Chapter 17. Spinning Black Hole

- *What's the difference between a spinning and a non-spinning black hole?*
- *How does one spinning black hole differ from another spinning black hole?*
- *How fast can a black hole spin?*
- *Does the spin of a black hole keep me from falling to the singularity?*
- *If I can fall to the singularity, will that fall take longer than my lifetime?*
- *What local inertial frames are useful near a spinning black hole?*

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CHAPTER

17 Spinning Black Hole

Edmund Bertschinger & Edwin F. Taylor *

Black holes are macroscopic [large-scale] objects with masses varying from a few solar masses to billions of solar masses. When stationary and isolated, they are all, every single one of them, described exactly by the Doran solution. This is the only instance we have of an exact description of a macroscopic object. The only elements in the construction of black holes are our basic concepts of space and time. They are thus the most perfect macroscopic objects in the universe. They are the simplest objects as well.

—Subrahmanyan ("Chandra") Chandrasekhar [edited]

17.1 ■ THE AMAZING SPINNING BLACK HOLE

Add spin, multiply consequences

This and the following chapters describe the spinning black hole, which displays spectacular effects that outstrip most science fiction:

Spectacular physical effects

Some Physical Effects Near the Spinning Black Hole

1. There is a region outside the event horizon in which no rocket—no matter how powerful—can keep a spaceship stationary in our chosen global coordinates.
2. There is a region inside the event horizon in which a spaceship does *not* inevitably move toward the center, but can be repelled away from it (Chapter 18).
3. Stable orbits that do not cross the event horizon reach smaller r than do stable orbits for a non-spinning black hole. This result leads to dramatic general relativistic effects on the so-called **accretion disk** that circles around the spinning black hole (Chapter 18).

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4. Unstable circular orbits exist in a region inside the event horizon and close to the singularity of the spinning black hole (Chapter 18).
5. Visual effects for the traveler near a spinning black hole are even wilder than those near the non-spinning black hole (Chapter 20).
6. The spinning black hole is an immense energy source, waiting to be tapped by an advanced civilization (Chapter 19).
7. The singularity of a spinning black hole is a ring through which a spaceship might pass undamaged (Chapter 21).
8. The spinning black hole may provide a gateway to other Universes (Chapter 21).

The present chapter sets the stage to describe these physical effects.

Why every black hole spins.

We expect every black hole to spin. Why? Because a group of stars or cloud of dust almost inevitably has *some* net spin angular momentum. When this system collapses to form a black hole, the spin rate increases in the same way that a spinning ice skater with arms extended rotates faster as she draws her arms inward. The skinnier the skater, the faster her final spin for a given initial angular momentum. The spinning black hole is the "skinniest possible astronomical skater." For this reason we expect (and have observational evidence) that black holes spin at a ferocious rate.

Apply the same toolkit to analyze the spinning black hole.

Comment 1. Have we wasted our time?
Since in Nature black holes spin, have we wasted our time studying the non-spinning black hole in the previous chapters of this book? Not at all! First, for most purposes the metric for the non-spinning black hole describes spacetime outside slowly rotating stars and planets such as Earth well enough so that we can use this metric to make predictions that are verified by observation. Second, we can generalize many of our non-spinning black hole tools to analyze the astonishing structure of the spinning black hole. Third, our analysis of the spinning black hole follows the same sequence as our analysis of the non-spinning black hole. Fourth, we can use our non-spinning black hole results as a limiting case to check predictions for the spinning black hole. Fifth—and most important—by now we have extensive experience using the power of the global metric plus the Principle of Maximal Aging to predict results of measurements and observations carried out near the spinning black hole.

Just two numbers: mass and spin

Spin parameter a

An isolated, uncharged spinning black hole is completely specified by just two quantities: its mass and its spin angular momentum. To avoid confusion between the rotational angular momentum of the spinning black hole (with mass M) and the orbital angular momentum of a stone (with mass m) around the black hole, we use the symbol J for the angular momentum of the spinning black hole and write J/M for this angular momentum per unit mass. The ratio J/M appears so often in the analysis that we define the lower-case italic a, called the **spin parameter**, which also has the unit of meters:

$$a \equiv \frac{J}{M} \qquad \text{(black hole spin parameter, unit of meters)} \qquad (1)$$

Note that the black hole spin parameter a has nothing to do with $a(t)$, the scale factor of the Universe defined in Section 15.2. We have run out of letters!

Think of an isolated star that collapses into a black hole while keeping its angular momentum constant. Its rotation rate will increase enormously. Look at the spinning black hole from either one side or the other. There is always a side for which the spin will be counterclockwise. We *choose* both J and a to be positive quantities for that counterclockwise spin direction. Now, the smallest value of J and a is zero. What is the largest possible value of each? In Query 5 you show that the ranges fit the following inequalities:

$$0 \leq J \leq M^2 \qquad \text{(range of spin angular momentum } J\text{, units of meters}^2\text{)} \qquad (2)$$

$$0 \leq a \leq M \qquad \text{(range of spin parameter } a\text{, units of meters)} \qquad (3)$$

17.2 ■ THE DORAN GLOBAL METRIC

Eighty-five years after Einstein's equations!

Karl Schwarzschild derived his global metric for the non-spinning black hole less than a month after Einstein published his field equations. In contrast, not until 1963—forty-eight years later—did Roy P. Kerr publish a paper with a title that begins, "Gravitational Field of a Spinning Mass . . .". Brandon Carter and others showed that Kerr's metric describes not just a spinning mass but a spinning black hole. Only in the year 2000—eighty-five years after Einstein derived his equations—did Chris Doran express Kerr's results in the global metric that we use to analyze the spinning black hole. As usual, we restrict global coordinates and their metric to a slice through the center of the black hole. The non-spinning black hole is spherically symmetric, so this slice through the center can have any orientation. For the spinning black hole, however, we choose the slice in the symmetry plane of the equator, perpendicular to the axis of rotation. In one of many tetrad forms—the sum and difference of squares (Section 7.6)—the **Doran metric** reads:

Doran global metric

$$d\tau^2 = dT^2 - \left[\left(\frac{r^2}{r^2+a^2}\right)^{1/2} dr + \left(\frac{2M}{r}\right)^{1/2}(dT - ad\Phi)\right]^2 - \left(r^2+a^2\right)d\Phi^2 \qquad (4)$$

$$-\infty < T < \infty, \quad 0 < r < \infty, \quad 0 \leq \Phi < 2\pi \qquad \text{(Doran, equatorial plane)}$$

In Query 1 you multiply out (4) to obtain the Doran metric in expanded form:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dT^2 - 2\left(\frac{2Mr}{r^2+a^2}\right)^{1/2} dTdr + 2a\left(\frac{2M}{r}\right) dTd\Phi$$
$$+ 2a\left(\frac{2Mr}{r^2+a^2}\right)^{1/2} drd\Phi - \left(\frac{r^2}{r^2+a^2}\right) dr^2 - R^2 d\Phi^2 \qquad (5)$$
$$-\infty < T < \infty, \quad 0 < r < \infty, \quad 0 \le \Phi < 2\pi \qquad \text{(Doran, equatorial plane)}$$

The expanded Doran metric (5) contains every possible cross term—sorry! It also contains a new expression R, a function of both r and a that we call the **reduced circumference**:

Define R

$$R^2 \equiv r^2 + a^2 + \frac{2Ma^2}{r} \qquad (R = \text{reduced circumference}) \qquad (6)$$

QUERY 1. Doran metric reduces to global rain metric for non-spinning black hole.

A. Let $a \to 0$ in the expanded Doran metric (5) for the spinning black hole and compare the result with the global rain metric for the non-spinning black hole, equation (32) in Section 7.5.

B. Now *demand* that the two global metrics of Item A be identical. Show that the result is that $d\Phi \to d\phi$ when $a \to 0$.

Figure 1 plots the reduced circumference R as a function of r for sample values of the spin parameter a. As $r \to \infty$ all curves converge asymptotically toward the curve for $a = 0$, the non-spinning black hole. Why do we call R the reduced circumference? Let $dr = dT = 0$. Then global metric (5) reduces to

$$d\tau^2 = -d\sigma^2 = -R^2 d\Phi^2 \qquad \text{(Doran: } dr = dT = 0) \qquad (7)$$

or $\sigma = 2\pi R$ for a complete circle at fixed r around the spinning black hole. This justifies calling R the *reduced circumference.*

?

Objection 1. *Why not use (6) to eliminate r from metrics (4) and (5) and use R exclusively?*

!

Because R violates the rule that global coordinates must label each event uniquely (Section 5.8). Figure 1 shows that for every value of R greater than its minimum there correspond two different values of r.

?

Objection 2. *Why in the world are there two values of r for each value of the reduced circumference? Geometry does not allow this!*

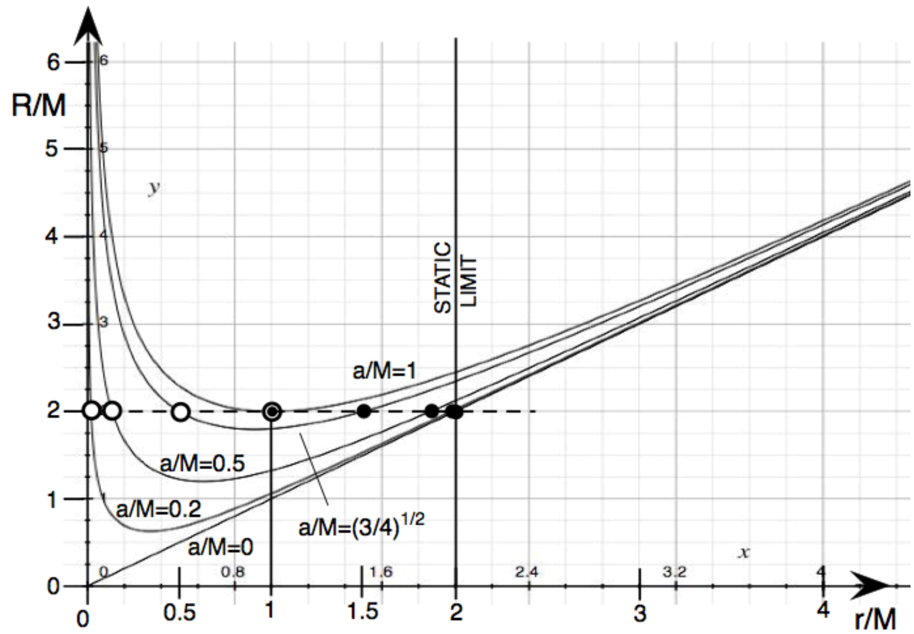

FIGURE 1 Plot of reduced circumference R vs. r for several values of the spin parameter a. Location of the static limit $r_S/M = 2$, equation (9), does not depend on spin. Section 17.3 and Figure 2 describe the significance of little filled and open circles along the dashed horizontal line $R/M = 2$.

Ah! You mean that *Euclidean geometry* does not allow this. Inside the static limit, especially, spacetime is radically distorted; Euclidean flat-space geometry simply does not apply there.

QUERY 2. Limiting cases of the Doran metric

A. Show that as $r \to \infty$ the Doran metric (4) becomes the metric for flat spacetime.

B. Write down the Doran metric (5) for the maximum-spin black hole ($a/M = 1$) and the expression for R_{max} in this case.

Comment 2. You do the math (if you wish).
At this point in the book some derivations become so algebraically complicated that we omit them, while leaving a skimpy trail to guide you if you choose to carry out these derivations yourself. Instead, we focus on results and predictions: What locations near the spinning black hole can we explore and still return home unharmed? What do we see and feel on the way? Which predictions can we verify now, and which must we leave to our descendants? Dive into the complications; enjoy the payoffs!

17.3 ■ A STONE'S THROW

Where you can go; how you can move

Now apply the Doran metric to two adjacent events that lie along the worldline of a stone. What commands does spacetime give to the stone through the metric? We examine two cases.

THE STONE AT REST IN DORAN COORDINATES

Where can the stone stand still in Doran coordinates?

The simplest possible motion of a stone is no motion at all: to stand still in global space coordinates. Where can the stone stand still? Expressed more carefully, can two adjacent events along the stone's worldline have $dr = d\Phi = 0$? To find out, put these conditions into the Doran metric:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dT^2 \qquad (dr = d\Phi = 0) \tag{8}$$

Wristwatch time must be real along the worldline of a stone, so both sides of (8) must be positive. This tells us that the stone cannot remain at rest in Doran global coordinates when $r < 2M$. Does this place the event horizon of the spinning black hole at $r = 2M$? No. In what follows we discover that, for the spinning black hole, the event horizon lies inside $r = 2M$. For the minute, simply ask what equation (8) does say: Inside $r = 2M$ the stone *must* move in either r or Φ or both; the stone cannot remain static in Doran coordinates. Therefore we give this value of r the label **static limit** with the subscript S. Equation (8) shows that the static limit has the same value $r_S = 2M$ for all values of the spin parameter a:

Static Limit at $r_S = 2M$

$$r_S = 2M \qquad (r\text{-coordinate of } \textbf{static limit} \text{ for all } a) \tag{9}$$

THE STONE WITH $dr = 0$ IN DORAN COORDINATES

Now loosen restrictions on the stone. Where can the stone remain at fixed r-value but move in Φ? To find out, set $dr = 0$ in the global metric (5) for two adjacent events along the stone's worldline:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dT^2 + 2\left(\frac{2Ma}{r}\right) dT d\Phi - R^2 d\Phi^2 \qquad (dr = 0) \tag{10}$$

We want a global metric in tetrad form—with no cross-term. Rewrite equation (10) as the sum and difference of squares on the right side. There are only two global coordinates in (10), so construct a linear combination of the form $dX = d\Phi - \omega dT$ and choose the function ω to eliminate the cross term in the metric. Substitute $d\Phi = dX + \omega dT$ into (10) and rearrange the result to obtain:

$$d\tau^2 = \left(1 - \frac{2M}{r} + \frac{4Ma\omega}{r} - \omega^2 R^2\right) + 2\left(\frac{2Ma}{r} - \omega R^2\right) dX dT - R^2 dX^2 \tag{11}$$

To eliminate the cross term, choose the function $\omega(r)$ to be

$$\omega(r) \equiv \frac{2Ma}{rR^2} \qquad \text{omega function} \qquad (12)$$

With this choice of $\omega(r)$, the global metric for constant-r motion takes the tetrad form:

$$d\tau^2 = \left[1 - \frac{2M}{r} + \frac{4M^2a^2}{r^2R^2}\right] dT^2 - R^2\left[d\Phi - \omega dT\right]^2 \qquad (dr = 0) \qquad (13)$$

Simplify the coefficient of dT^2 as follows:

$$\begin{aligned} 1 - \frac{2M}{r} + \frac{4M^2a^2}{r^2R^2} &\equiv \frac{\left(1 - \frac{2M}{r}\right)R^2 + \frac{4M^2a^2}{r^2}}{R^2} \\ &= \frac{\left(1 - \frac{2M}{r}\right)\left(r^2 + a^2 + \frac{2Ma^2}{r}\right) + \frac{4M^2a^2}{r^2}}{R^2} \\ &= \frac{r^2 + a^2 - 2Mr - \cancel{\frac{2Ma^2}{r}} + \cancel{\frac{2Ma^2}{r}} - \cancel{\frac{4M^2a^2}{r^2}} + \cancel{\frac{4M^2a^2}{r^2}}}{R^2} \\ &= \frac{r^2 - 2Mr + a^2}{R^2} = \left(\frac{rH}{R}\right)^2 \end{aligned} \qquad (14)$$

Define: **Horizon function** H.

where we define the **horizon function** $H(r)$ from the last line of equation (14):

$$H^2(r) \equiv \frac{r^2 - 2Mr + a^2}{r^2} = \frac{(r - r_{\rm EH})(r - r_{\rm CH})}{r^2} \qquad (H \equiv \text{horizon function}) (15)$$

Note that when $a \to 0$, then $H^2(r) \to (1 - 2M/r)$; so we can think of the common expression $(1 - 2M/r)$ for the non-spinning black hole to be a special case of $H^2(r)$.

Comment 3. Horizon function H is different from Hubble parameter.
The horizon function H defined in (15) has nothing to do with the Hubble parameter H defined in Chapter 15. There are only so many letters in any alphabet; in this case we recycle the symbol H.

Use the new horizon function H to give the Doran metric (13) with $dr = 0$ the simple form:

$$d\tau^2 = \left(\frac{rH}{R}\right)^2 dT^2 - R^2\left[d\Phi - \omega(r)dT\right]^2 \qquad (dr = 0) \qquad (16)$$

The roots of the numerator in expression (15) for H^2 introduce two special values of the r-coordinate, which we call the **event horizon** and the **Cauchy horizon**:

$$\frac{r_{\mathrm{EH}}}{M} \equiv 1 + \left(1 - \frac{a^2}{M^2}\right)^{1/2} \qquad \text{(event horizon)} \qquad (17)$$

$$\frac{r_{\mathrm{CH}}}{M} \equiv 1 - \left(1 - \frac{a^2}{M^2}\right)^{1/2} \qquad \text{(Cauchy horizon)} \qquad (18)$$

Comment 4. Augustin-Louis Cauchy
Mathematician Augustin-Louis Cauchy (1789 to 1852) derived results over the entire range of then-current mathematics and mathematical physics. Cauchy did not discover black holes or their horizons, but his work on differential equations is relevant to the properties of horizons.

Meaning of an horizon

How do we justify calling these special r-coordinates *horizons*? What do we mean by an horizon for the black hole? Look closely at the right side of equation (16). The second term is always negative unless $d\Phi = \omega dT$. Let's assume this equality, because it gives us the greatest possible latitude to have a worldline with $d\tau^2 > 0$ and $dr = 0$. The resulting equation tells us immediately that such a worldline is possible if and only if $(rH/R)^2 > 0$ or $H^2 > 0$. If this is not so, that is if $H^2 < 0$, then a stone *must* move in the r-coordinate. Why? Because if it does not move, that is if $dr/d\tau = 0$, then $d\tau^2 < 0$, which is forbidden along the worldline of a stone. (It will also move in the Φ-coordinate, because we just assumed that $d\Phi/dT = \omega$.) See Figure 2.

Question: How to define an event horizon?

How do we find an event horizon? A full definition of an event horizon involves examining the propagation of light, which we describe in Chapter 20. However a simplified (and in this case valid) definition can use the orbits of stones.

Answer: r-surface on one side of which nothing can remain at constant r.

We ask whether a stone can remain at constant r. The event horizon is the boundary where the answer changes from "Yes" to "No". For the *non-spinning* black hole, nothing can remain at constant r between $r = 2M$ and the singularity, so we label $r = 2M$ the event horizon. The *spinning* black hole is more complicated: Nothing can remain at constant r where $H^2 < 0$, which is the case between the upper event horizon and the lower Cauchy horizon. At r values between the Cauchy horizon and the singularity, amazingly, a stone can again remain at constant r-value. How can a free stone do this? One way is to travel in a circular orbit. Chapter 18 describes circular orbits of a stone, including circular orbits at r-values inside the Cauchy horizon and down almost to $r = 0$!

QUERY 3. Verify horizon equations
Solve the quadratic equation $r^2 - 2Mr + a^2 = 0$ from the numerator of equation (15). Show the roots are r_{EH} and r_{CH} in equations (17) and (18).

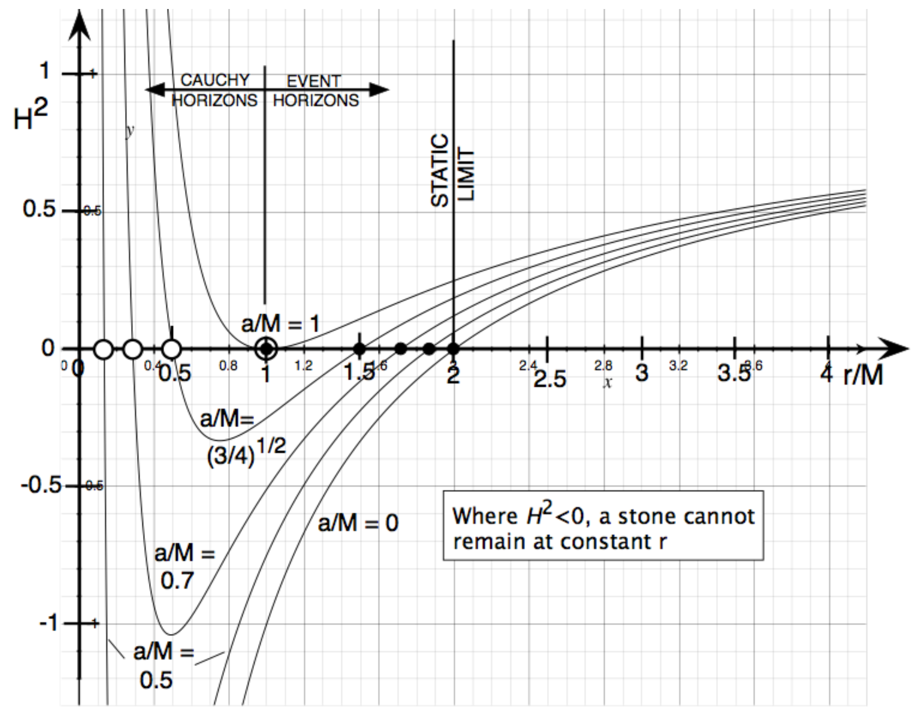

FIGURE 2 Plot of the function H^2 vs. r for selected values of a. Equation (16) says that when $d\Phi/dT = \omega(r)$, adjacent events along a stone's worldline are timelike—and that worldline is possible—only when $H^2 > 0$ in this plot. Little filled circles locate the event horizon for a given value of a, and little open circles locate the corresponding Cauchy horizons. For $a/M = 1$ these two horizons coincide at $r/M = 1$. Review similar symbols in Figure 1.

Sequence of horizons and static limit

Figure 3 plots r-values of event and Cauchy horizons for different spin parameters a. Equations (17) and (18) plus (9) lead to the following inequalities, also displayed in the figure:

$$0 \leq r_{\text{CH}} \leq M \leq r_{\text{EH}} \leq r_{\text{S}} = 2M \qquad (19)$$

QUERY 4. All horizons have reduced circumference $R = 2M$.
Substitute $r/M = 1 \pm (1 - a^2/M^2)^{1/2}$ from (17) and (18) into equation (6) for R^2 and verify that all horizons have reduced circumference $R = 2M$, as shown in Figure 1.

Prepare for local inertial frames

We can use any global metric expressed in tetrad form (Section 7.6) to define a local inertial frame. The next three sections prepare the way for us to

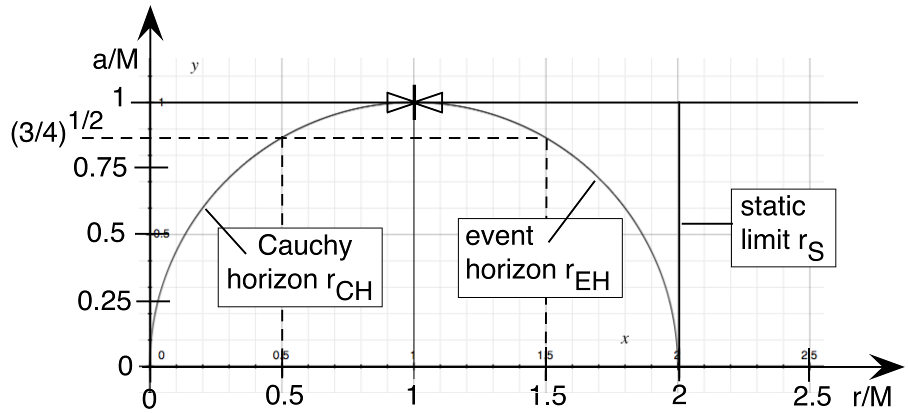

FIGURE 3 The r-values of the Cauchy and event horizons for different values of spin parameter a. Dashed lines are for $a/M = (3/4)^{1/2}$, for which $r_{\rm EH}/M = 1.5$ and $r_{\rm CH}/M = 0.5$. The static limit $r_{\rm S}/M = 2$ is independent of a. As the spin parameter a increases from zero, the event horizon drops from $r_{\rm EH}/M = 2$ to $r_{\rm EH}/M = 1$, while the Cauchy horizon emerges from the singularity and rises to the same final $r_{\rm CH}/M = 1$.

construct three useful local inertial frames from which to make measurements and observations near the spinning black hole.

QUERY 5. Horizons do not exist if $a > M$.

A. Show that if $a > M$, then $H^2(r) > 0$ everywhere.

B. Show that in this case, and for any given r, a stone can remain at that r while having $d\tau^2 > 0$ along its worldline.

C. Show that in this case a stone can move inward and outward from any r, while having $d\tau^2 > 0$.

D. Explain why this means that there is no event horizon.

Your analysis in this Query justifies the upper limit for a in relation (3).

We now describe the motion of a stone in the equatorial plane of the spinning black hole. For this we need global coordinate expressions for the stone's map energy and map angular momentum. Derivations of these expressions are closely similar to earlier derivations of similar quantities in Chapters 6, 8, and 9, so we relegate them to appendices in Sections 17.9 and 17.10. Here are the results:

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dT}{d\tau} - \left(\frac{2Mr}{r^2+a^2}\right)^{1/2}\frac{dr}{d\tau} + \frac{2Ma}{r}\frac{d\Phi}{d\tau} \qquad (20)$$

$$\boxed{\frac{L}{m} = R^2\frac{d\Phi}{d\tau} - \frac{2Ma}{r}\frac{dT}{d\tau} - a\left(\frac{2Mr}{r^2+a^2}\right)^{1/2}\frac{dr}{d\tau}} \qquad (21)$$

QUERY 6. Map energy and map angular momentum for the non-spinning black hole. For $a \to 0$, show that (20) reduces to equation (35) in Section 7.5 for E/m and (21) reduces to equation (10) in Section 8.2 for L/m for a stone near a non-spinning black hole.

17.4 ■ THE RAINDROP

A simple case that gives deep insight

Major equations in this chapter look complicated. In contrast, John Wheeler insisted that "everything important is utterly simple" (Appendix I. Wheeler's Rules). We now examine an important case, the raindrop, and find that its equations of motion are indeed utterly simple.

Definition of the **raindrop**

The raindrop, remember, is a free stone that drops from initial rest starting at very large r. "Initial rest" means that $dr/d\tau \to 0$ and $d\Phi/d\tau \to 0$ as $r \to \infty$. In addition, equation (8) says that $dT \to d\tau$ as $r \to \infty$, and from (20) and (21), the raindrop's map energy and map angular momentum become:

$$\frac{E}{m} = 1 \qquad \text{and} \qquad \frac{L}{m} = 0 \qquad \text{(raindrop)} \qquad (22)$$

Doran: Make raindrop equations simple.

In Query 2 you showed that in the limit $a \to 0$, the Doran metric for the spinning black hole reduces to the global rain metric for the non-spinning black hole. Exercise 2 in Section 7.10 analyzed the raindrop for the non-spinning black hole in global rain coordinates and found that $d\phi/d\tau = 0$ along its worldline. Chris Doran *chose* global coordinates Φ and T so that the raindrop worldline lies along constant Φ—that is $d\Phi/d\tau = 0$ along the raindrop worldline—and the raindrop wristwatch ticks at the same rate that global T passes—that is, $dT/d\tau = 1$ along the raindrop worldline. For the raindrop, then, equations (20), (21), and (22) lead to:

$$\frac{E}{m} = 1 = \left(1 - \frac{2M}{r}\right) - \left(\frac{2Mr}{r^2+a^2}\right)^{1/2}\frac{dr}{d\tau} \qquad \text{(raindrop)} \qquad (23)$$

$$\frac{L}{m} = 0 = -\frac{2Ma}{r} - a\left(\frac{2Mr}{r^2+a^2}\right)^{1/2}\frac{dr}{d\tau} \qquad \text{(raindrop)} \qquad (24)$$

You can solve either one of these equations to find the same expression for $dr/d\tau$:

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2}\left(\frac{r^2+a^2}{r^2}\right)^{1/2} \qquad \text{(raindrop)} \qquad (25)$$

Raindrop equations of motion

With Chris Doran's raindrop-related choice of global coordinates, the equations of motion for the raindrop become:

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2}\left(\frac{r^2+a^2}{r^2}\right)^{1/2} \qquad \text{(raindrop)} \qquad (26)$$

$$\frac{dT}{d\tau} = 1 \qquad \text{(raindrop)} \qquad (27)$$

$$\frac{d\Phi}{d\tau} = 0 \qquad \text{(raindrop)} \qquad (28)$$

Raindrop wristwatch time from r_0 to r

How much time does it take, on the raindrop's wristwatch, to fall from an initial global coordinate r_0 to a lower value r? (*Slogan:* "How many ticks of a raindrop clock if a raindrop could tick tock?") To answer this question, integrate equation (26):

$$\tau[r_0 \to r] = \left(\frac{1}{2M}\right)^{1/2}\int_r^{r_0}\left(\frac{r^{*2}}{r^{*2}+a^2}\right)^{1/2} r^{*1/2}dr^* \qquad \text{(raindrop)} \quad (29)$$

where r^* is a variable of integration. The right side of this equation does not have a closed-form solution, so we integrate it numerically. Figure 4 plots some results and compares these curves with one curve for $a = 0$ in Section 7.5.

QUERY 7. Arrive sooner at the singulariy From a quick examination of equation (29), show that as you ride a raindrop into a spinning black hole,

A. your wristwatch time to fall from a given r to the singularity is less than for a non-spinning black hole, and

B. your wristwatch time to fall from a higher r_0 to a lower r when both are far from the black hole is the same as for a non-spinning black hole.

From (26) through (28), it follows immediately that the "global coordinate displacement" of the raindrop has the components:

$$\frac{dr}{dT} \equiv \frac{dr}{d\tau}\frac{d\tau}{dT} = -\left(\frac{2M}{r}\right)^{1/2}\left(\frac{r^2+a^2}{r^2}\right)^{1/2} \qquad \text{(raindrop)} \qquad (30)$$

$$\frac{d\Phi}{dT} \equiv \frac{d\Phi}{d\tau}\frac{d\tau}{dT} = 0 \qquad \text{(raindrop)} \qquad (31)$$

Comment 5. Goodbye "radial"

Does the raindrop follow a "radial" path down to the singularity of a spinning black hole? No. The word "radial" no longer describes motion near the spinning black hole.

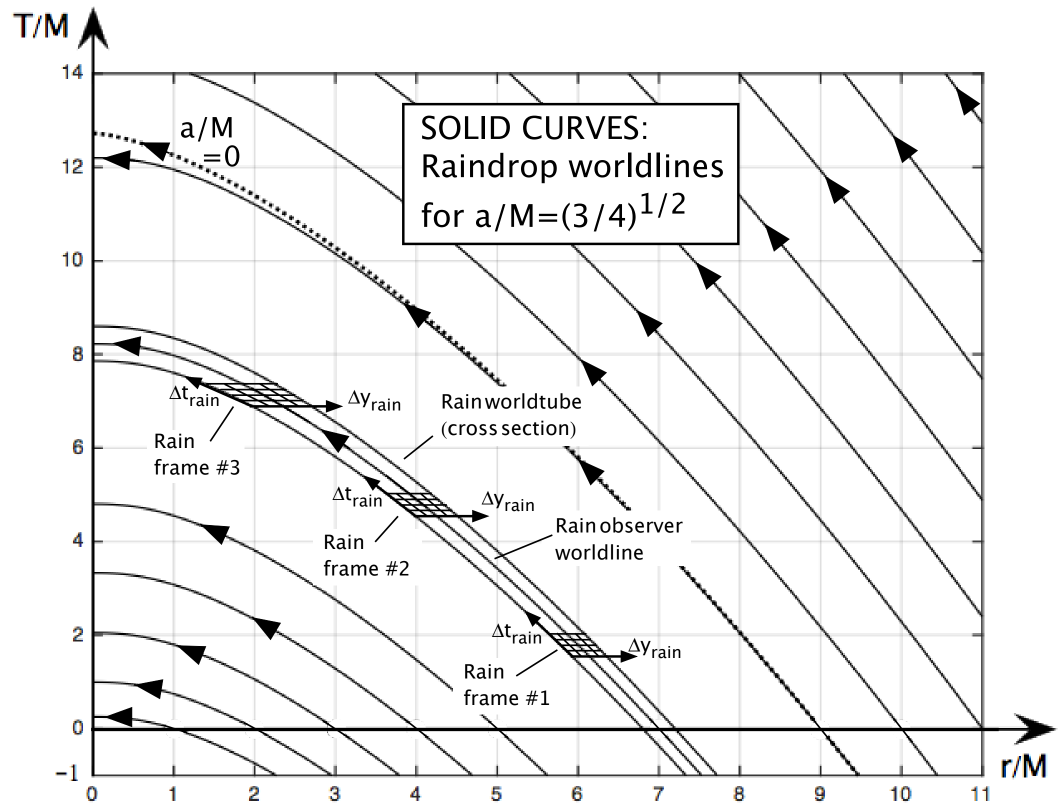

FIGURE 4 Solid curves: raindrop worldlines for a black hole with spin $a/M = (3/4)^{1/2}$, the numerical solution of equation (29), plotted on an $[r, T]$ slice. All these worldlines have the same shape and are simply displaced vertically with respect to one another. Note that these worldlines are continuous through the event and Cauchy horizons at $r_{\mathrm{EH}}/M = 1.5$ and $r_{\mathrm{CH}}/M = 0.5$. Around one of these worldlines we construct, in cross section, a worldtube that bounds local rain frames through which that rain observer passes. For local rain frame coordinates, see Section 17.7. Dotted curve for comparison: raindrop worldline for non-spinning black hole ($a/M = 0$); compare Figure 3, Section 7.5 for $a/M = 0$.

For the non-spinning black hole, we can still hang on to the intuitive term "radial," because the symmetry of that black hole demands that a raindrop—with zero map angular momentum—can veer neither clockwise nor counterclockwise as it descends.

Not so for the spinning black hole, which breaks the clockwise-counterclockwise symmetry. A stone with $dr/dT = d\Phi/dT = 0$ FINISH THIS COMMENT

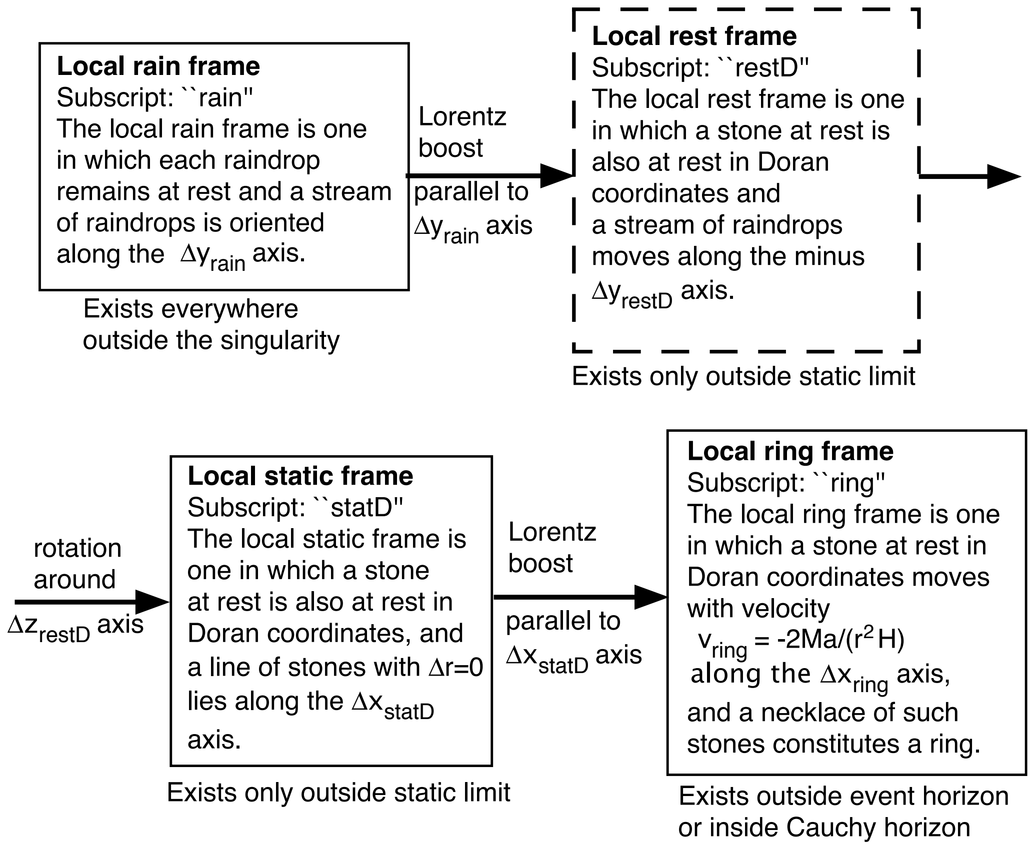

FIGURE 5 Definitions of several local inertial frames from which we choose to make measurements and observations near the spinning black hole. The so-called "local rest frame" (upper right box) serves mainly to connect the local rain frame to the local static frame, hence the dashed lines around the box that describes it.

17.5 ■ THE LOCAL RAIN FRAME

Take relaxed measurements as we fall

Choose local inertial frames for our measurements.

Thus far this chapter has introduced the Doran global metric and a few of its consequences for the motion of a free stone. As usual, our goal is to report measurements and observations made in local inertial frames; we now derive several of these from the Doran metric.

Figure 5 gives summary definitions of the local inertial frames we choose near the spinning black hole: local inertial rain, rest, static, and ring frames, described in this section and the following three sections. You will show that when $a \rightarrow 0$, the local rest, static, and ring frames all become the local shell frame (Section 5.7); and the local rain frame simply becomes the local rain frame for the non-spinning black hole (Section 7.5).

Comment 6. Generalized Lorentz transformation

The Lorentz transformations defined in Section 1.10 were limited to Lorentz boosts along the common $\Delta x_{\rm frame}$ axes of laboratory and rocket frames. In general, Lorentz boosts can take place along any direction in either frame. One way to do this is first to rotate the initial frame, then Lorentz-boost it to the desired final frame. Thus the general definition of **Lorentz transformation** also includes simple rotation of one frame with respect to the other. Look at labels on the arrows in Figure 5. Each of these labels describes a Lorentz transformation.

Initially Figure 5 may seem strange and perplexing; this section and the next three sections describe each of these frames in more detail.

Local rain frame from equation (4)

The right side of Doran metric (4) is in tetrad form—the sum and difference of squares (introduced in Section 7.6). Therefore its approximate form gives us *some* local inertial frame coordinates expressed in Doran global coordinates. Which particular local inertial frame? We will find that it earns the name **local inertial rain frame**; so the coordinates for the local rain frame in terms of Doran coordinates are:

Local rain frame coordinates

$$\Delta t_{\rm rain} \equiv \Delta T \qquad (32)$$

$$\Delta y_{\rm rain} \equiv \left[\left(\frac{\bar{r}^2}{\bar{r}^2+a^2}\right)^{1/2}\Delta r - \left(\frac{2M}{\bar{r}}\right)^{1/2} a\Delta\Phi\right] + \left(\frac{2M}{\bar{r}}\right)^{1/2}\Delta T \qquad (33)$$

$$\Delta x_{\rm rain} \equiv \left(\bar{r}^2+a^2\right)^{1/2}\Delta\Phi \qquad (34)$$

The expression in square brackets in equation (33) appears also in equations for some later local inertial frames. Figure 5 contains a definition of the local rain frame.

Local rain frame: valid everywhere.

Expressions on the right sides of (32) through (34) are all real outside $r = 0$, so the local inertial rain frame exists everywhere outside the singularity. These three equations plus the approximate form of (4) guarantee that the local rain frame metric has the usual form:

$$\Delta\tau^2 \approx \Delta t_{\rm rain}^2 - \Delta y_{\rm rain}^2 - \Delta x_{\rm rain}^2 \qquad (35)$$

Comment 7. The rain tetrad

Equations (32) through (34) express local rain coordinates in Doran coordinates when the global metric is in *tetrad* form. Notice that two of the components, $\Delta t_{\rm rain}$ and $\Delta x_{\rm rain}$, depend on a single global coordinate difference, while $\Delta y_{\rm rain}$ depends on all three: ΔT, Δr, and $\Delta\Phi$. This result, due to black hole spin, generalizes the rain tetrad for a non-spinning black hole, where $\Delta y_{\rm rain}$ depends on two coordinate differences—equation (43) in Section 7.5.

QUERY 8. Compare rain frame coordinates for spinning and non-spinning black holes.

Compare local rain coordinate expressions (32) through (34) with those for the non-spinning black hole in Box 4 of Section 7.5. Under what assumption or assumptions do the spinning black hole expressions reduce to those for the non-spinning black hole when $a \to 0$?

The worldtube projected on the $[r, T]$ slice in Figure 4 embraces rain frames through which the rain observer passes. The time axis of a local inertial frame is always tangent to the worldline of a stone at rest in that frame. The raindrop is at rest in the local rain frame; therefore the Δt_{rain} axis is tangent to the raindrop worldline in Figure 4. What is the direction of the Δy_{rain} axis on the $[r, T]$ slice? The Δy_{rain} axis is a line along which $\Delta t_{\text{rain}} = \Delta x_{\text{rain}} = 0$. With these conditions, equation (33) tells us that the Δy_{rain} axis lies along the global Δr direction, as shown in Figure 4.

?

Objection 3. *Figure 4 is all wrong! Equation (32) clearly says that $\Delta t_{\text{rain}} = \Delta T$, so the Δt_{rain} axis must point along the vertical T/M axis in Figure 4.* More: *Equation (33) says that Δy_{rain} has contributions from all three global coordinates, so cannot point along the horizontal r/M axis in the figure.*

!

You *are* observant! To answer your objection, start with the Δy_{rain} axis: Note, first, that Figure 4 displays an $[r, T]$ slice. On that slice $\Delta\Phi = 0$. Second, for events simultaneous in the rain frame, $\Delta t_{\text{rain}} = 0$ so $\Delta T = 0$ from (32). That leaves the Δy_{rain} axis pointing along the r-direction, from (33). Now for the Δt_{rain} axis: By definition, raindrops lie at rest in the local rain frame. Setting $\Delta y_{\text{rain}} = \Delta x_{\text{rain}} = 0$ in (33) and (34) yields the worldline equation (30)—in its approximate form—so the local Δt_{rain} axis must lie along the raindrop worldline.

Stone's velocity in local rain frame

Equations (32) through (34) relate local measurement to global coordinates. An example is the velocity of a stone. Equations (32) through (34) lead to the following relation between global coordinate expressions $dr/dT, d\Phi/dT$ and the stone's velocity measured in the local rain frame:

$$v_{\text{rain,y}} \equiv \lim_{\Delta t_{\text{rain}} \to 0} \frac{\Delta y_{\text{rain}}}{\Delta t_{\text{rain}}} = \left(\frac{r^2}{r^2 + a^2}\right)^{1/2} \frac{dr}{dT} + \left(\frac{2M}{r}\right)^{1/2} \left(1 - a\frac{d\Phi}{dT}\right) \quad (36)$$

$$v_{\text{rain,x}} \equiv \lim_{\Delta t_{\text{rain}} \to 0} \frac{\Delta x_{\text{rain}}}{\Delta t_{\text{rain}}} = \left(r^2 + a^2\right)^{1/2} \frac{d\Phi}{dT} \quad (37)$$

In the limit-taking process the local frame shrinks to a point (event) in spacetime, which removes the superscript bars that show average values.

Raindrop velocity in local rain frame

Now let the stone be a raindrop and verify its velocity components in the local rain frame. To do this, substitute for the raindrop from (30) and (31) into (36) and (37):

$$v_{\text{rain,y}} = v_{\text{rain,x}} = 0 \qquad \text{(raindrop)} \quad (38)$$

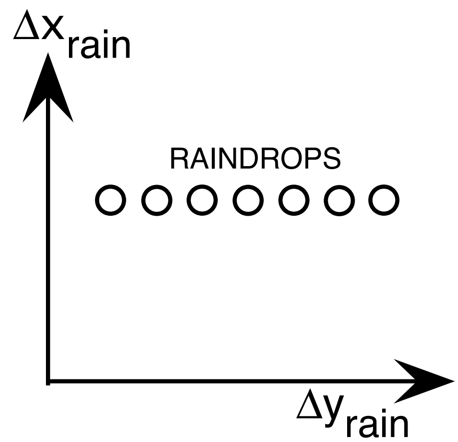

FIGURE 6 A snapshot ($\Delta t_{\rm rain} = 0$) shows a line of raindrops, which are at rest in each local rain frame (Figure 4). Equations (36), (37), and (38) show that in Doran coordinates these raindrops have identical Φ and T but different r.

which shows that the raindrop is at rest in the local inertial rain frame. This justifies the name *rain frame*.

A line of raindrops

But the raindrop has more to tell us about the local rain frame. Consider a line of raindrops, for example a sequence of drops from a faucet, all with the same value of Φ but released in sequence so that a snapshot ($\Delta t_{\rm rain} = 0$) shows the raindrops at slightly different r-values. Then equations (33) and (34) tell us that this line of raindrops (with $\Delta T = \Delta\Phi = 0$ but with slightly different values of Δr) all have the same $\Delta x_{\rm rain}$ but different values of $\Delta y_{\rm rain}$. Therefore raindrops of equal Φ lie at rest in the rain frame and a line of raindrops lies parallel to the $\Delta y_{\rm rain}$ axis (Figure 6).

17.6 ■ THE LOCAL REST FRAME

At rest in Doran global coordinates

We want more choices for measurement than just a suicide raindrop trip to the singularity. For example, it is convenient to have a local frame in which a stone at rest has constant r.

Frame stands still in Doran coordinates

To find such constant-r frames, start with the rain frame, then apply a Lorentz boost in the $\Delta y_{\rm rain}$ direction so that a stone with $dr/dT = 0$ and $d\Phi/dT = 0$ has zero velocity in the new frame. Label this the **local inertial rest frame**, with the subscript "restD" to remind us that it is at rest in Doran global coordinates. The required Lorentz boost between rain and rest frames has the form of equation (40) in Section 1.10:

$$\Delta t_{\text{restD}} = \gamma_{\text{rel}}\left(\Delta t_{\text{rain}} - v_{\text{rel}}\Delta y_{\text{rain}}\right) \tag{39}$$

$$\Delta y_{\text{restD}} = \gamma_{\text{rel}}\left(\Delta y_{\text{rain}} - v_{\text{rel}}\Delta t_{\text{rain}}\right) \tag{40}$$

$$\Delta x_{\text{restD}} = \Delta x_{\text{rain}} \tag{41}$$

What is the value of v_{rel}, the relative speed between the rest and rain frame? We want a stone with $\Delta r = \Delta\Phi = 0$ to have zero velocity in the new frame, that is $\Delta y_{\text{restD}} = \Delta x_{\text{restD}} = 0$. Now from (41) and (34) we already have $\Delta x_{\text{restD}} = \Delta x_{\text{rain}} = 0$ for a stone with $\Delta\Phi = 0$, and from equations (32) and (33):

$$\begin{aligned}\Delta y_{\text{rain}} - v_{\text{rel}}\Delta t_{\text{rain}} &= \left(\frac{\bar{r}^2}{\bar{r}^2 + a^2}\right)^{1/2}\Delta r - \left(\frac{2M}{\bar{r}}\right)^{1/2} a\Delta\Phi \\ &\quad + \left(\frac{2M}{\bar{r}}\right)^{1/2}\Delta T - v_{\text{rel}}\Delta T\end{aligned} \tag{42}$$

v_{rel} between rest and rain frames

We want this expression to be zero when $\Delta r = \Delta\Phi = 0$. This will be the case if the last two terms on the right side of (42) cancel. That is, we need a Lorentz boost such that:

$$v_{\text{rel}} = \left(\frac{2M}{\bar{r}}\right)^{1/2} \qquad \text{so} \qquad \gamma_{\text{rel}} = \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \tag{43}$$

Local rest frame coordinates

Now substitute equations (43) and (32) through (34) into (39) through (41) to obtain local rest frame coordinates in global Doran coordinates:

$$\begin{aligned}\Delta t_{\text{restD}} &= \left(1 - \frac{2M}{\bar{r}}\right)^{1/2}\Delta T \\ &\quad - \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2}\left(\frac{2M}{\bar{r}}\right)^{1/2}\left[\left(\frac{\bar{r}^2}{\bar{r}^2 + a^2}\right)^{1/2}\Delta r - \left(\frac{2M}{\bar{r}}\right)^{1/2} a\Delta\Phi\right]\end{aligned} \tag{44}$$

$$\Delta y_{\text{restD}} = \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2}\left[\left(\frac{\bar{r}^2}{\bar{r}^2 + a^2}\right)^{1/2}\Delta r - \left(\frac{2M}{\bar{r}}\right)^{1/2} a\Delta\Phi\right] \tag{45}$$

$$\Delta x_{\text{restD}} = \left(\bar{r}^2 + a^2\right)^{1/2}\Delta\Phi \tag{46}$$

The two square-bracket expressions are the same as the one in (33). Figure 5 contains a definition of the local rest frame.

Equations (44) and (45) show that the local inertial rest frame exists only outside the static limit, because these local coordinates are imaginary for $r < 2M$. This result reinforces the interpretation of the static limit defined in Section 17.3.

Stone's velocity in local rest frame.

From equations (44) through (46) we derive expressions for the stone's velocity in the local inertial rest frame:

$$v_{\rm restD,y} \equiv \lim_{\Delta t_{\rm restD}\to 0} \frac{\Delta y_{\rm restD}}{\Delta t_{\rm restD}} \tag{47}$$

$$= \frac{\left(\dfrac{r^2}{r^2+a^2}\right)^{1/2}\dfrac{dr}{dT} - \left(\dfrac{2M}{r}\right)^{1/2} a\dfrac{d\Phi}{dT}}{\left(1-\dfrac{2M}{r}\right) - \left(\dfrac{2M}{r}\right)^{1/2}\left[\left(\dfrac{r^2}{r^2+a^2}\right)^{1/2}\dfrac{dr}{dT} - \left(\dfrac{2M}{r}\right)^{1/2} a\dfrac{d\Phi}{dT}\right]}$$

$$v_{\rm restD,x} \equiv \lim_{\Delta t_{\rm restD}\to 0} \frac{\Delta x_{\rm restD}}{\Delta t_{\rm restD}} \tag{48}$$

$$= \frac{\left(1-\dfrac{2M}{r}\right)^{1/2}\left(r^2+a^2\right)^{1/2}\dfrac{d\Phi}{dT}}{\left(1-\dfrac{2M}{r}\right) - \left(\dfrac{2M}{r}\right)^{1/2}\left[\left(\dfrac{r^2}{r^2+a^2}\right)^{1/2}\dfrac{dr}{dT} - \left(\dfrac{2M}{r}\right)^{1/2} a\dfrac{d\Phi}{dT}\right]}$$

In the limit-taking process the local frame shrinks to a point (event) in spacetime, which removes the superscript bars that specify average values.

The right sides of these equations are a mess, but the computer does not care and translates between global coordinate velocities and velocities in the local rest frame. For example, to find the speed of the raindrop in the local rest frame, substitute into these equations from (30) and (31). The result is:

$$v_{\rm restD,y} = -\left(\frac{2M}{r}\right)^{1/2} = -v_{\rm rel} \qquad \text{(raindrop)} \tag{49}$$

$$v_{\rm restD,x} = 0 \qquad \text{(raindrop)} \tag{50}$$

The last step in (49) is from (43); since a raindrop is at rest in the rain frame and we Lorentz boost $+v_{\rm rel}$ in the $\Delta y_{\rm rain}$ direction, therefore the raindrop must have velocity $-v_{\rm rel}$ in the new frame.

Stone at rest in Doran coordinates is at rest in local rest frame.

Now check that we are consistent: To verify that a stone at rest in Doran coordinates is indeed at rest in the local rest frame, substitute $dr/dT = d\Phi/dT = 0$ into (47) and (48) to obtain

$$v_{\rm restD,y} = v_{\rm restD,x} = 0 \qquad (\text{stone: } dr/dT = d\Phi/dT = 0) \tag{51}$$

The stone at rest in global Doran coordinates is also at rest in the local rest frame.

QUERY 9. Local rest frame coordinates when $a \to 0$ Show that when $a \to 0$ for the non-spinning black hole, equations (44) through (46) recover expressions for the local shell frame in global rain coordinates, Box 2 in Section 7.4.

17.7 ■ THE LOCAL STATIC FRAME

Lining up with the string of stones in a necklace.

Figure 6 shows a sequence of raindrops at rest in the local rain frame and lined up along the $\Delta y_{\rm rain}$ axis. The Lorentz boost from rain to rest frame takes place along the same $\Delta y_{\rm rain}$, so the line of raindrops also lies along the $\Delta y_{\rm restD}$ axis, as shown in Figure 7. But in this local frame they are moving in the global inward direction shown in that figure.

For the non-spinning black hole we made observations from local shell frames outside the event horizon. On the symmetry slice through the center of a non-spinning black hole, each shell is a ring. The spinning black hole permits shell-rings only outside the static limit (see the exercises). More useful for the spinning black hole is a set of concentric rings that rotate with respect to global Doran coordinates. Think of each ring as composed of a necklace of stones at a given value of r that move in the Φ direction, as shown in Figure 7.

Rotating rings for $a > 0$ replace shell-rings for $a = 0$.

?

Objection 4. *In Figure 7 your Φ and r axes are not perpendicular. This violates the Pythagorean Theorem. It's illegal!*

!

Pythagoras was aware of what was later called Euclidean geometry in flat space, in which, for orthogonal coordinates,

$$\Delta s^2 = A\Delta r^2 + B\Delta\Phi^2 \qquad \text{(Phythagoras)} \qquad (52)$$

for some positive constants A and B. In contrast, you can show from (5) that, for $\Delta T = 0$,

$$\Delta s^2 = A\Delta r^2 + B\Delta\Phi^2 + C\Delta r\Delta\Phi \qquad \text{(Doran space)} \qquad (53)$$

that is, there is a cross term in the metric that signals non-orthogonality.

For every local inertial frame, we *demand* that spatial coordinates be orthogonal, so that

$$\Delta s^2 = \Delta x^2_{\rm frame} + \Delta y^2_{\rm frame} \qquad \text{(every local inertial frame)} \qquad (54)$$

Hence we force the Pythagorean Theorem to apply for space coordinates of every local inertial frame. It *need not* apply to global coordinates; Figure 7 is an example.

In the present section we start toward the rotating ring by finding a local inertial frame at fixed Doran global coordinates but with its local x-coordinate axis lying along the Φ-direction. We call it the **local static frame**, (subscript: "statD"). The local static frame is rotated with respect to the local rest frame (Figure 7).

local static frame

The rotation formulas between local rest and local static frames are:

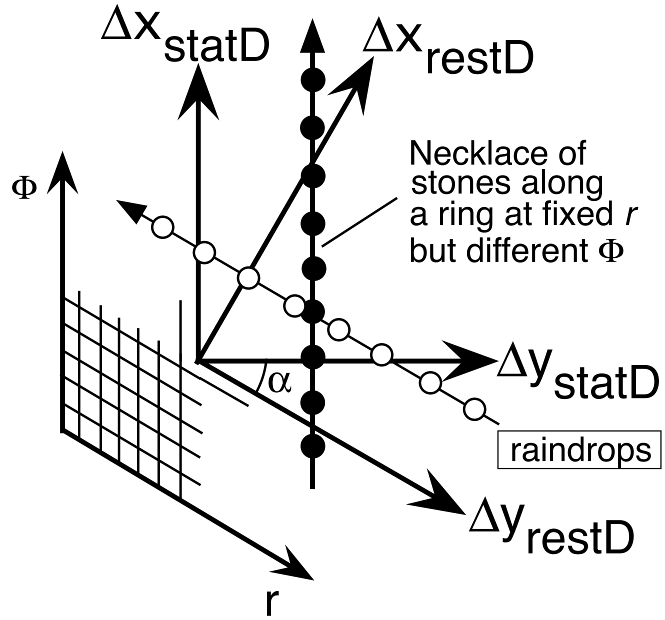

FIGURE 7 Three coordinate systems—local static and local rest plus global r-Φ—plotted on a single flat patch at a fixed global coordinate T. The line of raindrops lies along the global r-direction and moves in the negative r-direction. The necklace of stones around the spinning black hole forms a ring that lies along the global Φ-direction; stones in the necklace move in the positive Φ-direction. The relation between the local rest and static frames is a simple rotation through the angle α—equations (55) through (57). *Important:* This is a two-dimensional figure, not a perspective figure.

$$\Delta t_{\text{statD}} = \Delta t_{\text{restD}} \tag{55}$$

$$\Delta y_{\text{statD}} = \Delta y_{\text{restD}} \cos\alpha + \Delta x_{\text{restD}} \sin\alpha \tag{56}$$

$$\Delta x_{\text{statD}} = \Delta x_{\text{restD}} \cos\alpha - \Delta y_{\text{restD}} \sin\alpha \tag{57}$$

We choose the angle α so that Δy_{statD} has no terms that contain $\Delta\Phi$. In other words, orient the rotated frame so that a *ring* of stones with the same r but with different Φ-values all have $\Delta y_{\text{statD}} = 0$; the ring lies locally parallel to the Δx_{statD} axis. Equations (56), (45), and (46) yield:

$$\Delta y_{\text{statD}} = \left(1 - \frac{2M}{r}\right)^{-1/2} \left[\left(\frac{\bar{r}^2}{\bar{r}^2 + a^2}\right)^{1/2} \Delta r - \left(\frac{2M}{\bar{r}}\right)^{1/2} a\Delta\Phi\right] \cos\alpha \tag{58}$$
$$+ \left(\bar{r}^2 + a^2\right)^{1/2} \Delta\Phi \sin\alpha$$

Rearrange this equation to combine coefficients of $\Delta\Phi$:

$$\begin{aligned}\Delta y_{\text{statD}} = & \left(1-\frac{2M}{\bar{r}}\right)^{-1/2}\left(\frac{\bar{r}^2}{\bar{r}^2+a^2}\right)^{1/2}\Delta r\cos\alpha \\ & - \left[\left(\frac{2M}{\bar{r}}\right)^{1/2}\left(1-\frac{2M}{\bar{r}}\right)^{-1/2} a\cos\alpha - \left(\bar{r}^2+a^2\right)^{1/2}\sin\alpha\right]\Delta\Phi\end{aligned} \tag{59}$$

To eliminate $\Delta\Phi$ from the second line of equation (59), set the contents of the square bracket equal to zero. This determines angle α:

$$\frac{\sin\alpha}{\cos\alpha} \equiv \tan\alpha = \left(\frac{2M}{\bar{r}}\right)^{1/2}\left(1-\frac{2M}{\bar{r}}\right)^{-1/2}\left(\frac{a^2}{\bar{r}^2+a^2}\right)^{1/2} \tag{60}$$

In Query 10 you verify the following expressions for $\sin\alpha$ and $\cos\alpha$:

$$\sin\alpha = \left(\frac{2M}{\bar{r}}\right)^{1/2}\frac{a}{\bar{r}\bar{H}} \tag{61}$$

$$\cos\alpha = \left(1-\frac{2M}{\bar{r}}\right)^{1/2}\frac{\left(\bar{r}^2+a^2\right)^{1/2}}{\bar{r}\bar{H}} \tag{62}$$

The angle α should be written $\alpha(r)$ to remind us that it is a function of the r-coordinate, but we will not bother with this more complicated notation.

QUERY 10. Check expressions for $\sin\alpha$ and $\cos\alpha$.

A. Divide corresponding sides of (61) and (62) to check that the result gives $\tan\alpha$ in (60).

B. Confirm that $\sin^2\alpha + \cos^2\alpha = 1$.

C. Show that when $r \to \infty$, then $\alpha \to 0$.

D. Show that when $r \to 2M^+$ (that is, when $r \to 2M$ while $r > 2M$), then $\alpha \to \pi/2$.

E. Show that α is undefined for $r < 2M$. *Prediction:* The static frame exists only outside the static limit.

Local static frame coordinates

When we substitute (61) and (62) into (59), the second line on the right side of this equation goes to zero and the first line yields the simple expression for Δy_{statD} in (64). For rotation, $\Delta t_{\text{restD}} = \Delta t_{\text{statD}}$. Then substitution into (57) finds Δx_{statD}, which completes the coordinates of the static frame in global Doran coordinates:

$$\Delta t_{\rm statD} \equiv \left(1-\frac{2M}{\bar{r}}\right)^{1/2}\Delta T \tag{63}$$
$$- \left(1-\frac{2M}{\bar{r}}\right)^{-1/2}\left(\frac{2M}{\bar{r}}\right)^{1/2}\left[\left(\frac{\bar{r}^2}{\bar{r}^2+a^2}\right)^{1/2}\Delta r - \left(\frac{2M}{\bar{r}}\right)^{1/2} a\Delta\Phi\right]$$
$$\Delta y_{\rm statD} \equiv \frac{\Delta r}{\bar{H}} \tag{64}$$
$$\Delta x_{\rm statD} \equiv -\left(1-\frac{2M}{\bar{r}}\right)^{-1/2}\left[\left(\frac{2M}{\bar{r}}\right)^{1/2}\left(\frac{\bar{r}^2}{\bar{r}^2+a^2}\right)^{1/2}\frac{a}{\bar{r}\bar{H}}\Delta r - \bar{r}\bar{H}\Delta\Phi\right] \tag{65}$$

These equations show that, like the local rest frame, the local static frame exists only outside the static limit. Figure 5 contains a summary definition of the local static frame.

Stone's velocity in local static frame.

Now we derive expressions for the stone's velocity in the local inertial static frame:

$$v_{\rm statD,y} \equiv \lim_{\Delta t_{\rm statD}\to 0}\frac{\Delta y_{\rm statD}}{\Delta t_{\rm statD}} \tag{66}$$
$$= \frac{H^{-1}\left(1-\dfrac{2M}{r}\right)^{1/2}\dfrac{dr}{dT}}{\left(1-\dfrac{2M}{r}\right) - \left(\dfrac{2M}{r}\right)^{1/2}\left[\left(\dfrac{r^2}{r^2+a^2}\right)^{1/2}\dfrac{dr}{dT} - \left(\dfrac{2M}{r}\right)^{1/2} a\dfrac{d\Phi}{dT}\right]}$$
$$v_{\rm statD,x} \equiv \lim_{\Delta t_{\rm statD}\to 0}\frac{\Delta x_{\rm statD}}{\Delta t_{\rm statD}} \tag{67}$$
$$= \frac{(rH)^{-1}\left[r^2H^2\dfrac{d\Phi}{dT} - \left(\dfrac{2M}{r}\right)^{1/2}\left(\dfrac{r^2}{r^2+a^2}\right)^{1/2} a\dfrac{dr}{dT}\right]}{\left(1-\dfrac{2M}{r}\right) - \left(\dfrac{2M}{r}\right)^{1/2}\left[\left(\dfrac{r^2}{r^2+a^2}\right)^{1/2}\dfrac{dr}{dT} - \left(\dfrac{2M}{r}\right)^{1/2} a\dfrac{d\Phi}{dT}\right]}$$

In the limit-taking process the local frame shrinks to a point (event) in spacetime, which removes the superscript bars that show average values.

The right sides of these equations are a mess, but the computer does not care and translates between global coordinate velocities and velocities in the local static frame. For example, for the static frame components of a raindrop's velocity use equations (30) and (31):

$$\begin{aligned} v_{\text{statD,y}} &= -H^{-1}\left(\frac{2M}{r}\right)^{1/2}\left(1-\frac{2M}{r}\right)^{1/2}\left(\frac{r^2+a^2}{r^2}\right)^{1/2} \qquad (68)\\ &= -\left(\frac{2M}{r}\right)^{1/2}\cos\alpha \qquad \text{(raindrop)}\\ v_{\text{statD,x}} &= H^{-1}\left(\frac{2M}{r}\right)\frac{a}{r} \qquad (69)\\ &= \left(\frac{2M}{r}\right)^{1/2}\sin\alpha \qquad \text{(raindrop)} \end{aligned}$$

Figure 7 shows us that the raindrop moves inward at an angle α with respect to the Δy_{statD} axis, in agreement with equations (68) and (69).

QUERY 11. Raindrop in the local static frame

A. Show that the speed of the raindrop in the static frame is $(2M/r)^{1/2}$.
B. Show that at large r, the raindrop moves slowly in the local static frame and in the direction $\alpha \to 0$ in that frame.
C. Show that as $r \to 2M^+$, the raindrop moves sideways at angle $\alpha \to \pi/2$ with respect to the Δy_{statD} axis at a speed approaching light speed in that frame.

Stone at rest in Doran coordinates is at rest in local static frame.

Finally, a consistency check: We verify that a stone at rest in Doran coordinates is indeed at rest in the local static frame. For this, substitute $dr/dT = d\Phi/dT = 0$ into (66) and (67) to obtain

$$v_{\text{statD,y}} = v_{\text{statD,x}} = 0 \qquad \text{(stone: } dr/dT = d\Phi/dT = 0) \qquad (70)$$

QUERY 12. Local static frame coordinates when $a \to 0$ Show that when $a \to 0$ for the non-spinning black hole, equations (63) through (65) recover expressions for the local shell frame in global rain coordinates, Box 2 in Section 7.4. Compare the results of Query 9: when $a \to 0$, both rest frames and static frames become shell frames!

Objection 5. *Why are the line of raindrops and the string of necklace stones not perpendicular in Figure 7? You cannot tell me this is due to the non-measurability of global coordinates; These are real objects!*

Right you are: in a local frame the line of raindrops and the string of necklace stones are not perpendicular, regardless of the global

Dragging of inertial frames

coordinates that we use. The reason is subtle, but can be understood in analogy to raindrops that fall on Earth. Let a horizontal wind blow each raindrop sideways, so the line of raindrops deviates from the vertical. The spin of the black hole has a similar effect, a phenomenon sometimes called **dragging of inertial frames**. How big is the effect? Angle α in Figure 7 measures the size of this effect. In Query 10 you showed that far from the spinning black hole, $r \to \infty$, the angle $\alpha \to 0$. In contrast, as $r \to 2M^+$ the angle $\alpha \to \pi/2$ and the raindrop speed approaches that of light. At the static limit the "spinning black hole winds" are so great that raindrops are blown horizontal at the speed of light. Hurricanes on Earth are gentle beasts compared to the spinning black hole!

17.8 ■ THE LOCAL RING FRAME

Relax on a ring that circles around the black hole.

Necklace of stones becomes a ring.

The local static frame derived in Section 17.7 exists only outside the static limit. But we know from Section 17.3 that a stone can exist with no r motion all the way down to the event horizon if it has some tangential motion.

We give the name **ring** to a necklace of stones, all at the same r, that have $dr/dT = 0$ with $d\Phi/dT = \omega(r)$; then we seek a corresponding set of **local inertial ring frames** that exist down to the event horizon. Each local inertial ring frame is at rest on the ring. We will discover, to our surprise, that the ring—and local ring frames—can exist also between the Cauchy horizon and the singularity.

To find a local inertial ring frame in which the necklace of stones are at rest, we perform a Lorentz boost in the Δx_{statD} direction.

$$\Delta t_{\text{ring}} = \gamma_{\text{rel}} \left(\Delta t_{\text{statD}} - v_{\text{rel}} \Delta x_{\text{statD}}\right) \tag{71}$$

$$\Delta y_{\text{ring}} = \Delta y_{\text{statD}} \tag{72}$$

$$\Delta x_{\text{ring}} = \gamma_{\text{rel}} \left(\Delta x_{\text{statD}} - v_{\text{rel}} \Delta t_{\text{statD}}\right) \tag{73}$$

Values of v_{rel} and γ_{rel} in these equations are *not* the same as the corresponding values in equations (39) and (40).

How do we find the value of v_{rel}? We choose v_{rel} to fulfill our demand that $\Delta x_{\text{ring}} = 0$ in (73) when $\Delta r = 0$ and $\Delta\Phi = \bar{\omega}(r)\Delta T$, where equation (12) defines $\omega(r)$. In Query 13 you show that this demand leads to:

$$v_{\text{rel}} = \frac{2Ma}{\bar{r}^2 \bar{H}} \qquad \text{(ring frame speed in stat frame)} \tag{74}$$

from which

$$\gamma_{\text{rel}} \equiv \left(1 - v_{\text{rel}}^2\right)^{-1/2} = \frac{\bar{r}\bar{H}}{\bar{R}} \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2} \tag{75}$$

QUERY 13. Find v_{rel}

A. Demand that $\Delta x_{\text{ring}} = 0$ in equation (73) when $\Delta r = 0$ and $\Delta\Phi = \bar{\omega}\Delta T$. Show that this yields

$$v_{\rm rel} = \frac{\bar{r}\bar{H}\bar{\omega}}{1 - \dfrac{2M}{\bar{r}} + \dfrac{2M}{\bar{r}}a\bar{\omega}} \tag{76}$$

B. Substitute for $\bar{\omega}$ from (12) into (76) and manipulate the result to verify (74).

Local ring frame coordinates

Now we can complete Lorentz boost equations (71) through (73) using equations (63) through (65) plus equations (74) and (75). *Result:* coordinates of the local ring frame in global coordinates:

$$\Delta t_{\rm ring} \equiv \frac{\bar{r}\bar{H}}{\bar{R}}\Delta T - \frac{\bar{\beta}}{\bar{H}}\Delta r \tag{77}$$

$$\Delta y_{\rm ring} \equiv \frac{\Delta r}{\bar{H}} \tag{78}$$

$$\Delta x_{\rm ring} \equiv \bar{R}\left(\Delta\Phi - \bar{\omega}\Delta T\right) - \frac{\bar{\omega}\bar{r}}{\bar{\beta}}\Delta r \tag{79}$$

Definition of β

where

$$\beta \equiv \left(\frac{2M}{r}\right)^{1/2}\left(\frac{r^2 + a^2}{R^2}\right)^{1/2} \tag{80}$$

The average $\bar{\beta}$ is the same expression with $r \rightarrow \bar{r}$ and $R \rightarrow \bar{R}$.

The unitless symbol β stands for a bundle of constants and global coordinates similar (but not equal) to dr/dT for a raindrop in equation (30). Box 1 summarizes useful functions defined in this chapter.

Ring frames valid for $r > r_{\rm EH}$ and $0 < r < r_{\rm CH}$

Equations (77) through (79) tell us that the local ring frame can exist wherever H is real, which from (15) is down to the event horizon. The function H is imaginary between the two horizons, so ring frames cannot exist there. Inside the Cauchy horizon, however, H is real again. This astonishing result predicts that local ring frames can exist between the Cauchy horizon and the singularity. *Question:* How can this possibly be? *Answer:* Close to the singularity of a spinning black hole our intuition fails. Recall our paraphrase of *Wheeler's radical conservatism*, Comment 1 in Section 7.1: Follow what the equations tell us, no matter how strange the results. Then develop a new intuition!

Figure 5 contains a definition of the local ring frame.

QUERY 14. Local ring frame coordinates when $a \rightarrow 0$ Show that when $a \rightarrow 0$ for the non-spinning black hole, equations (77) through (79) recover expressions for the local shell frame in global rain coordinates, Box 2 in Section 7.4.

Box 1. Useful Relations for the Spinning Black Hole

Many derivations manipulate these expressions.

Static limit from Section 17.3:

$$r_S = 2M \tag{81}$$

Reduced circumference from Section 17.2:

$$R^2 \equiv r^2 + a^2 + \frac{2Ma^2}{r} \tag{82}$$

Horizon function from Section 17.3:

$$H^2 \equiv \frac{1}{r^2}\left(r^2 - 2Mr + a^2\right) \tag{83}$$

$$= \frac{1}{r^2}\left(r - r_{EH}\right)\left(r - r_{CH}\right) \tag{84}$$

where r_{EH} and r_{CH} are r-values of the event and Cauchy horizons, respectively, from Section 17.3.

$$\frac{r_{EH}}{M} \equiv 1 + \left(1 - \frac{a^2}{M^2}\right)^{1/2} \quad \text{(event horizon)} \tag{85}$$

$$\frac{r_{CH}}{M} \equiv 1 - \left(1 - \frac{a^2}{M^2}\right)^{1/2} \quad \text{(Cauchy horizon)} \tag{86}$$

Ring omega from Section 17.3:

$$\omega \equiv \frac{2Ma}{rR^2} \tag{87}$$

An equivalence from Section 17.3:

$$1 - \frac{2M}{r} + R^2\omega^2 = \left(\frac{rH}{R}\right)^2 \tag{88}$$

Definition of α from Section 17.7:

$$\alpha \equiv \arcsin\left[\left(\frac{2M}{r}\right)^{1/2}\frac{a}{rH}\right] \tag{89}$$

$$(0 \leq \alpha \leq \pi/2), \text{ namely } (r \geq 2M)$$

Definition of β from Section 17.8:

$$\beta \equiv \left(\frac{2M}{r}\right)^{1/2}\left(\frac{r^2 + a^2}{R^2}\right)^{1/2} \tag{90}$$

Stone velocity in local ring frame

Now suppose that a stone moves in the local ring frame. Equations (77) through (79) lead to the following relation between components of global coordinate velocities dr/dT and $d\Phi/dT$ and components of the stone's velocity measured in the local ring frame:

$$v_{\text{ring,y}} \equiv \lim_{\Delta t_{\text{ring}} \to 0} \frac{\Delta y_{\text{ring}}}{\Delta t_{\text{ring}}} = \frac{\dfrac{dr}{dT}}{\dfrac{rH^2}{R} - \beta\dfrac{dr}{dT}} \tag{91}$$

$$v_{\text{ring,x}} \equiv \lim_{\Delta t_{\text{ring}} \to 0} \frac{\Delta x_{\text{ring}}}{\Delta t_{\text{ring}}} = \frac{R\left(\dfrac{d\Phi}{dT} - \omega\right) - \dfrac{\omega r}{\beta}\dfrac{dr}{dT}}{\dfrac{rH}{R} - \dfrac{\beta}{H}\dfrac{dr}{dT}} \tag{92}$$

In the limit-taking process the local frame shrinks to a point (event) in spacetime, which removes the superscript bars that show average values.

Stone at rest in Doran coordinates moves in local ring coordinates.

Suppose that a stone remains at rest in Doran coordinates. What is its velocity in the local ring frame? Recall from Section 7.3 that at or inside the static limit a stone cannot be at rest in Doran coordinates, so we require that $r \geq 2M$. But what goes wrong with observations at and inside the static limit? The trouble is different for different r-values there. Substitute $dr/dT = d\Phi/dT = 0$ into (91) and (92) to obtain

$$v_{\rm ring,y} = 0 \qquad \text{(stone at rest in Doran coordinates, } r \geq 2M) \tag{93}$$

$$v_{\rm ring,x} = -\frac{2Ma}{r^2 H} \qquad \text{(ditto)} \tag{94}$$

QUERY 15. Velocity in ring frame of stone at rest in Doran coordinates
Analyze equation (94) with the following steps:

A. For $r = 2M$, show that $v_{\rm ring,x} = -1$, the speed of light.
B. For $r_{\rm EH} < r < 2M$, show that $v_{\rm ring,x} < -1$, greater than light speed.
C. For $r_{\rm CH} < r < r_{\rm EH}$ show that no ring frame exists and $v_{\rm ring,x}$ is imaginary.
D. For $r < r_{\rm CH}$, show that $v_{\rm ring,x} < -1$, greater than light speed.

QUERY 16. Velocity of necklace stones in static frame With a symmetry argument, show that the velocity of the necklace stones measured in the static frame has the same y component as (93) but the negative of the x-component in (94).

Now let us find the velocity of the raindrop in the local ring frame. Into equations (91) and (92) substitute dr/dT from (30) and $d\Phi/dT = 0$ from (31).

QUERY 17. Denominator of (91). Show that for the raindrop, the denominator of the right side of (91) becomes R/r.

The result of Query 17 plus (30) and (90) lead to an expression for $v_{\rm ring,y}$:

$$v_{\rm ring,y} = -\left(\frac{2M}{r}\right)^{1/2}\left(\frac{r^2+a^2}{R^2}\right)^{1/2} = -\beta \qquad \text{(raindrop)} \tag{95}$$

QUERY 18. Numerator of (92). Show that for the raindrop, the numerator of the right side of (92) is equal to zero.

Query 18 shows that:

$$v_{\rm ring,x} = 0 \qquad \text{(raindrop)} \tag{96}$$

Raindrop falls vertically in ring frame.

Surprising result: Every raindrop falls vertically through every local ring frame. Compare this result with parts B and C in Query 11; in the local static frame, raindrops move sideways. The local ring frame compensates for this

TABLE 17.1 Measured velocity of raindrop in several local inertial frames

Frame	Valid Region	$v_{\rm frame,y}$	$v_{\rm frame,x}$
Rain	Everywhere, $r > 0$	0	0
Rest	$r > r_{\rm S}$	$-(2M/r)^{1/2}$	0
Static	$r > r_{\rm S}$	$-(2M/r)^{1/2}\cos\alpha$	$+(2M/r)^{1/2}\sin\alpha$
Ring	$r \leq r_{\rm CH}$ & $r \geq r_{\rm EH}$	$-\beta$	0

sideways motion with a Lorentz boost, so raindrops fall vertically through the ring frame.

Table 1 summarizes the velocity components of the raindrop in the four local inertial frames we have set up.

Comment 8. Goodbye local rest frame.
We can construct an infinite number of local inertial frames at any point (event) in spacetime. From this infinite number, we choose a few frames that are useful for our purpose of making observations near a spinning black hole. The local rest frame (subscript: restD) helped to get us from the rain frame to the local static frame (subscript: statD), but has little further usefulness. Therefore we do not include the local rest frame in the exercises of this chapter or in later chapters about the spinning black hole.

In Query 19 you predict results of some measurements that observers can make in the local rain, static, and ring frames.

QUERY 19. Observations from local frames.

A. A stone is at rest in the local rain frame. What are the components of its velocity in the local static frame and in the local ring frame? What is its (scalar) speed in each of these frames?

B. A stone is at rest in the local static frame. What are the components of its velocity in the local rain frame and in the local ring frame? What is its (scalar) speed in each of these frames?

C. A stone is at rest in the local ring frame. What are the components of its velocity in the local rain frame and in the local static frame? What is its (scalar) speed in each of these frames?

D. Think of a static *ray* of stones, that is a set of stones with different r values but the same Φ values. Is this ray vertical in the local ring frame (with $\Delta x_{\rm ring} = 0$ but $\Delta y_{\rm ring} \neq 0$)? Is this ray vertical in the local rain frame (with $\Delta x_{\rm rain} = 0$ but $\Delta y_{\rm rain} \neq 0$)? Is it vertical in the local static frame (with $\Delta x_{\rm statD} = 0$ but $\Delta y_{\rm statD} \neq 0$)?

17.9 ■ APPENDIX A: MAP ENERGY OF A STONE IN DORAN COORDINATES

Derived using the Principle of Maximal Aging

We now show that the free stone has two global constants of motion: map energy and map angular momentum, just as the stone has as it moves around the non-spinning black hole. Happily we already have a well-honed routine for finding these constants of motion, most recently for the non-spinning black hole in Sections 6.2 and 8.2.

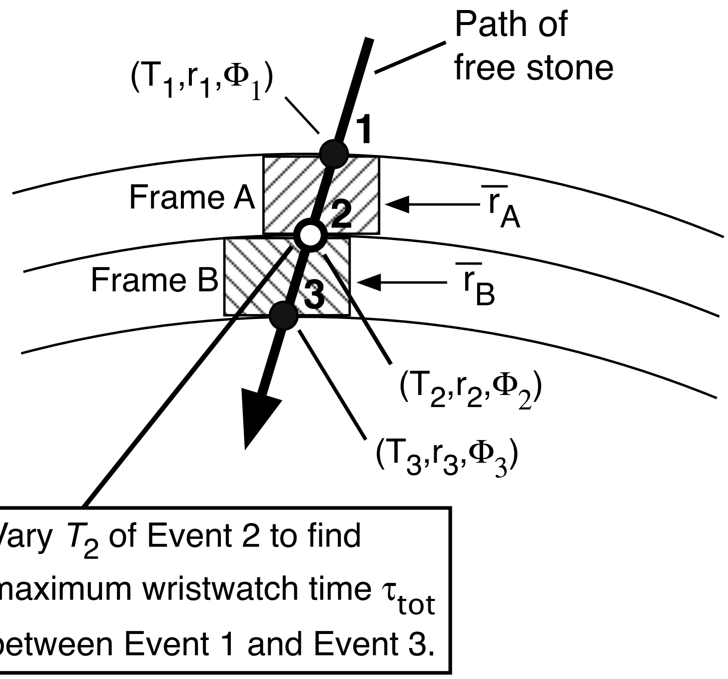

FIGURE 8 Use the Doran metric plus the Principle of Maximal Aging to derive the expression for map energy. Adaptation of Figure 3 in Section 6.2. Why does this arrow point at an angle, rather than vertically downward? See Objection 6.

Derive E and L using the Principle of Maximal Aging.

As usual, to derive map energy and map angular momentum we apply the Principle of Maximal Aging to the motion of the stone across two adjacent local inertial frames. This section adapts the procedure carried out for a non-spinning black hole in Section 6.2.

PREVIEW OF MAP ENERGY DERIVATION (Figure 8)

1. The stone enters the above local inertial Frame A at Event 1 with map coordinates (T_1, r_1, Φ_1).
2. The stone moves straight across the above inertial Frame A in time lapse τ_A measured on its wristwatch.
3. The stone crosses from the above inertial Frame A to the below inertial Frame B at Event 2 with map coordinates (T_2, r_2, Φ_2).
4. The stone moves straight across the below inertial Frame B in time lapse τ_B measured on its wristwatch.
5. The stone exits the below inertial frame at Event 3 with map coordinates (T_3, r_3, Φ_3) .
6. Use the Principle of Maximal Aging to define map energy of the stone: Vary *only* the value of T_2 at the boundary between above and below frames to maximize the total wristwatch time τ_{tot} across both frames.

The total wristwatch time $\tau_{\rm tot}$ across both local frames is the sum of wristwatch times across the above and below frames:

$$\tau_{\rm tot} \equiv \tau_{\rm A} + \tau_{\rm B} \tag{97}$$

To find the path of maximal aging, set to zero the derivative of $\tau_{\rm tot}$ with respect to T_2:

$$\frac{d\tau_{\rm tot}}{dT_2} = \frac{d\tau_{\rm A}}{dT_2} + \frac{d\tau_{\rm B}}{dT_2} = 0 \tag{98}$$

or

$$\frac{d\tau_{\rm A}}{dT_2} = -\frac{d\tau_{\rm B}}{dT_2} \tag{99}$$

Write approximate versions of metric (5) for the above and below patches; spell out only those terms that contain T. In the following, ZZ means "terms that do not contain T."

$$\tau_{\rm A} \approx \left[\left(1-\frac{2M}{\bar{r}_{\rm A}}\right)(T_2-T_1)^2 - 2\left(\frac{2M\bar{r}_{\rm A}}{\bar{r}_{\rm A}^2+a^2}\right)^{1/2}(T_2-T_1)(r_2-r_1) \right. \tag{100}$$
$$\left. +2\left(\frac{2Ma}{\bar{r}_{\rm A}}\right)(T_2-T_1)(\Phi_2-\Phi_1) + ZZ\right]^{1/2}$$

$$\tau_{\rm B} \approx \left[\left(1-\frac{2M}{\bar{r}_{\rm B}}\right)(T_3-T_2)^2 - 2\left(\frac{2M\bar{r}_{\rm B}}{\bar{r}_{\rm B}^2+a^2}\right)^{1/2}(T_3-T_2)(r_3-r_2) \right. \tag{101}$$
$$\left. +2\left(\frac{2Ma}{\bar{r}_{\rm B}}\right)(T_3-T_2)(\Phi_3-\Phi_2) + ZZ\right]^{1/2}$$

All coordinates are fixed except T_2. When we take the derivative of these two expressions with respect to T_2, the resulting denominators are simply $\tau_{\rm A}$ and $\tau_{\rm B}$, respectively:

$$\frac{d\tau_{\rm A}}{dT_2} \approx \frac{\left(1-\frac{2M}{\bar{r}_{\rm A}}\right)(T_2-T_1) - \left(\frac{2M\bar{r}_{\rm A}}{\bar{r}_{\rm A}^2+a^2}\right)^{1/2}(r_2-r_1) + \left(\frac{2Ma}{\bar{r}_{\rm A}}\right)(\Phi_2-\Phi_1)}{\tau_{\rm A}} \tag{102}$$

$$\frac{d\tau_{\rm B}}{dT_2} \approx -\frac{\left(1-\frac{2M}{\bar{r}_{\rm B}}\right)(T_3-T_2) - \left(\frac{2M\bar{r}_{\rm B}}{\bar{r}_{\rm B}^2+a^2}\right)^{1/2}(r_3-r_2) + \left(\frac{2Ma}{\bar{r}_{\rm B}}\right)(\Phi_3-\Phi_2)}{\tau_{\rm B}} \tag{103}$$

Note the initial minus sign on the right side of the second equation.

Now substitute these two equations into (99). The minus signs cancel to yield expressions of similar form on both sides of the equation. *Result:* The

Map energy in Doran coordinates

expression on the left side of (99) depends only on $\bar{r}_{\rm A}$ plus differences in the global coordinates across that local inertial frame. The expression on the right side of (99) depends only on $\bar{r}_{\rm B}$ plus corresponding differences in the global coordinates across that frame. In other words, we have found an expression in global coordinates that has the same form and the same value in two adjacent frames; it is a **map constant of the motion** (Comment 6, Section 1.11). We call this expression **map energy:** E/m. Shrink the differences to differentials (Comment 4, Section 1.7). Map energy becomes:

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dT}{d\tau} - \left(\frac{2Mr}{r^2+a^2}\right)^{1/2}\frac{dr}{d\tau} + \frac{2Ma}{r}\frac{d\Phi}{d\tau} \qquad (104)$$

QUERY 20. Cleanup questions for map energy of a stone.

A. Why do we give the name E/m to the expression on the right side of (20)? Verify that for $r \gg 2M$, that is in flat spacetime, this expression reduces to $E/m = dt/d\tau$, the special relativity expression for energy—equation (23) in Section 1.7.

B. Show that for the non-spinning black hole equation (20) for E/m reduces to equation (35) in Section 7.5.

Map energy

The map energy E of a free stone on the left side of (20) is a constant of motion whose numerical value is independent of the global coordinate system. The *form* of the right side, however, looks different when expressed in different global coordinate systems.

?

Objection 6. *In your derivation of map energy for the non-spinning black hole in Section 6.2, the arrow pointed vertically downward. Why does the arrow in Figure 8 in the present chapter point in another direction?*

!

A perceptive question! The term *ZZ* in both equations (100) and (101) represents "terms that do not contain T." Now look at the fourth term on the right side of global metric (5). This term does not contain dT, but it does contain $d\Phi$, so this term would be eliminated if the arrow in Figure 8 pointed vertically downward (for which $d\Phi = 0$). With this error, equation (20) for map energy would be incomplete; it would not contain the term that ends with $d\Phi/d\tau$. You can show that this complication does not exist in the earlier derivation of map energy for the non-spinning black hole (Section 6.2).

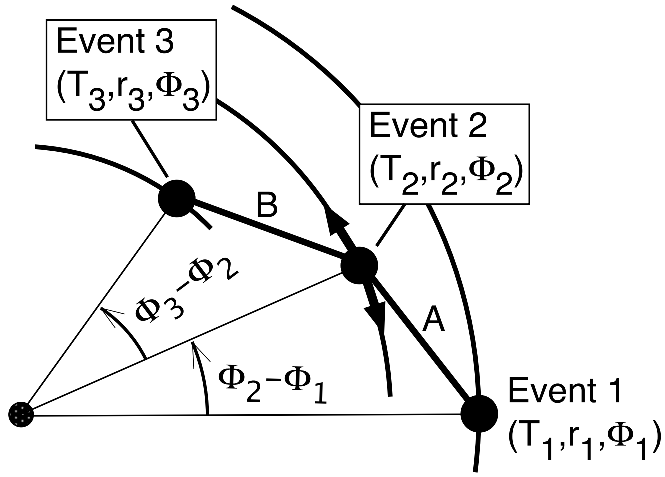

FIGURE 9 Use the Principle of Maximal Aging to derive the expression for map angular momentum in Doran coordinates. Vary Φ_2 of Event 2 to find the Φ-coordinate that leads to maximum τ_{tot} along worldline segments A and B between Events 1 and 3. Adaptation of Figure 2 in Section 8.2.

17.10 ■ APPENDIX B: MAP ANGULAR MOMENTUM OF A STONE IN DORAN COORDINATES

Again, use the Principle of Maximal Aging

To derive the expression for map angular momentum in Doran coordinates, our overall strategy closely follows that of the derivation of E/m in Section 17.9, with the notation shown in Figure 9. Run your finger down the Summary of Map Energy Derivation in Section 17.9 to preview the parallel derivation here.

In this case let the adjacent local inertial frames straddle the straight segments A and B in Figure 9. Write approximate versions of metric (5); spell out only those terms that contain Φ. In the following equations, YY stands for "terms that do not contain Φ."

$$\tau_{\text{A}} \approx \left[2\left(\frac{2Ma}{\bar{r}_{\text{A}}}\right)(T_2 - T_1)(\Phi_2 - \Phi_1) \right. \tag{105}$$

$$\left. +2a\left(\frac{2M\bar{r}_{\text{A}}}{\bar{r}_{\text{A}}^2 + a^2}\right)^{1/2}(r_2 - r_1)(\Phi_2 - \Phi_1) - \bar{R}_{\text{A}}^2(\Phi_2 - \Phi_1)^2 + YY\right]^{1/2}$$

$$\tau_{\text{B}} \approx \left[2\left(\frac{2Ma}{\bar{r}_{\text{B}}}\right)(T_3 - T_2)(\Phi_3 - \Phi_2) \right. \tag{106}$$

$$\left. +2a\left(\frac{2M\bar{r}_{\text{B}}}{\bar{r}_{\text{B}}^2 + a^2}\right)^{1/2}(r_3 - r_2)(\Phi_3 - \Phi_2) - \bar{R}_{\text{B}}^2(\Phi_3 - \Phi_2)^2 + YY\right]^{1/2}$$

All event coordinates are fixed except for Φ_2. To apply the Principle of Maximal Aging, take the derivatives of both these expressions with respect to Φ_2 and set the resulting sum equal to zero:

$$\frac{d\tau_{\rm tot}}{d\Phi_2} = \frac{d\tau_{\rm A}}{d\Phi_2} + \frac{d\tau_{\rm B}}{d\Phi_2} = 0 \tag{107}$$

or

$$\frac{d\tau_{\rm A}}{d\Phi_2} = -\frac{d\tau_{\rm B}}{d\Phi_2} \tag{108}$$

Take these derivatives with respect to Φ_2 of each expression in (105) and (106). The resulting two equations have $\tau_{\rm A}$ and $\tau_{\rm B}$ in the denominator, respectively:

$$\frac{d\tau_{\rm A}}{d\Phi_2} \approx \frac{\left(\frac{2Ma}{\bar{r}_{\rm A}}\right)(T_2 - T_1) + a\left(\frac{2M\bar{r}_{\rm A}}{\bar{r}_{\rm A}^2 + a^2}\right)^{1/2}(r_2 - r_1) - \bar{R}_{\rm A}^2(\Phi_2 - \Phi_1)}{\tau_{\rm A}} \tag{109}$$

$$\frac{d\tau_{\rm B}}{d\Phi_2} \approx -\frac{\left(\frac{2Ma}{\bar{r}_{\rm B}}\right)(T_3 - T_2) + a\left(\frac{2M\bar{r}_{\rm B}}{\bar{r}_{\rm B}^2 + a^2}\right)^{1/2}(r_3 - r_2) - \bar{R}_{\rm B}^2(\Phi_3 - \Phi_2)}{\tau_{\rm B}} \tag{110}$$

Note the initial minus sign on the right side of the second equation.

Now substitute these two equations into (108). The minus signs cancel, yielding expressions of similar form on both sides of the equation. *Result:* The left side of (108) depends only on $\bar{r}_{\rm A}$ plus differences in the global coordinates across that frame. The right side of (108) depends only on $\bar{r}_{\rm B}$ plus corresponding differences in the global coordinates across that frame. In other words, we have found an expression in global coordinates that—in this approximation—has the same form and the same value in two adjacent frames. Shrink to differentials and the expression becomes exact. It is another constant of motion, which we call **map angular momentum**:

Map angular momentum in Doran coordinates

$$\frac{L}{m} = R^2\frac{d\Phi}{d\tau} - \frac{2Ma}{r}\frac{dT}{d\tau} - a\left(\frac{2Mr}{r^2 + a^2}\right)^{1/2}\frac{dr}{d\tau} \tag{111}$$

Comment 9. The sign of L/m: our choice

Notice that the right side of (21) is the negative of what we would expect, given its derivation from (109) and (110). The sign of L/m is arbitrary, our choice because either way L/m is constant for a free stone. We choose the minus sign so that when r becomes large, L/m is positive when the tangential component of motion is in the positive (counterclockwise) Φ direction. Recall the discussion after equation (1).

Map angular momentum

The map angular momentum L/m of a free stone, on the left side of (21), is a constant of motion whose numerical value is independent of the global coordinate system. The *form* of the right side, however, will look different when expressed in different global coordinate systems.

QUERY 21. Cleanup questions for map angular momentum of a stone.
Why do we give the name L/m to the expression on the right side of (21)? Verify that *either* for $r \gg 2M$ (far from the spinning black hole) *or* for $a \rightarrow 0$ (the non-spinning black hole) this expression reduces to $L/m = r^2 d\phi/d\tau$, the expression for the non-spinning black hole—equation (10) in Section 8.2.

17.11 ■ PROJECT: BOYER-LINDQUIST GLOBAL COORDINATES

Metric in Boyer-Lindquist coordinates

In 1963 Roy Kerr published his paper that first contained a global metric for the spinning black hole. In 1967 R. H. Boyer and R. W. Lindquist published a global metric that simplifies the form of Kerr's original metric. Here it is, expressed in so-called **Boyer-Lindquist global coordinates**. As usual, for simplicity we restrict global coordinates and their metric to a slice through the equatorial plane of the black hole, perpendicular to its axis of rotation.

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 + \frac{4Ma}{r} dt d\phi - \frac{dr^2}{H^2} - R^2 d\phi^2 \quad \text{(Boyer-Lindquist...} \qquad (112)$$
$$-\infty < t < \infty, \quad 0 < r < \infty, \quad 0 \leq \phi < 2\pi \qquad \text{...on the equatorial slice)}$$

Box 2 defines H^2 and R^2. Global ϕ has the same meaning as it does in the global rain metric for the non-spinning black hole, equation (32) in Section 7.5.

Comment 10. Why not use Boyer-Lindquist coordinates?
The Boyer-Lindquist metric (112) has only one cross term instead of all possible cross terms in the Doran metric (5). Why does this chapter use and develop the consequences of this complicated Doran metric? The first term on the right of (112) tells why: this term goes to zero as $r \rightarrow 2M^+$. As a result, Boyer-Lindquist map time t increases without limit along the worldline of a descending stone as it approaches $r = 2M$. This is the same inconvenience we found in the Schwarzschild metric for the non-spinning black hole. To avoid this problem, in Chapter 7 we converted from Schwarzschild coordinates to global rain coordinates. We could have carried out the same sequence in the present chapter: begin with the Boyer-Lindquist metric, then convert to the Doran metric. But this conversion is an algebraic mess (with the simple result given in the following exercise). Instead, we chose to start immediately with the Doran metric and to relegate investigation of the Boyer-Lindquist metric to these exercises.

BL-1. Conversion from Doran coordinates to Boyer-Lindquist global coordinates

Substitute the following expressions into the Doran global metric and simplify the results to show that the outcome is the Boyer-Lindquist metric (112):

$$dT = dt + \frac{R\beta}{rH^2}dr \tag{113}$$

$$d\Phi = d\phi + \frac{\omega R}{rH^2\beta}dr \tag{114}$$

BL-2. Limiting cases of the Boyer-Lindquist metric

A. Show that for zero spin angular momentum ($a = 0$), the Boyer-Lindquist metric (112) reduces to the Schwarzschild metric, equation (6) in Section 3.1.

B. Show that the Boyer-Lindquist metric for a maximum-spin black hole ($a = M$) takes the form

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 + \frac{4M^2}{r}dtd\phi - \frac{dr^2}{H^2_{\text{max}}} - R^2_{\text{max}}d\phi^2 \quad (a = M) \tag{115}$$

BL-3. Tetrad form of the Boyer-Lindquist metric

To put the Boyer-Lindquist metric into a tetrad form, eliminate the $dtd\phi$ cross term by completing the square: Add and subtract a function $G(r)d\phi^2$ to terms on the right side of the metric, then define $G(r)$ to eliminate the cross term.

Show that the resulting tetrad form of the Boyer-Lindquist metric is:

$$\begin{aligned} d\tau^2 = &\left(1 - \frac{2M}{r}\right)^{-1}\left[\left(1 - \frac{2M}{r}\right)dt + \frac{2Ma}{r}d\phi\right]^2 \\ &- \frac{dr^2}{H^2} - \left(1 - \frac{2M}{r}\right)^{-1}\left[R^2\left(1 - \frac{2M}{r}\right) + \frac{4M^2a^2}{r^2}\right]d\phi^2 \qquad \text{(Boyer-Lindquist)} \end{aligned} \tag{116}$$

BL-4. Local shell frame in Boyer-Lindquist coordinates

A. Adapt equation (14) to simplify the coefficient of $d\phi^2$ in (116).

B. Use the results of Item A and exercise 2 to derive the following local shell coordinates in Boyer-Lindquist coordinates.

$$\Delta t_{\text{shell}} \equiv \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2}\left[\left(1 - \frac{2M}{\bar{r}}\right)\Delta t + \frac{2Ma}{\bar{r}}\Delta\phi\right] \tag{117}$$

$$\Delta y_{\text{shell}} \equiv \frac{\Delta r}{\bar{H}} \qquad \text{(Boyer-Lindquist)} \tag{118}$$

$$\Delta x_{\text{shell}} \equiv \left(1 - \frac{2M}{\bar{r}}\right)^{-1/2}\bar{r}\bar{H}\Delta\phi \tag{119}$$

C. How do we know that equations (117) through (119) define a local *shell* frame and not, for example, a local ring frame or rain frame?

E. Show that as $a \to 0$ equations (117) through (119) recover shell frame expressions in global rain coordinates (Section 7.5).

Comment 11. Shell frame in Doran coordinates.
You can use conversion equations (113) and (114) to express local shell coordinates in Doran global coordinates. Like equations (117) and (119), the resulting equations show that shell frames exist only outside the static limit.

BL-5. Local ring frame in Boyer-Lindquist coordinates

A. Show that the following tetrad form reduces to the Boyer-Lindquist metric (112):

$$d\tau^2 = \left(\frac{rH}{R}\right)^2 dt^2 - \frac{dr^2}{H^2} - R^2\left[d\phi - \omega(r)dt\right]^2 \quad \text{(Boyer-Lindquist)} \tag{120}$$

where Box 1 defines $\omega(r) \equiv 2Ma/(rR^2)$.

B. Individual terms in (120) allow us to define the local ring frame:

$$\Delta t_{\text{ring}} \equiv \frac{\bar{r}\bar{H}}{\bar{R}}\Delta t \qquad \text{(Boyer-Lindquist)} \tag{121}$$

$$\Delta y_{\text{ring}} \equiv \frac{\Delta r}{\bar{H}} \tag{122}$$

$$\Delta x_{\text{ring}} \equiv \bar{R}\left(\Delta\phi - \bar{\omega}\Delta t\right) \tag{123}$$

C. Use transformations (113) and (114) to show that Boyer-Lindquist ring equations (121) through (123) imply Doran ring equations (77) through (79).

D. What is the measurable relative velocity, call it v_{ring}, between local ring coordinates and local shell coordinates?

E. Show that as $a \to 0$ equations (121) through (123) recover shell frame expressions in global rain coordinates (Section 7.5).

BL-6. Local rain frame in Boyer-Lindquist coordinates

A. Substitute the Δ forms of equations (113) and (114) into equations (32) through (34) to obtain the following expressions for local rain coordinates in Boyer-Lindquist coordinates:

$$\Delta t_{\text{rain}} = \Delta t + \beta\frac{\bar{R}}{\bar{r}\bar{H}^2}\Delta r \tag{124}$$

$$\Delta y_{\text{rain}} = \frac{\bar{R}}{\bar{r}\bar{H}^2}\Delta r + \beta\Delta t \tag{125}$$

$$\Delta x_{\text{rain}} = \Delta x_{\text{ring}} = \bar{R}\left(\Delta\phi - \bar{\omega}\Delta t\right) \tag{126}$$

B. Use these equations to write the Boyer-Lindquist metric in tetrad form.

BL-7. Not "at rest" in both global coordinates

Show that a stone at rest in Boyer-Lindquist global coordinates ($dr = d\phi = 0$) is not at rest in Doran global coordinates; in particular, $d\Phi \neq 0$ for that stone.

BL-8. Boyer-Lindquist metric for $M = 0$.

Show that when the mass of the spinning black hole gets smaller and smaller, $M \to 0$ in (112), but the angular momentum parameter a keeps a constant value, then the Boyer-Lindquist metric becomes equal to the Doran metric under the same limits, as examined in Exercies 3.

17.12 ■ EXERCISES

1. Our Sun as a black hole

Suppose that our Sun collapses into a spinning black hole without blowing off any mass. What is the value of its spin parameter a/M? The magnitude of the Sun's angular momentum is approximately:

$$J_{\text{Sun}} \approx 1.63 \times 10^{41} \quad \text{kilogram meters}^2/\text{second} \qquad (127)$$

A. Use equation (10) in Section 3.2 to convert kilograms to meters. The result to one significant digit is $J = 1 \times 10^{14}$ meters^3/second. Derive the answer to three significant digits. [My answer: 1.21×10^{14} meters^3/second]

B. Divide your answer to Item A by c to find the angular momentum of the Sun in units of meters^2.

C. Divide the result of Item B by the square of the mass of our Sun in meters (inside the front cover) to show that $a_{\text{Sun}}/M_{\text{Sun}} = 0.185$.

2. Ring frame time for one rotation

How does someone riding in the ring frame know that she is revolving around the spinning black hole? She can tell because the same pattern of stars overhead repeats sequentially, separated by ring frame time we can call Δt_{ring1}. Derive an expression for Δt_{ring1} using the following outline or some other method:

A. The observer is stationary in the ring frame. Show that this means that $\Delta r = 0$ and $\Delta\Phi = \bar{\omega}\Delta T$.

B. Show from equation (77) and results of Item A that, for one rotation, that is for $\Delta\Phi = 2\pi$:

$$\Delta t_{\text{ring1}} = \frac{\bar{r}\bar{H}}{\bar{R}}\Delta T = \frac{2\pi(\bar{r}\bar{H})}{\bar{R}\bar{\omega}} \qquad \text{(in meters)} \qquad (128)$$

C. Substitute for the various factors in (128) to obtain

$$\Delta t_{\text{ring1}} = \frac{\pi \bar{R}\bar{r}}{Ma}\left(\bar{r} - r_{\text{EH}}\right)^{1/2}\left(\bar{r} - r_{\text{CH}}\right)^{1/2} \quad \text{(meters)} \qquad (129)$$

$$= \frac{\pi M}{a^*} R^* r^* \left[\left(r^* - r^*_{\text{EH}}\right)\left(r^* - r^*_{\text{CH}}\right)\right]^{1/2} \quad \text{(meters)} \qquad (130)$$

Equation (130) uses unitless variables, for example $r^* \equiv r/M$, and for simplicity we have deleted the average value bar over the symbols.

D. For a spinning black hole of mass $M = 10M_{\text{Sun}}$ and spin $a^* = a/M = (3/4)^{1/2}$, find the ring rotation times for one rotation at ring r-values given in items (b) through (f) in the following list. Express your results in both meters and seconds.

(a) Show that $\pi M/a^* = 5.369 \times 10^4$ meters.
(b) $r^* = 10^3$
(c) $r^* = 10$
(d) $r^* = 3$
(e) $r^* = 1.51$
(f) $r^* = 0.25$

Notice that each of these short times is measured in the local inertial ring frame.

E. For the spinning black hole in Item D, what is the value of Δt_{ring1} for a ring at the radius of Mercury around our Sun? Use Mercury orbit values in Chapter 10. Compare this value of Δt_{ring1} for our spinning black hole with the orbital period of Mercury around our Sun.

F. Equation (130) tells us that, for a given value of a^*, the ring frame time for one rotation of the ring is proportional to the mass M of the black hole. As a result, you can immediately write down the corresponding times Δt_{ring1} for Item D around the spinning black hole at the center of our galaxy whose mass $M = 4 \times 10^6 M_{\text{Sun}}$. Assume that the (unknown) value of its spin parameter $a^* = (3/4)^{1/2}$.

3. Distance between rings measured by a rain observer

A rain observer measures the distance between two adjacent concentric rings around a spinning black hole. The two rings are separated by dr in Doran r-coordinate. The rain observer their distance in two distinct ways:

[1] As she travels past the two rings, she measures, on her wristwatch, the time d? it takes her to get from the outer ring to the inner ring. She knows her speed vrel relative to the two adjacent rings. She then calculates the distance between the two adjacent rings from these two numbers.

[2] During her short travel through the two adjacent rings she is in a local inertial rain frame. She considers two events along the yrain axis in this frame: one takes place on the inner ring, the other on the outer ring, and they

simultaneous as measured in her local inertial rain frame. She then determines the distances between the rings as the separation of yrain-coordinates between these two events.

A. Write an expression for distance ds between the two adjacent rings, according to her first measurement technique? [Hint: Use (26) through (28) and (43).]

B. What is the distance ds between the two adjacent rings, according to her second measurement technique? [Hint: Use (32) through (34).] Show that the two techniques give the same result for the distance between the two rings as measured by a rain observer.

C. Take the limit of ds as $a \to 0$, and compare the result with Box 5 in Chapter 7 which suggested that for a non-spinning black hole the distance between two adjacent shells as measured by a rain observer is $ds = dr$, where dr is the incremental difference in Schwarzschild r-coordinate between the two shells.?

4. Raindrop speed measured in local inertial ring frame

Use (95) and your favorite plotting program to plot the speed of a raindrop measured in a local inertial ring frame, as a function of the Doran r-coodinate of that ring frame, for each of the following black hole spin parameters:

- (a) $a/M = 0$ (non-spinning black hole). Compare this plot with Figure 2 in Chapter 6.

- (b) $a/M = (3/2)^{1/2}$

- (c) $a/M = 1$ (maximally spinning black hole)

Show that wherever a local inertial ring frame can be constructed, the speed of the raindrop measured in that frame does not exceed the speed of light. At what r-values does the measured speed of the raindrop reach the speed of light?

5. Relative orientation of local ring frame and local rest frame axes

Table 1 shows that the velocity of a raindrop measured in the local ring frame points along the Δy_{rain} axis. Table 1 also tells us that the velocity of the same raindrop measured in the local rest frame points along the Δy_{rest} axis. Does this mean that the spatial axes in the local ring frame have the same orientation as the spatial axes in the local rest frame? Isn?t this in contradiction with Figure 7, which implies that the orientation of the spatial axes in the local ring frame matches the orientation of spatial axes in the local static frame?

6. Stone released from rest on a local ring frame

Release a stone from rest in a local ring frame at Doran coordinate r_0. Derive an expression for the velocity v_{ring} of the stone measured in a local ring frame as a function of the Doran r-coordinate of that ring frame ($r < r_0$). Show that in the limit in which the stone drops from rest far away ($r_0 \to \infty$), the expression for the velocity of the stone reduces to expression (95) for a raindrop.

7. Stone hurled inward from a local ring frame far away

Hurl a stone inward with velocity components $v_{\text{ring},x} = 0$ and $v_{\text{ring},y} = -v_{\text{far}}$ from a local inertial ring frame far away from a spinning black hole.

A. Derive an expression for the velocity components of the stone measured in a local ring frame as a function of the Doran r-coordinate of that ring frame.

B. Show that in the limit in which the stone drops *from rest* in a ring frame far away ($v_{\text{far}} \to 0$), the expression for the velocity of the stone reduces to expression (95) for a raindrop.

8. Tetrad form of the Doran global metric

A. From equations (77) through (79), write down the corresponding tetrad form of the Doran global metric.

B. Multiply out the resulting global metric to verify that the result is Doran metric (5).

9. Doran metric for $M \to 0$

Let the mass of the spinning black hole get smaller and smaller, $M \to 0$, while the angular momentum parameter a retains a a constant value. Then metric (5) becomes:

$$d\tau^2 = dT^2 - \frac{r^2}{r^2 + a^2} dr^2 - \left(r^2 + a^2\right) d\Phi^2 \qquad (M = 0) \qquad (131)$$

Does metric (131) represent flat spacetime? To find out we show a coordinate transformation that reduces (131) to an inertial metric in flat spacetime. Let

$$\rho \equiv \left(r^2 + a^2\right)^{1/2} \qquad (132)$$

The last term in metric (131) becomes $\rho^2 d\Phi^2$ and ρ is the reduced circumference.

A. Take the differential of both sides of (132) and substitute the result for the second term on the right side of (131). Show that the outcome is the metric

$$d\tau^2 = dt^2 - d\rho^2 - \rho^2 d\Phi^2 \qquad (M = 0) \tag{133}$$

The global metric (131) has been transformed to the globally flat form (133). This is *not* the metric of a local frame; it is a *global metric*—but with a strange exclusion, discussed in the following Items.

B. Does the spatial part of the metric (133) describe the Euclidean plane? To describe Euclidean space, that spatial part of the metric

$$ds^2 = d\rho^2 + \rho^2 d\Phi^2 \qquad \text{(Euclid)} \tag{134}$$

must, by definition, be valid for the full range of ρ, the radial coordinate in equation (134), namely $0 \leq \rho < \infty$. But this is not so: Definition (132) tells us that $\rho = a$, when $r = 0$. So global metric (131) is undefined for $0 < \rho < a$. Can we "do science"—that is, carry out measurements—in the region $0 < \rho < a$?

C. Is $\rho = 0$ *actually* a point or a ring? What is the meaning of the word *actually* when we describe spacetime with (arbitrary!) map coordinates.

D. Does the Doran metric for $M \to 0$ but $a > 0$ reduce to the flat spacetime metric of special relativity? Show that the answer is no, that the black hole spin remains imprinted on spacetime like the Cheshire cat's grin after its body—the mass—fades away.

10. Free stone vs. powered spaceship vs. light

Review Section 17.3, A stone's throw. Which formulas in that section describe only a free stone? Which formulas apply generally to any object with nonzero mass (free stone, powered spaceship, etc.)? Which formulas apply to light also? [*Hint:* The metric describes nearby events along the worldline of any object: free stone, powered spaceship, or light ray. The Principle of Maximal Aging is valid only for objects that move freely.]

11. Toy model of a pulsar

A **pulsar** is a spinning neutron star that emits electromagnetic radiation in a narrow beam. We observe the pulsar only if the beam sweeps across Earth. Box 5 in Section 3.3 tells us that "General relativity significantly affects the structure and oscillations of the neutron star." In particular, the neutron star has a maximum spin rate related to a_{max} for a black hole—equation (3). Let the neutron star have the mass of our Sun with the surface at $R = 10$ kilometers. Use Newtonian mechanics to make a so-called *toy model* of a pulsar—that is, a rough first approximation to the behavior of a non-Newtonian system. The pulsar PSR J1748-2446, located in the globular cluster called Terzan 5, rotates at 716 hertz $\equiv$ 716 revolutions per second. Set the neutron star's angular momentum to that of a uniform sphere rotating at that rate and call the result "our pulsar." Then the angular momentum, as a function of the so-called **moment of inertia** I_{sphere} and spin rate ω radians per second is:

$$J \equiv I_{\text{sphere}}\omega = \left(\frac{2M}{5} M_{\text{kg}} R^2\right)\omega \qquad \text{(Newton, conventional units)} \quad (135)$$

Our pulsar spins once in Newton universal time $t = 1.40$ millisecond. Use numerical tables inside the front cover to answer the following questions:

A. What is the value of our pulsar's angular momentum in conventional units?

B. Express the our pulsar's angular momentum in meters^2.

C. Find the value of $J/(Ma_{\text{max}}) = J/M^2$ for our pulsar, where M is in meters.

D. Suppose that our pulsar collapses to a black hole. Explain why it would have to blow off some of its mass to complete the process.

12. Spinning baseball a naked singularity?

A standard baseball has a mass $M = 0.145$ kilogram and radius $r_{\text{b}} = 0.0364$ meter. The Newtonian expression for the spin angular momentum of a sphere of uniform density is, in conventional units

$$J_{\text{conv}} = I_{\text{conv}}\omega = \frac{2}{5}M_{\text{kg}} r_{\text{b}}^2 \omega = \frac{4\pi M_{\text{kg}} r_{\text{b}}^2}{5} f \qquad \text{(Newton)} \qquad (136)$$

where ω is the rotation rate in radians per second. The last step makes the substitution $\omega = 2\pi f$, where f is the frequency in rotations per second. We want to find the value of the angular momentum parameter $a = J/M$ in meters. Begin by dividing both sides of (136) by the baseball's mass M_{kg}:

$$\frac{J_{\text{conv}}}{M_{\text{kg}}} = \frac{4\pi r_{\text{b}}^2}{5} f \qquad \text{(Newton: conventional units)} \qquad (137)$$

The units of the right side of (137) are meters^2/second. Convert to meters by dividing through by c, the speed of light, to obtain an expression for a:

$$a \equiv \frac{J}{M} = \frac{4\pi r_{\text{b}}^2}{5c} f \qquad \text{(Newton: units of meters)} \qquad (138)$$

A. Insert numerical values to show the result in the unit meter:

$$a = 1.1 \times 10^{-11} \text{ second} \times f \qquad \text{(Newton: units of meters)} \quad (139)$$

B. We want to know if a is greater than the mass of the baseball. What is the mass M of the baseball in meters? [My answer: 1.1×10^{-28} meter.]

C. Suppose that a pitched or batted baseball spins at 4 rotations per second. What is the value of a for this flying ball? [My answer: 4.4×10^{-11} meter.] Does this numerical value violate the limits on the spin angular momentum parameter a for a spinning black hole? [My answer: And how!]

QUESTION: Is this baseball a naked singularity?
ANSWER: No, because the Doran metric is valid only in curved *empty* space; it does not apply inside a baseball. ("Outside of a dog, a book is man's best friend. Inside of a dog it's too dark to read." –Groucho Marx)

D. What is the value of r/M at the surface of the baseball, that is, what is the value of r_b/M? Calculate the resulting value of H^2 at the surface of the baseball. What is the value of R^2/M^2 at this surface?

E. Divide Doran metric (5) through by M^2 to make it unitless. At the surface of the baseball, determine how much each term in the resulting metric differs from the corresponding term for flat spacetime:

$$\left(\frac{d\tau}{M}\right)^2 = \left(\frac{dT}{M}\right)^2 - \left(\frac{dr}{M}\right)^2 - \left(\frac{r}{M}\right)^2 d\Phi^2 \qquad \text{(flat spacetime)} \quad (140)$$

F. Will the gravitational effects of the baseball's spin be noticeable to the fielder who catches the spinning ball?

G. Use equation (12) and the values of M and a calculated in Items B and C to calculate the $\omega_\text{framedragging}$ function that expresses the "frame dragging effect" of this baseball at its surface. How many orders of magnitude is this greater or less than ω_rotation, the angular speed of the spinning baseball.

13. Spinning electron a naked singularity?

The electron is a quantum particle; Einstein's classical (non-quantum) general relativity cannot predict results of experiments with the electron. Ignore these limitations in this exercise; treat the electron as a classical particle.

The electron has mass $m_\text{e} = 9.12 \times 10^{-31}$ kilogram and spin angular momentum $J_\text{e} = \hbar/2$, where the value of "h-bar," $\hbar = 1.05 \times 10^{-34}$ kilogram-meter^2/second. Calculate the numerical value of the quantity a/m_e for the electron. If the electron is a point particle, then the Doran metric describes the electron all the way down to (but not including) $r = 0$. *Questions:* Is the electron a spinning black hole? Is the electron a naked singularity?

17.13 ■ REFERENCES

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