# Chapter 16. Gravitational Waves



- *What are gravitational waves?*
- *How do gravitational waves differ from ocean waves?*
- *How do gravitational waves differ from light waves?*
- *What is the source (or sources) of gravitational waves?*
- *Why has it taken us so long to detect gravitational radiation?*
- *Why is the Laser Interferometer Gravitational-Wave Observatory (LIGO) so big?*
- *Why are LIGOs located all over the Earth?*
- *What will the next generation of gravitational wave detectors look like?*

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# CHAPTER 16

# Gravitational Waves

**Edmund Bertschinger & Edwin F. Taylor** *

*If you ask me whether there are gravitational waves or not, I must answer that I do not know. But it is a highly interesting problem.*

—Albert Einstein

## 16.1 ■ THE PREDICTION AND DISCOVERY OF GRAVITATIONAL WAVES

*Gravitational wave: a tidal acceleration that propagates through spacetime.*

General relativity predicts black holes with properties utterly foreign to Newtonian and quantum physics. And general relativity predicts gravitational waves, also foreign to Newtonian and quantum physics.

Newton: Gravity propagates instantaneously.

Without quite saying so, Newton assumed that gravitational interaction propagates instantaneously: When the Earth moves around the Sun, the Earth's gravitational field changes all at once everywhere. When Einstein formulated special relativity and recognized its requirement that no information can travel faster than the speed of light in a vacuum, he realized that Newtonian gravity would have to be modified. Not only would static gravitational effects differ from the Newtonian prediction in the vicinity of compact masses, but also gravitational effects would propagate as waves that move with the speed of light.

Einstein: No signal propagates faster than light.

Einstein's conceptual prototype for gravitational waves was electromagnetic radiation. In 1873 James Clerk Maxwell demonstrated that the laws of electricity and magnetism predicted electromagnetic radiation. Einstein was born in 1879. Heinrich Hertz demonstrated electromagnetic waves experimentally in 1888. The adult Einstein realized that a general relativity theory would not look like Maxwell's electromagnetic theory, but he and others were able to formulate the corresponding gravitational wave equations.

Compare gravitational waves to electromagnetic waves.

What do we mean by a "gravitational wave"? The gravitational wave is a tidal acceleration that propagates; that is all it is. As a gravitational wave passes over you, you are alternately stretched and compressed in ways that

Gravitational waves propagate tidal accelerations.

*Draft of Second Edition of *Exploring Black Holes: Introduction to General Relativity*

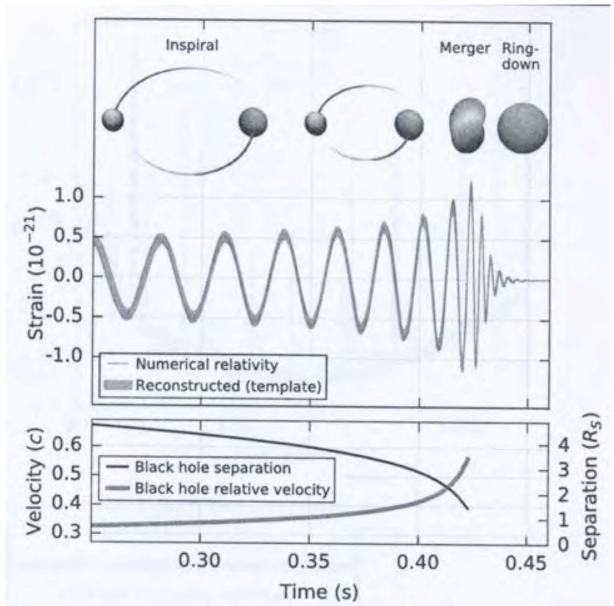


**FIGURE 1** Predicted "chirp" of the gravitational wave as two black holes in a binary system merge. Frequency and amplitude increase, followed by a "ring down" due to oscillation of the merged black hole. The present chapter explains details of this figure.

depend on the particular form of the wave. In principle there is no limit to the amplitude of a gravitational wave. In the vicinity of the coalescence, gravity-wave-induced tidal forces would be lethal. Far from such a source, gravitational waves are tiny, which makes them difficult to detect.

In 2015, the most sensitive gravitational wave detector is the **Laser Interferometer Gravitational Wave Observatory**, or **LIGO** for short. Gravitational waves were first detected on 14 September 2015 with two LIGO detectors, one at Hanford, Washington state USA, the other at Livingston, Louisiana state. These detections give us confidence that gravitational waves from various sources continually sweep over us on Earth. Sections 16.3 and 16.7 describe some of these sources.

Gravitational wave on Earth: An extremely small traveling tidal effect

Basically we observe gravitational waves by detecting changes in separation between two test masses suspended near to one another—changes in gravitational-wave tidal effects. Changes in this separation are *extremely* small for gravitational waves detected on Earth.

Gravitational wave detectors are interferometers.

Current gravitational wave detectors on Earth are interferometers in which light reflects back and forth between "free" test masses (mirrors) positioned at the ends of two perpendicular vacuum chambers. A passing gravitational wave changes the relative number of wavelengths along each leg, with a resulting change in interference between the two returning waves. The "free" test masses are hung from wires that are in turn supported on elaborate shock-absorbers to minimize the vibrations from passing trucks and even ocean waves crashing on a distant shore. The pendulum-like motions of these test masses are free enough to permit measurement of their change in separation due to tidal effects of a passing gravitational wave, caused by some remote gigantic distant event such as the coalescence of two black holes modeled in Figure **??**.

?

**Objection 1.** *Does the change in separation induced by gravitational waves affect everything, for example a meter stick or the concrete slab on which a gravitational wave detector rests?*

!

The structure of a meter stick and a concrete slab are determined by electromagnetic forces mediated by quantum mechanics. The two ends of a meter stick are not freely-floating test masses. The tidal force of a passing gravitational wave is much weaker than the internal forces that maintain the shape of a meter stick—or the concrete slab supporting the vacuum chamber of a gravitational-wave observatory; these are stiff enough to be negligibly affected by a passing gravitational wave.

**Comment 1. Why not "gravity wave"?**
Why do we use the five-syllable *gravitational* to describe this waves, and not the three-syllable *gravity*? Because the term *gravity wave* is already taken. *Gravity wave* describes the disturbance at an interface—for example between the sea and the atmosphere—where gravity provides the restoring force.

## 16.2 ■ GRAVITATIONAL WAVE METRIC

*Tiny but significant departure from the inertial metric*

Our analysis examines effects of a particular gravitational wave: a plane wave from a distant source that moves in the $z$-direction. Every gravitational wave we discuss in this chapter (except those shown in Figure **??**) represents a very small deviation from flat spacetime. Here is the metric for a gravitational plane wave that propagates along the $z$-axis.

Gravitational wave metric

$$d\tau^2 = dt^2 - (1+h)dx^2 - (1-h)dy^2 - dz^2 \qquad (h \ll 1) \tag{1}$$

First, for light $d\tau = 0$. Then, as usual, no experiment or observation is global; every one is local. At the LIGO detector the local metric has the form:

$$\begin{aligned} 0 &\approx \Delta t_{\mathrm{LIGO}}^2 - [(1+h)^{1/2}\Delta x_{\mathrm{LIGO}}]^2 - [(1-h)^{1/2}\Delta y_{\mathrm{LIGO}}]^2 - \Delta z_{\mathrm{LIGO}}^2 \\ &\approx \Delta t_{\mathrm{LIGO}}^2 - [(1+h/2)\Delta x_{\mathrm{LIGO}}]^2 - [(1-h/2)\Delta y_{\mathrm{LIGO}}]^2 - \Delta z_{\mathrm{LIGO}}^2 \qquad (h \ll 1) \end{aligned} \tag{2}$$

In this metric $h/2$ is the tiny fractional deviation from the flat-spacetime coefficients of $dx^2$ and $dy^2$. The technical name for fractional deviation of length is **strain**, so $h/2$ is also called the **gravitational wave strain**. Metric (1) describes a transverse wave, since $h$ is a perturbation in the $x$ and $y$ directions transverse to the $z$-direction of propagation. The metric guarantees that $t$ will vary, along with $x$ and $y$.

$h/2 =$ gravitational wave strain

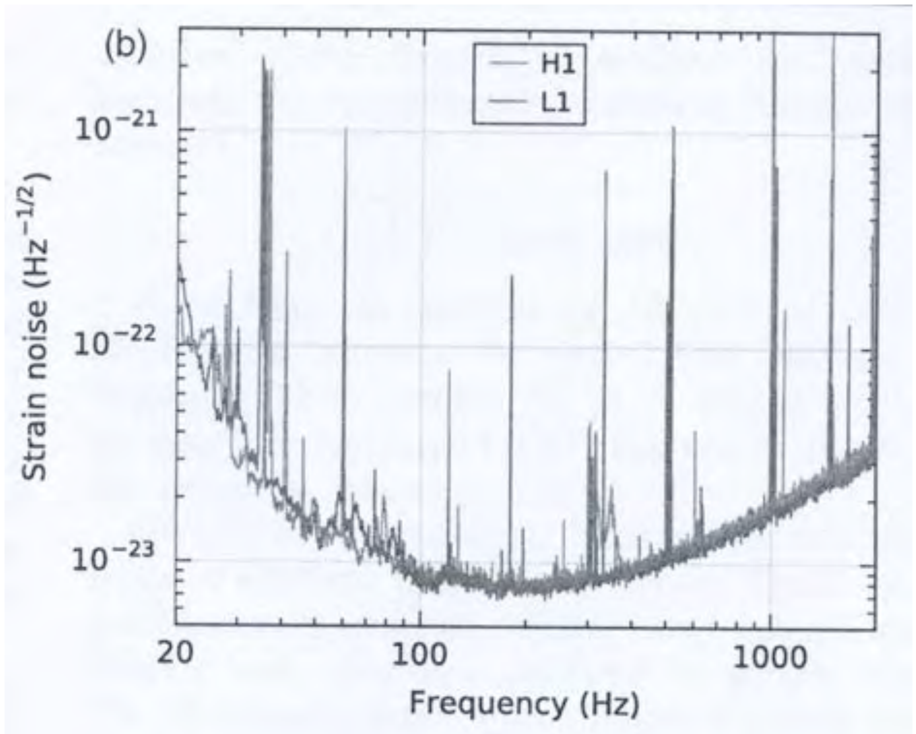


**FIGURE 2** Strain noise of LIGO detectors at Hanford, Washington state (curve H1) and at Livingston, Louisiana state (curve L1) at the first detection of a gravitational wave on 14 September, 2015. On the vertical axis $h = 10^{-23}$ , for example, means a fractional change in separation of $10^{-23}$ between test masses. Spikes occur at frequencies of electrical or acoustical noise. To be detectable, gravitational wave signals must cause fractional change above these noise curves.

Let two free test masses be at rest $D$ apart in the $x$ or $y$ direction. When a $z$-directed gravitational wave passes over them, the change in their separation, called the **displacement**, equals $h/2 \times D$, which follows from the definition of $h/2$ as a "fractional deviation."

?

**Objection 2.** *Awkward! Why define the strain as $h/2$ instead of simply $h$?*

!

*Response:* This results from squared values of separation in both global and local metrics. We could use $(1-2h)$ instead of $(1-h)$ in global metric (1), but that would be awkward in another way. As usual, we get to choose the awkwardness, but cannot eliminate or ignore it!

LIGO gravity wave detector

Einstein's field equations yield predictions about the magnitude of the function $h$ in equation (1) for various kinds of astronomical phenomena. Current gravity wave detectors use laser interferometry and go by the full name **Laser Interferometer Gravitational Wave Observatory**, or **LIGO** for short.

Various kinds of noise

LIGO sensitivity

Figure 2 shows the noise spectrum of the two LIGO instruments that were the first to detect a gravitational wave. The displacement sensitivity is expressed in the units of meter/(hertz)$^{1/2}$ because the amount of noise limiting the measurement grows with the frequency range being sampled. Note that the instruments are designed to be most sensitive near 150 hertz. This frequency is determined by the different kinds of noise faced by experimenters: Quantum noise ("shot noise") limits the sensitivity at high frequencies, while seismic noise (shaking of the Earth) is the largest problem at low frequencies. If the range of sampled frequencies—*bandwidth*—is 100 hertz, then LIGO's best sensitivity is about $10^{-21} \times 100^{1/2} = 10^{-23}$. This means that along a length of 4 kilometers $= 4 \times 10^3$ meters, the change in length is approximately $10^{-21} \times 4 \times 10^3 = 4 \times 10^{-18}$ meters, which is one thousandth the size of a proton, or a hundred million times smaller than a single atom!

?

**Objection 3.** *Your gravitational wave detector sits on Earth's surface, but equation (1) says nothing about curved spacetime described, for example, by the Schwarzschild metric. The expression $2M/r$ measures departure from flatness in the Schwarzschild metric. At Earth's surface, $2M/r \approx 1.4 \times 10^{-9}$, which is $10^{13}$—ten million million!—times greater than the corresponding gravitational wave factor $h \sim 10^{-22}$. Why doesn't the quantity $2M/r$—which is much larger than $h$—appear in (1)?*

!

The factor $2M/r$ is essentially constant across the structure of LIGO, so we can ignore its change as the gravitational wave sweeps over it. LIGO is totally insensitive to the *static* curvature introduced by the factor $2M/r$ at Earth's surface. Indeed, the LIGO detector is "tuned" to detect gravitational wave frequencies near 150 hertz. For this reason, we simply omit static curvature factors from equation (1), effectively describing gravitational waves "in free space" for the predicted $h \ll 1$.

Einstein's equations become a wave equation.

In flat spacetime and for small values of $h$, Einstein's field equations reduce to a wave equation for $h$. For the most general case, this wave has the form $h = h(t, x, y, z)$. When $t, x, y, z$ are all expressed in meters, this wave equation takes the form:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \frac{\partial^2 h}{\partial t^2} \qquad \text{(flat spacetime and } h \ll 1) \qquad (3)$$

For simplicity, think of a plane wave moving along the $z$-axis. The most general solution to the wave equation under these circumstances is

$$h = h_{+z}(z - t) + h_{-z}(z + t) \qquad (4)$$

Assume gravity wave moves in $+z$ direction.

The expression $h_{+z}(z - t)$ means a function $h$ of the single variable $z - t$. The function $h_{+z}(z - t)$ describes a wave moving in the positive $z$-direction and the function $h_{-z}(z + t)$ describes a wave moving in the negative $z$-direction. In this chapter we deal only with a gravitational wave propagating in the positive $z$-direction (Figure 5) and hereafter set

$$h \equiv h(z - t) \equiv h_{+z}(z - t) \qquad \text{(wave moves in } +z \text{ direction)} \qquad (5)$$

The argument $z - t$ means that $h$ is a function of *only* the combined variable $z - t$. Indeed, $h$ can be *any function whatsoever* of the variable $(z - t)$. The form of this variable tells us that, whatever the profile of the gravitational wave, that profile displaces itself in the positive $z$-direction with the speed of light (local light speed = one in our units).

LIGO sensitive 75 to 500 hertz

Figure 2 shows that the LIGO gravitational wave detector has maximum sensitivity for frequencies between 75 and 500 hertz, with a peak sensitivity at around 150 hertz. Even at 500 hertz, the wavelength of the gravitational wave is very much longer than the overall 4-kilometer dimensions of the LIGO detector. Therefore *we can assume in the following that the value of $h$ is spatially uniform over the entire LIGO detector.*

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**QUERY 1. Uniform $h$?**

Using numerical values, verify the claim in the preceding paragraph that $h$ is effectively uniform over the LIGO detector.

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Analogy: draw global map coordinates on rubber sheet.

It is important to understand that coordinates in metric (1) are global and to recall that global coordinates are arbitrary; we choose them to help us visualize important aspects of spacetime. For $h \neq 0$, these global coordinates are invariably distorted. Think of the three mutually perpendicular planes formed by $(x, y)$, $(y, z)$, and $(z, x)$ pairs. Draw a grid of lines on a rubber sheet lying in each corresponding plane. By analogy, the passing gravitational wave distorts these rubber sheets.

Gravitational wave distorts rubber sheet.

Glue map clocks to intersections of these grid lines on a rubber sheet so that they move as the rubber sheet distorts. A gravitational wave moving in the $+z$ direction (Figure 3) passes through a rubber sheet and acts in different

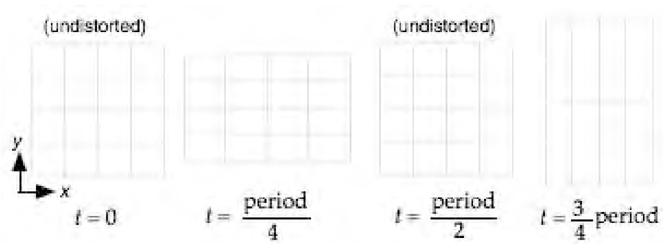


**FIGURE 3** Change in shape (greatly exaggerated!) of the map coordinate grid at the same $x, y$ location at four sequential $t$-values as a periodic gravitational wave passes through in the $z$-direction (perpendicular to the page). NOTE carefully: The $x$-axis is stretched while the $y$-axis is compressed and vice versa. The areas of the panels remain the same.

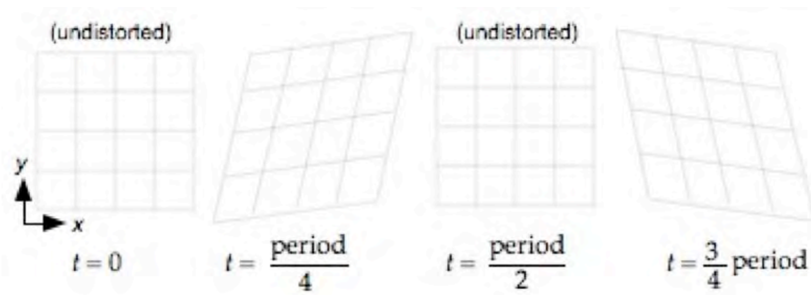


**FIGURE 4** Effects of a periodic gravitational wave with polarization "orthogonal" to that of Figure 3 on the map grid in the $xy$ plane. Note that the axes of compression and expansion are at 45 degrees from the $x$ and $y$ axes. All grids stay in the $xy$ plane as they distort. As in Figure 3, the areas of the panels are all the same.

directions within the plane of the sheet (Figures 3 and 4). The map clocks glued at intersections of map coordinate grid lines ride along with the grid as the sheet distorts, so the map coordinates of any clock do not change.

Map $t$ read on clocks glued to the rubber sheet.

Think of two ticks on a single map clock. Between ticks the map coordinates of the clock do not change: $dx = dy = dz = 0$. Therefore metric (1) tells us that the wristwatch time $d\tau$ between two ticks is also map $dt$ between ticks. Map $t$ corresponds to the time measured on the clocks glued to the rubber sheet, even when the strain $h/2$ varies at their locations.

Figure 3 represents the map distortion of the rubber sheet with $t$ at a given location due to a particular polarization of the gravitational wave. Although gravitational waves are transverse like electromagnetic waves, the polarization forms of gravitational waves are different from those of electromagnetic waves. Figure 4 shows the distortion caused by a polarization "orthogonal" to that shown in Figure 3.

## 16.3 ■ SOURCES OF GRAVITATIONAL WAVES

*Many sources; only one type leads to a clear prediction*

No linear "antenna" for gravitational waves

Sources of gravitational waves include collapsing stars, exploding stars, stars in orbit around one another, and the Big Bang itself. Neither an electromagnetic wave nor a gravitational wave results from a spherically symmetric distribution of charge (for electromagnetic waves) or matter (for gravitational waves), even when that spherical distribution pulses symmetrically in and out (Birkhoff's Theorem, Section 6.5). Therefore, a *symmetric* collapse or explosion emits no waves, either electromagnetic or gravitational. The most efficient source of electromagnetic radiation, for example along an antenna, is oscillating pairs of electric charges of opposite sign moving back and forth along the antenna, the resulting waves technically called **dipole radiation**. But mass has only one "polarity" (there is no negative mass), so there is no gravity dipole radiation from masses that oscillate back and forth along a line. Emission of gravitational waves requires *asymmetric* movement or oscillation; the technical name for the simplest result is **quadrupole radiation**. Happily, most collapses and explosions are asymmetric; even the motion in a binary system is sufficiently asymmetric to emit gravitational waves.

Binary system emits gravity waves . . .

We study here gravitational waves emitted by a binary system consisting of two black holes orbiting about one another (Section 16.7). The pair whose gravitational waves were detected are a billion light-years distant, so are not visible to us. As the two objects orbit, they emit gravitational waves, so the orbiting objects gradually spiral in toward one another. These orbits are well described by Newtonian mechanics until about one millisecond before the two objects coalesce.

. . . whose amplitude is predictable.

Emitted gravitational waves are nearly periodic during the Newtonian phase of orbital motion. As a result, these particular gravitational waves are easy to predict and hence to search for. When the two objects coalesce, they emit a burst of gravitational waves (Figures **??** and 1). After coalescence the resulting black hole vibrates ("rings down"), emitting additional gravitational waves as it settles into its final state.

**Comment 2. Amplitude, not intensity of gravitational waves**
The gravitational wave detector measures the *amplitude* of the wave. The wave amplitude received from a small source decreases as the inverse $r$-separation. In contrast, our eyes and other detectors of light respond to its *intensity,* which is proportional to the square of its amplitude, so the received intensity of light decreases as the inverse $r$-separation.

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**QUERY 2. Increased volume containing detectable sources**
If LIGO sensitivity is increased by a factor of two, what is the increased volume ratio from which it can detect sources?

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From other sources: hard to predict.

Binary coalescence is the only source for which we can currently make a clear prediction of the signal. Other possible sources include supernovae and the collapse of a massive star to form a black hole—the event that triggers a so-called **gamma-ray burst**. We can only speculate about how far away any of these can be and still be detectable by LIGO.

**Comment 3. Detectors do not affect gravitational waves**
We know well that metal structures can distort or reduce the amplitude of electromagnetic waves passing across them. Even the presence of a receiving antenna can distort an electromagnetic wave in its vicinity. The same is not true of gravitational waves, whose generation requires massive moving structures. Gravitational wave detectors have negligible effect on the waves they detect.

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**QUERY 3. Electromagnetic waves *vs.* gravitational waves. Discussion.**
What property of electromagnetic waves makes their interaction with conductors so huge compared with the interaction of gravitational waves with matter of any kind?

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## 16.4 ■ MOTION OF LIGHT IN MAP COORDINATES

*Light reflected back and forth between mirrored test masses*

LIGO is an interferometer.

Currently the LIGO detector system consists of two *interferometers* that employ mirrors mounted on "test masses" suspended at rest at the ends of an L-shaped vacuum cavity. The length of each leg $L = 4$ kilometers for interferometers located in the United States. Gravitational wave detection measures the changing interference of light waves round-trip *time delays*sent down the two legs of the detector.

Suppose that a gravitational wave of the polarization illustrated in Figure 3 moves in the $z$-direction as shown in Figure 5 and that one leg of the detector along the $x$-direction and the other leg along the $y$-direction. In order to analyze the operation of LIGO, we need to know (a) how light propagates along the $x$ and $y$ legs of the interferometer and (b) how the test masses at the ends of the legs move when the $z$-directed gravitational wave passes over them.

Motion of light in map coordinates.

With what map speed does light move in the $x$-direction in the presence of a gravitational wave implied by metric (1)? To answer this question, set $dy = dz = 0$ in that equation, yielding

$$d\tau^2 = dt^2 - (1+h)dx^2 \tag{6}$$

As always, the wristwatch time is zero between two adjacent events on the worldline of a light pulse. Set $d\tau = 0$ to find the map speed of light in the $x$-direction.

$$\frac{dx}{dt} = \pm(1+h)^{-1/2} \qquad \text{(light moving in } x \text{ direction)} \tag{7}$$

The plus and minus signs correspond to a pulse traveling in the positive or negative $x$-direction, respectively—that is, in the plane of LIGO in Figure 5.

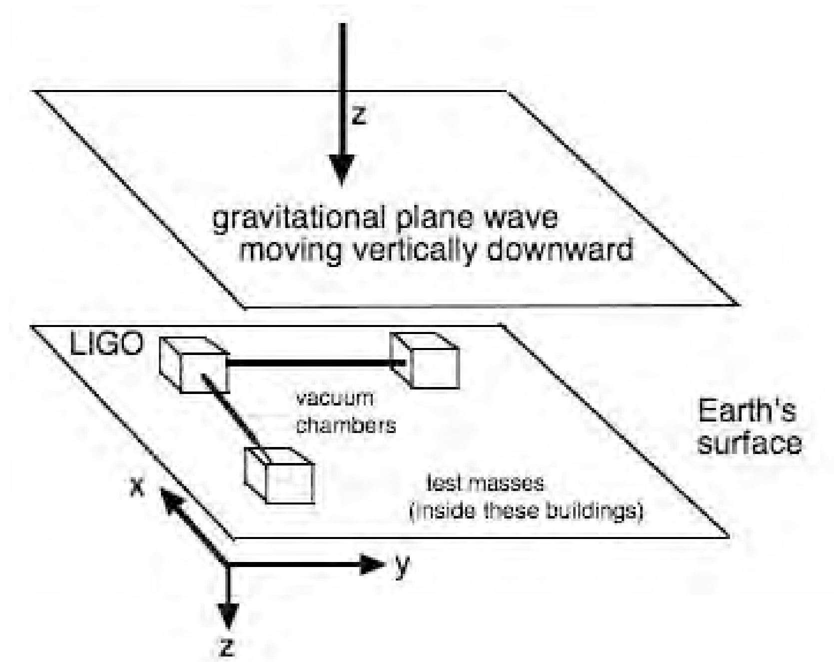


**FIGURE 5** Perspective drawing of the relative orientation of legs of the LIGO interferometer lying in the $x$ and $y$ directions on the surface of Earth and the $z$-direction of the incident gravitational wave descending vertically. [Illustrator: Rotate lower plate and contents CCW 90 degrees, so corner box is above the origin of the coordinate system. Same for Figure 10.]

Remember that the magnitude of $h$ is very much smaller than one, so we use the approximation inside the front cover. To first order:

$$(1+\epsilon)^n \approx 1 + n\epsilon \qquad |\epsilon| \ll 1 \text{ and } |n\epsilon| \ll 1 \tag{8}$$

Apply this approximation to (7) to obtain

$$\frac{dx}{dt} \approx \pm(1 - \frac{h}{2}) \qquad \text{(light moving in } x \text{ direction)} \tag{9}$$

Gravitational wave modifies map speed of light.

In words, the map speed of light changes (slightly!) in the presence of our gravitational wave. Since $h$ is a function of $t$ as well as $x$ and $y$, the map speed of light in the $x$-direction is not constant, but varies as the wave passes through. (Should we worry that the speed in (9) does not have the standard value one? No! This is a *map speed*—a mythical beast—measured directly by no one.)

By similar arguments, the map speeds of light in the $y$ and $z$ directions for the wave described by the metric (1) are:

$$\frac{dy}{dt} \approx \pm(1 + \frac{h}{2}) \qquad \text{( light moving in } y \text{ direction)} \tag{10}$$

$$\frac{dz}{dt} = \pm 1 \qquad \text{( light moving in } z \text{ direction)} \tag{11}$$

## 16.5 ■ ZERO MOTION OF LIGO TEST MASSES IN MAP COORDINATES

*"Obey the Principle of Maximal Aging!"*

Consider two test masses with mirrors suspended at opposite ends of the $x$-leg of the detector. The signal of the interferometer due to the motion of light along this leg will be influenced only by the $x$-motion of the test masses due to the gravitational wave. In this case the metric is the same as (6).

How does a test mass move as the gravitational wave passes over it? As always, to answer this question we use the Principle of Maximal Aging to maximize the wristwatch time of the test mass across two adjoining segments of its worldline between fixed end-events. In what follows we verify the surprising result, anticipated in Section 16.2, that a test mass initially at rest in map coordinates rides with the expanding and contracting map coordinates drawn on the rubber sheet, so this test mass does not move with respect to map coordinates as a gravitational wave passes over it. This result comes from showing that an out-and-back jog in the vertical worldline in map coordinates leads to smaller aging and therefore does not occur for a free test mass.

How does the test mass move?

Figure 6 pictures the simplest possible round-trip excursion: an incremental linear deviation from a vertical worldline from origin 0 to the event at $t = 2t_0$. Along Segment A the displacement $x$ increases linearly with $t$: $x = v_0 t$, where $v_0$ is a constant. Along segment B the displacement returns to zero at the same constant rate. Twice the strain $h$ has average values $\bar{h}_{\mathrm{A}}$ and $\bar{h}_{\mathrm{B}}$ along segments A and B respectively. We use the Principle of Maximal Aging to find the value of the speed $v_0$ that maximizes the wristwatch time along this worldline. We will find that $v_0 = 0$. In other words, the free test mass initially at rest in map coordinates stays at rest in map coordinates; it does not deviate from the vertical worldline in Figure 6. Now for the details.

Idealized case: Linear jogs out and back.

Write the metric (6) in approximate form for one of the segments:

$$\Delta\tau^2 \approx \Delta t^2 - (1 + \bar{h})\Delta x^2 \tag{12}$$

where $\bar{h}$ is an average value of $h$ across that segment. Apply (12) first to Segment A in Figure 6, then to Segment B. We are going to take derivatives of these expressions, which will look awkward applied to $\Delta$ symbols. Therefore we temporarily ignore the $\Delta$ symbols in (12) and let $\tau$ stand for $\Delta\tau$, $t$ for $\Delta t$, and $x$ for $\Delta x$, holding in mind that these symbols represent increments, so equations in which they appear are approximations.

With these substitutions, equation (12) becomes, for the two adjoining worldline segments:

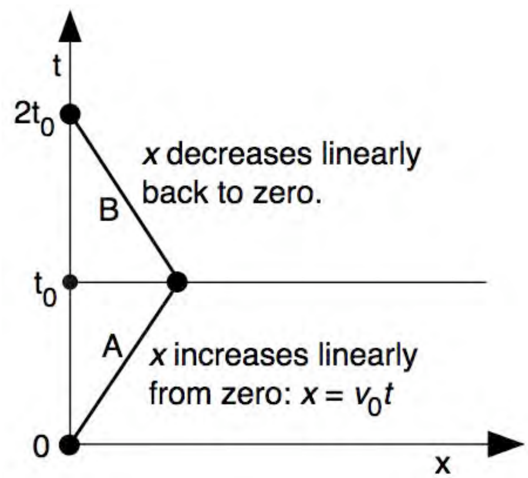


**FIGURE 6** Trial worldline for a test mass; incremental departure from vertical line of a particle at rest. Segments A and B are very short.

$$\tau_\text{A} \approx \left[t_0^2 - \left(1+\bar{h}_\text{A}\right)\left(v_0 t_0\right)^2\right]^{1/2} \qquad \text{Segment A} \tag{13}$$
$$\tau_\text{B} \approx \left[t_0^2 - \left(1+\bar{h}_\text{B}\right)\left(v_0 t_0\right)^2\right]^{1/2} \qquad \text{Segment B}$$

so that the total wristwatch time along the bent worldline from $t = 0$ to $t = 2t_0$ is the sum of the right sides of equations (13).

We want to know what value of $v_0$ (the out-and-back speed of the test mass) will lead to a maximal value of the total wristwatch time. To find this, take the derivative with respect to $v_0$ of the sum of individual wristwatch times and set the result equal to zero.

$$\frac{d\tau_\text{A}}{dv_0} + \frac{d\tau_\text{B}}{dv_0} \approx -\frac{(1+\bar{h}_\text{A})v_0 t_0^2}{\tau_\text{A}} - \frac{(1+\bar{h}_\text{B})v_0 t_0^2}{\tau_\text{B}} = 0 \tag{14}$$

so that

$$\frac{(1+\bar{h}_\text{A})v_0 t_0^2}{\tau_\text{A}} = -\frac{(1+\bar{h}_\text{B})v_0 t_0^2}{\tau_\text{B}} \tag{15}$$

Initially at rest in map coordinates? Then stays at rest in map coordinates.

Worldline segments A and B in Figure 6 are identical except in the direction of motion in $x$. In equation (15), $v_0$ is our proposed speed in global coordinates, a positive quantity. The only way that (15) can be satisfied is if $v_0 = 0$. *The test mass initially at rest does not change its map x-coordinate as the gravitational wave passes over.*

Our result seems rather specialized in two senses: First, it treats only the vertical worldline in Figure 6 traced out by a test mass at rest. Second, it deals only with a very short segment of the worldline, along which $\bar{h}$ is considered to be nearly constant. Concerning the second point, you can think of (14) as a tiny out-and-back "jog" *anywhere* on a much longer vertical worldline. Then our result implies that *any* jog in the vertical worldline does not lead to an increased value of the wristwatch time, even if $h$ varies a lot over a longer stretch of the worldline.

Not at rest in map coordinates? Maybe kink in map worldline.

The first specialization, the vertical worldline in Figure 6, is important: The gravitational wave does not cause a kink in a *vertical* map worldline. The same is typically *not* true for a particle that is moving in map coordinates before the gravitational wave arrives. (We say "typically" because the kink may not appear for some directions of motion of the test mass and for some polarization forms and directions of propagation of the gravitational wave.) In this more general case, a kink in the worldline corresponds to a change of velocity. In other words, a passing gravitational wave can change the map velocity of a moving particle just as if it were a velocity-dependent force. If the particle velocity is zero, then the force is zero: a particle at rest in map coordinates remains at rest.

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**QUERY 4. Disproof of relativity? (optional)**

"Aha!" exclaims Kristin Burgess. "Now I can disprove relativity once and for all. If the test mass *moves*, a passing gravitational wave can cause a kink in the worldline of the test mass as observed in the local inertial Earth frame. No kink appears in its worldline if the test mass is at rest. But if a worldline has a kink in it as observed in one inertial frame, it will have a kink in it as observed in all overlapping relatively-moving inertial frames. An observer in any such frame can detect this kink. So the *absence* of a kink tells me *and every other inertial observer* that the test mass is 'at rest'? We have found a way to determine absolute rest using a local experiment. Goodbye relativity!" Is Kristin right? (A detailed answer is beyond the scope of this book, but you can use some relevant generalizations drawn from what we already know to think about this paradox. As an analogy from flat-spacetime electromagnetism, think of a charged particle at rest in a purely magnetic field: The particle experiences no magnetic force. In contrast, when the same charged particle moves in the same frame, it may experience a magnetic force for some directions of motion.)

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At rest in map coordinates? Still can move in Earth coordinates.

In this book we make every measurement in a local inertial frame, not using differences in global map coordinates. So of what possible use is our result that a particle at rest in global coordinates does not move in those coordinates when a gravitational wave passes over it? Answer: Just because something is at rest in map coordinates does not mean that it is at rest in local inertial Earth coordinates. In the following section we find that a gravitational wave *does* move a test mass as observed in the Earth coordinates. LIGO—attached to the Earth—can detect gravitational waves!

## 16.6 ■ DETECTION OF A GRAVITATIONAL WAVE BY LIGO

*Make measurement in the local Earth frame.*

Suppose that the gravitational wave that satisfies metric (1) passes over the LIGO detector oriented as in Figure 5. We know how the test masses at the two ends of the legs of the detector respond to the gravitational wave: they remain at rest in map coordinates (Section 16.5). We know how light propagates along both legs: as the gravitational wave passes through, the map speed of light varies slightly from the value one, as given by equations (9) through (11) in Section 16.4.

Earth frame tied to LIGO slab

The trouble with map coordinates is that they are arbitrary and typically do not correspond to what an observer measures. Recall that we require all measurements to take place in a local inertial frame. So think of a local inertial frame anchored to the concrete slab on which LIGO rests. (Section 16.1 insisted that the gravitational wave has essentially no effect on this slab.) Call the coordinates in the resulting local coordinate system **Earth coordinates**. Earth coordinates are analogous to shell coordinates for the Schwarzschild black hole: useful only locally but yielding the numbers that predict results of measurements. The metric for the local inertial frame then has the form:

$$\Delta\tau^2 \approx \Delta t_{\rm Earth}^2 - \Delta x_{\rm Earth}^2 - \Delta y_{\rm Earth}^2 - \Delta z_{\rm Earth}^2 \qquad (16)$$

Compare this with the approximate version of (1):

$$\Delta\tau^2 \approx \Delta t^2 - (1+h)\Delta x^2 - (1-h)\Delta y^2 - \Delta z^2 \qquad (h \ll 1) \qquad (17)$$

Earth frame coordinate differences

Legalistically, in order to make the coefficients in (17) constants we should use the symbol $\bar{h}$, with a bar over the $h$, to indicate the average value of the gravitational wave amplitude over the detector. However, in Query 1 you showed that for the frequencies at which LIGO is sensitive, the wavelength is very much greater than the dimensions of the detector, so the amplitude $h$ of the gravitational wave is effectively uniform across the LIGO detector. Therefore it is not necessary to take an average, and we use the symbol $h$ without a superscript bar.

Compare (16) with (17) to yield:

$$\Delta t_{\rm Earth} = \Delta t \qquad (18)$$

$$\Delta x_{\rm Earth} = (1+h)^{1/2}\Delta x \approx (1+\frac{h}{2})\Delta x \qquad h \ll 1 \qquad (19)$$

$$\Delta y_{\rm Earth} = (1-h)^{1/2}\Delta y \approx (1-\frac{h}{2})\Delta y \qquad h \ll 1 \qquad (20)$$

$$\Delta z_{\rm Earth} = \Delta z \qquad (21)$$

where we use approximation (8). Notice, first, that the lapse $\Delta t_{\rm Earth}$ between two events is identical to their lapse $\Delta t$ and the $z$ component of their separation in Earth coordinates, $\Delta z_{\rm Earth}$, is identical to the $z$ component of their separation in map coordinates, $\Delta z$.

Test masses move in Earth coordinates.

Now for the differences! Let $\Delta x$ be the map $x$-coordinate separation between the pair of mirrors in the $x$-leg of the LIGO interferometer and $\Delta y$ be the map separation between the corresponding pair of mirrors in the $y$-leg. As the $z$-directed wave passes through the LIGO detector, the test masses at rest at the ends of the legs stay at rest in map coordinates, as Section 16.5 showed. Therefore the value of $\Delta x$ remains the same during this passage, as does the value of $\Delta y$. But the presence of varying $h(t)$ in (19) and (20) tell us that these test masses move when observed in Earth coordinates. *More:* When $\Delta x_{\text{Earth}}$ between test masses increases (say) along the Earth $x$-axis, it decreases along the perpendicular $\Delta y_{\text{Earth}}$; and vice versa. Perfect for detection of a gravitational wave by an interferometer!

Light speed $= 1$ in local Earth frame.

Earth metric (16) is that of an inertial frame in which the speed of light has the value one in whatever direction it moves. With light we have the opposite weirdness to that of the motion of test masses initially at rest: In map coordinates light moves at map speeds different from unity in the presence of this gravitational wave—equations (9) through (11)—but in Earth coordinates light moves with speed one. This is reminiscent of the corresponding case near a Schwarzschild black hole: In Schwarzschild map coordinates light moves at speeds different from unity, but in local inertial shell coordinates light moves at speed one.

Different Earth times along different legs

*In summary* the situation is this: As the gravitational wave passes over the LIGO detector, the speed of light propagating down the two legs of the detector has the usual value one as measured by the Earth observer. However, for the Earth observer the separations between the test masses along the $x$-leg and the $y$-leg change: one increases while the other decreases, as given by equations (19) and (20). The result is a $t$-difference in the round-trip of light along the two legs. It is this difference that LIGO is designed to measure and thereby to detect the gravitational wave.

What will be the value of this difference in round-trip $t$ between light propagation along the two legs? Let $D$ be the Earth-measured length of each leg in the absence of the gravitational wave. The round-trip $t$ is twice this length divided by the speed of light, which has the value one in Earth coordinates. Equations (19) and (20) tell us that the difference in round-trip $t$ between light propagated along the two legs is

$$\Delta t_{\text{Earth}} = 2D\left(\frac{h}{2}+\frac{h}{2}\right) = 2Dh \qquad \text{(one round trip of light)} \qquad (22)$$

Time difference after $N$ round trips.

Using the latest interferometer techniques, LIGO reflects the light back and forth down each leg approximately $N = 300$ times. That is, light executes approximately 300 round trips, which multiplies the detected delay, increasing the sensitivity of the detector by the same factor. Equation (22) becomes

$$\Delta t_{\text{Earth}} = 2NDh \qquad (N \text{ round trips of light}) \qquad (23)$$

Quantities $N$ and $h$ have no units, so the unit of $\Delta t_{\text{Earth}}$ in (23) is the same as the unit of $D$, for example meters.

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**QUERY 5. LIGO fast enough?**
Do the 300 round trips of light take place much faster than one period of the gravitational wave being detected? (If it does not, then LIGO detection is not fast enough to track the *change* in $h$.)

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**QUERY 6. Application to LIGO.**
Each leg of the LIGO interferometer is of length $D = 4$ kilometers. Assume that the laser emits light of wavelength 1064 nanometer, $\approx 10^{-6}$ meter (infrared light from a NdYAG laser). Suppose that we want LIGO to reach a sensitivity of $h = 10^{-23}$. For $N = 300$, find the corresponding value of $\Delta t_{\text{Earth}}$. Express your answer as a decimal fraction of the period $T$ of the laser light used in the experiment.

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**QUERY 7. Faster derivation?**
In this book we insist that global map coordinates are arbitrary human choices and do not treat map coordinate differences as measurable quantities. However, the value of $h$ in (1) is so small that the metric differs only slightly from an inertial metric. This once, therefore, we treat map coordinates as directly measurable and ask you to redo the derivation of equations (22) and (23) using only map coordinates.

Remember that test masses initially at rest in map coordinates do not change their coordinates as the gravitational wave passes over them (Section 16.4), but the gravitational wave alters the map speeds of light, differently in the $x$-direction, equation (9), and in the $y$-direction, equation (10). Assume that each leg of the interferometer has the length $D_{\text{map}}$ in map coordinates.

A. Find an expression for the difference $\Delta t$ between the two legs for one round trip of the light.

B, How great do you expect the difference to be between $\Delta t$ and $\Delta t_{\text{Earth}}$ and the difference between $D$ (in Earth coordinates) and $D_{\text{map}}$? Taken together, will these differences be great enough so that the result of your prediction and that of equation (23) can be distinguished experimentally?

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**QUERY 8. Different directions of propagation of the gravitational wave**
Thus far we have assumed that the gravitational plane wave of the polarization described by equation (1) descends vertically onto the LIGO detector, as shown in Figure 5. Of course the observers cannot prearrange in what direction an incident gravitational wave will move. Suppose that the wave propagates along the direction of, say, the $y$-leg of the interferometer, while the $x$-direction lies along the other leg, as before. What is the equation that replaces (23) in this case?

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**QUERY 9. LIGO fails to detect a gravitational wave?**

Think of various directions of propagation of the gravitational wave pictured in Figure 3, together with different directions of $x$ and $y$ in equation (1) with respect to the LIGO detector. Give the name **orientation** to a given set of directions $x$ and $y$—the transverse directions in (1)—plus $z$ (the direction of propagation) in (1) relative to the LIGO detector. How many orientations are there for which LIGO will detect *no signal whatever*, even when its sensitivity is 10 times better than that needed to detect the wave arriving in the orientation shown in Figure 5? Are there zero such orientations? one? two? three? some other number less than 10? an unlimited number?

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## 16.7 ■ BINARY SYSTEM AS A SOURCE OF GRAVITATIONAL WAVES

*"Newtonian" source of gravitational waves*

Unequal masses, each in circular orbit

The gravitational wave detected on 15 September 2015 came from the merging of two black holes; assume that each is initially in a circular orbit around their center of mass. The binary system is the only known example for which we can explicitly calculate the emitted gravitational waves. Let the $M_1$ and $M_2$ represent the masses of these two black holes that initially orbit at a value $r$ apart, as shown in Figure 7.

Energy of the system.

The basic parameters of the orbit are adequately computed using Newtonian mechanics, according to which the energy of the system in conventional units is given by the expression:

$$E_{\text{conv}} = -\frac{GM_{1,\text{kg}}M_{2,\text{kg}}}{2r} \qquad \text{(Newtonian circular orbits)} \qquad (24)$$

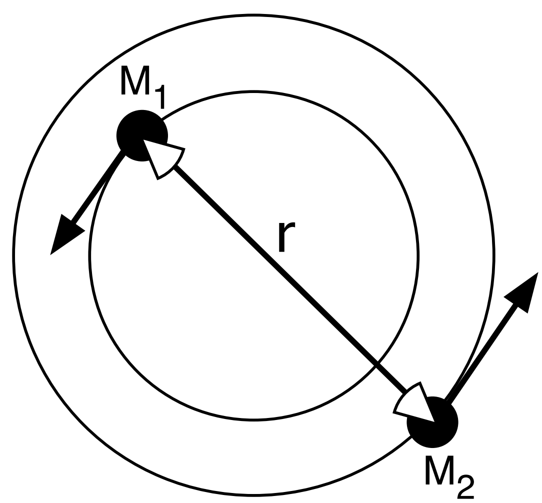


**FIGURE 7** A binary system with each object in a circular path.

Rate of energy loss . . .

As these black holes orbit, they generate gravitational waves. General relativity predicts the rate at which the orbital energy is lost to this radiation. In conventional units, this rate is:

$$\frac{dE_{\text{conv}}}{dt_{\text{conv}}} = -\frac{32G^4}{5c^5r^5}\left(M_{1,\text{kg}}M_{2,\text{kg}}\right)^2\left(M_{1,\text{kg}} + M_{2,\text{kg}}\right) \qquad \text{(Newtonian circular orbits)} \tag{25}$$

. . . derived from Einstein's equations.

Equation (25) assumes that the two orbiting black holes are separated by much more than the $r$-values of their event horizons and that they move at nonrelativistic speeds. Deriving equation (25) involves a lengthy and difficult calculation starting from Einstein's field equations. The same is true for the derivation of the metric (1) for a gravitational wave. These are two of only three equations in this chapter that we simply quote from a more advanced treatment.

---

**QUERY 10. Energy and rate of energy loss**

Convert Newton's equations (24) and (25) to units of meters to be consistent with our notation and to get rid of the constants $G$ and $c$. Use the sloppy professional shortcut, "Let $G = c = 1$."

A. Show that (24) and (25) become:

$$E = -\frac{M_1M_2}{2r} \qquad \text{(Newton: units of meters)} \tag{26}$$

$$\frac{dE}{dt} = -\frac{32}{5r^5}\left(M_1M_2\right)^2\left(M_1 + M_2\right) \qquad \text{(Newton: units of meters)} \tag{27}$$

B. Verify that in both of these equations $E$ has the unit of length.

C. Suppose you are given the value of $E$ in meters. Show how you would convert this value first to kilograms and then to joules.

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**QUERY 11. Rate of change of radius**

Derive a Newtonian expression for the rate at which the radius changes as a result of this energy loss. Show that the result is:

$$\frac{dr}{dt} = -\frac{64}{5r^3}M_1M_2\left(M_1 + M_2\right) \qquad \text{(Newton: circular orbits)} \tag{28}$$

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## 16.8 ■ GRAVITATIONAL WAVE AT EARTH DUE TO DISTANT BINARY SYSTEM

*How far away from a binary system can we detect its emitted gravitational waves?*

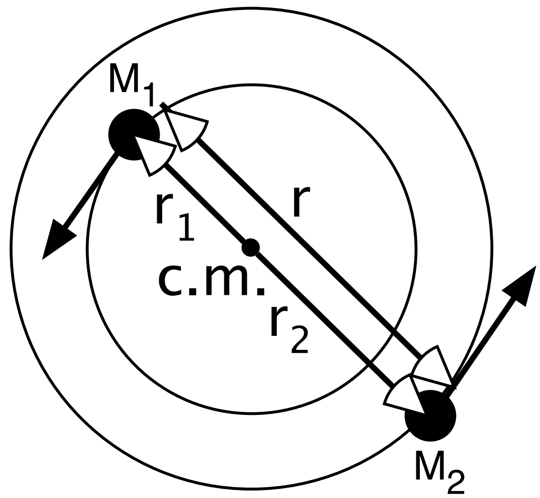


**FIGURE 8** Figure 7 augmented to show the center of mass (c.m.) and orbital $r$-values of individual masses in the binary system.

LIGO on Earth's surface detects the gravitational waves emitted by the distant binary system of two black holes of Figure 7, augmented in Figure 8 to show the center of mass and individual $r_1$ and $r_2$ of the two black holes.

Gravitational waveform

What is the amplitude of gravitational waves from this source measured on Earth? Here is the third and final result of general relativity quoted without proof in this chapter. The function $h(z,t)$ is given by the equation (in conventional units)

$$h(z,t) = -\frac{4G^2 M_1 M_2}{c^4 r z} \cos\left[\frac{2\pi f(z-ct)}{c}\right] \qquad \text{(conventional units)} \qquad (29)$$

where $r$ is the separation of orbiters in Figures 7 through 9. Here $z$ is the separation between source to detector, and—surprisingly—$f$ is twice the frequency of the binary orbit (see Query 15). Convert (29) to units of meters by setting $G = c = 1$. Note that $h(z,t)$ is a function of $z$ and $t$.

Figure 9 schematically displays the notation of equation (29), along with relative orientations and relative magnitudes assumed in the equation. This equation makes the Newtonian assumptions that

(a) the $r$ separation between two the circulating black holes is much larger than either Schwarzschild $r$-value, and

(b) they move at nonrelativistic speeds.

Additional assumptions are:

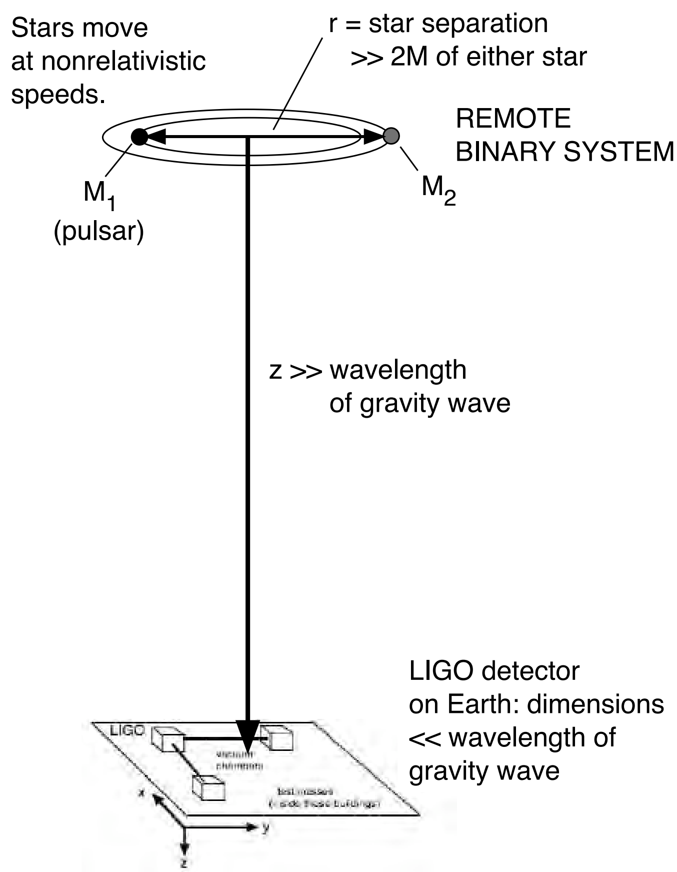


**FIGURE 9** Schematic diagram, *not to scale*, showing notation and relative magnitudes for equation (29). The binary system and the LIGO detector lie in parallel planes.[Illustrator: See note in caption to Figure 5.]

(c) Separation $z$ between the binary system and Earth is very much greater than a wavelength of the gravitational wave. This assumption assures that the radiation at Earth constitutes the so-called “far radiation field” where it assumes the form of a plane wave given in equation (5).

(d) The wavelength of the gravitational wave is much longer than the dimensions of the LIGO detector.

(e) The binary stars are orbiting in the $xy$ plane, so that from Earth the orbits would appear as circles if we could see them (which we cannot).

. . . for one case

Equation (29) describes only one linear polarization at Earth, the one generated by metric (1) and shown in Figure 3. The orthogonal polarization

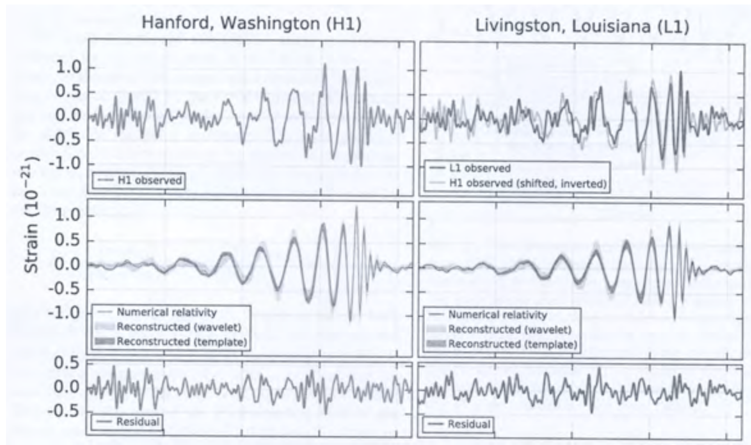


**FIGURE 10** Detected "chirps" of the gravitational wave at two locations. The top row shows detected waveforms (superposed in the right-hand panel). The second row shows the cleaned-up image (again superposed). The bottom row displays "residuals," the noise deducted from images in the first row.

shown in Figure 4 is also transverse and equally strong, with components proportional to $(1 \pm h)$. The formula for the magnitude of $h$ in that orthogonally polarized wave is identical to (29) with a sine function replacing the cosine function. We have not displayed the metric for that orthogonal polarization.

Detection requirements

In order for LIGO to detect a gravitational wave, two conditions must be met: (a) the amplitude $h$ of the gravitational wave must be sufficiently large, and (b) the frequency of the wave must be in the range in which LIGO is most sensitive (100 to 400 hertz). Query 14 deals with the amplitude of the wave. The frequency of gravitational waves, discussed in Query 15, contains a surprise.

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**QUERY 12. Amplitude of gravitational wave at Earth**

A. Use (29) to calculate the maximum amplitude of $h$ at Earth due to the radiation from our "idealized circular-orbit" binary system.

B. Can LIGO detect the gravitational waves whose amplitude is given in part A?

C. What is the maximum amplitude of $h$ at Earth just before coalescence, when the orbiting black holes are separated by $r = 20$ kilometers (but with orbits still described approximately by Newtonian mechanics)?

---

**QUERY 13. Frequency of emitted gravitational waves**

A. In order LIGO to detect the gravitational waves whose amplitude is given in Query 14, the frequency of the gravitational wave must be in the range 100 to 400 hertz. In Figure 9 the point

C. M. is the stationary center of mass of the pulsar system. Using the symbols in this figure, fill in the steps to complete the following derivation.

$$\frac{v_1^2}{r_1} = \frac{GM_1}{r_1^2} \qquad \text{(for } M_1\text{, Newton, conventional units)} \tag{30}$$

$$\frac{v_2^2}{r_1} = \frac{GM_2}{r_2^2} \qquad \text{(for } M_2\text{, Newton, conventional units)} \tag{31}$$

$$M_1 r_1 = M_2 r_2 \qquad \text{(center-of-mass condition)} \tag{32}$$

$$f_{\text{orbit}} \equiv \frac{1}{T_{\text{orbit}}} = \frac{v_1}{2\pi r_1} = \frac{v_2}{2\pi r_2} \qquad \text{(common orbital frequency)} \tag{33}$$

where $f_{\text{orbit}}$ and $T_{\text{orbit}}$ are the frequency and period of the orbit, respectively. From these equations, show that for $r \equiv r_1 + r_2$ the frequency of the orbit is

$$f_{\text{orbit}} = \frac{1}{2\pi}\left[\frac{G\,(M_1+M_2)}{r^3}\right]^{1/2} \qquad \text{(conventional units)} \tag{34}$$

$$= \frac{1}{2\pi}\left[\frac{M_1+M_2}{r^3}\right]^{1/2} \qquad \text{(metric units)} \tag{35}$$

B. Next is a surprise: The frequency $f$ of the gravitational wave generated by this binary pair and appearing in (29) is twice the orbital frequency.

$$f_{\text{gravity wave}} = 2 f_{\text{orbit}} \tag{36}$$

Why this doubling? Essentially it is because gravitational waves are waves of tides. Just as there are two high tides and two low tides per day caused by the moon's gravity acting on the Earth, there are two peaks and two troughs of gravitational waves generated per binary orbit.

C. Approximate the average of the component masses in (34) by the value $M = 30M_{\text{Sun}}$. Find the $r$-value between the binary stars when the orbital frequency is 75 hertz, so that the frequency of the gravitational wave is 150 hertz.

D. Use results quoted earlier in this chapter to find an approximate expression for the time for the binary system to decay from the current radial separation to the radial separation calculated in part C.
ANS: $t_2 - t_1 = 5(r_2^4 - r_1^4)/(256M^3)$, every symbol in unit meter.

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“Chirp” at coalescence

Newtonian mechanics predicts the motion of the binary system surprisingly accurately until the two components touch, a few milliseconds before they coalesce. Newton tells us that as the separation $r$ between the orbiting masses decreases, their orbiting frequency increases. As a result the gravitational wave sweeps upward in both frequency and amplitude in what is called a **chirp**. Figure 1 is the predicted wave form for such a chirp.

## 16.9 ■ RESULTS FROM GRAVITATIONAL WAVE DETECTION; FUTURE PLANS

*Unexpected details*

Investigators milked a surprising amount of information from the first detection of gravitational waves. For example:

1. The initial binary system consisted of two black holes of mass $M_1 = (36 + 5/-4)M_{\text{Sun}}$ (that is, uncertainty of $+5M_{\text{Sun}}$ and $-4M_{\text{Sun}}$) and $M_2 = (29 \pm 4)M_{\text{Sun}}$.
2. The mass of the final black hole was $(62 \pm 4)M_{\text{Sun}}$.
3. Items 1 and 2 mean that the total energy of emitted gravitational radiation was about $3M_{\text{Sun}}$. A cataclysmic event indeed!
4. The two detection locations are separated by 10 milliseconds of light-travel time, or 3000 kilometers.
5. The signals were separated by $6.9 + 0.5/-0.4$ milliseconds, which means that they did not come from overhead.

How did observations lead to these results?

Item 1 derives from two equations in two unknowns (27) and (34), with validation in the small separation $r$-value at which merging takes place.
Item 2 follows from the frequency of ringing in the merged black hole.
Item 3 follows from Item 2.
Item 4 results from standard surveying.
Item 5 follows from direct comparison of synchronized clocks.

What are plans for future gravitational wave detections?

A. Increased sensitivity of each LIGO system
B. Increased number of LIGO detectors across the Earth, to measure the source direction more accurately.
C. Installation of LISA (Laser Interferometer Space Antenna Project) in space, which removes seismic noise at low frequencies in Figure 2).

## 16.10 ■ REFERENCES

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Correcting an error in Einstein's first paper: Albert Einstein, "Näherungsweise Integration der Feldgleichungen der Gravitation", Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften, Berlin (1916), 688–696.

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