeoff or can you refuel at your destination star? From your answers to these questions, plan your trip and find the resulting fractions of spaceship mass to initial mass for different stages of your trip.

i Discussion question: From your results for this exercise, what are your conclusions about the technical possibilities of human flight to the stars?

References: Adapted from A. P. French, *Special Relativity* (W.W. Norton, New York, 1968), pages 183–184. See also J. R. Pierce, *Proceedings of the IRE,* Volume 47, pages 1053–1061 (1959).