Chapter 18. Circular Orbits around the Spinning Black Hole

- *How do circular orbits around the spinning black hole differ from those around the non-spinning black hole?*
- *Can a stone in orbit close to the black hole move in a direction opposite to the black hole spin?*
- *Can circular orbits exist inside the event horizon?*
- *Does black hole spin make orbiters go faster? slower?*
- *What happens to material that circles in the accretion disk of a spinning black hole?*
- *Are quasars associated in some way with spinning black holes? If so, how can these structures emit so much radiation?*

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CHAPTER 18 Circular Orbits around the Spinning Black Hole

Edmund Bertschinger & Edwin F. Taylor *

The Mevlevi Order, founded in 1273 by Jalal ad-Din Muhammad Rumi's followers, perform their "dance" and musical ceremony known as the Sama, which involves the whirling from which the order acquires its nickname, Whirling Dervish. The Sama represents a mystical journey of humanity's spiritual ascent. Turning towards the truth, the follower grows through love, deserts ego, finds the truth, and arrives at the "Perfect."

—Wikipedia, The Free Encyclopedia [edited]

18.1 ■ REPRISE: THE DORAN METRIC

Prepare for a trip into the spinning black hole

Prepare to fall into a spinning black hole.

"What's it like to fall into a black hole?" Our first twelve chapters developed answers to this question for the *non-spinning* black hole. We could not give details until Chapter 12, because we needed the background provided by earlier chapters. "What is it like to fall into a *spinning* black hole?" Again, we cannot give details until Chapter 21, because we need the background provided by Chapters 17 through 20.

This chapter: circular orbits

But we can say this now: Falling into the spinning black hole has many more possibilities—and is much more interesting—than falling into the non-spinning black hole. To reach this conclusion we study orbits of stones and light. The present chapter examines *circular orbits* of a stone around the spinning black hole.

Most circular orbits unstable

We find that around the spinning black hole, most of the circular orbits are unstable. An unpowered spaceship can perch temporarily in an unstable circular orbit on its way to a stable circular orbit (Section 18.8).

Blazing accretion disk: a sequence of stable circular orbits

In the *accretion disk* (Section 18.9), gas and dust slowly cascade down through a series of (semi-)stable circular orbits of decreasing r, each successive orbit with slightly smaller orbital energy. Electromagnetic radiation carries away the energy difference between orbits (Section 18.9). We can detect this

*Draft of Second Edition of *Exploring Black Holes: Introduction to General Relativity* emitted energy at our location far from the black hole. Eventually however, no circular orbit exists for smaller r, and the accreted material spirals inward across the event horizon.

Doran global metric

To begin, recall the Doran global metric in the equatorial plane of the isolated spinning black hole—equations (4) and (5) in Section 17.2:

$$d\tau^2 = dT^2 - \left[\left(\frac{r^2}{r^2+a^2}\right)^{1/2} dr + \left(\frac{2M}{r}\right)^{1/2}(dT - ad\Phi)\right]^2 - \left(r^2+a^2\right)d\Phi^2 \qquad (1)$$

$$-\infty < T < \infty, \quad 0 < r < \infty, \quad 0 \leq \Phi < 2\pi \qquad \text{(Doran, equatorial plane)}$$

The black hole spin parameter $a \equiv J/M$, with J the angular momentum of the black hole (Section 17.2). The spin parameter a has the unit meter. In Query 1 of Section 17.2 you multiplied out (1) to obtain:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dT^2 - 2\left(\frac{2Mr}{r^2+a^2}\right)^{1/2} dTdr + 2a\left(\frac{2M}{r}\right)dTd\Phi \qquad (2)$$

$$+ 2a\left(\frac{2Mr}{r^2+a^2}\right)^{1/2} drd\Phi - \left(\frac{r^2}{r^2+a^2}\right)dr^2 - R^2 d\Phi^2$$

$$-\infty < T < \infty, \quad 0 < r < \infty, \quad 0 \leq \Phi < 2\pi \quad \text{(Doran, equatorial plane)}$$

Equation (6) in Box 1 defines the symbol R.

Comment 1. Heavy algebra
This chapter requires a great deal of algebra to derive many of its equations, algebra that we mostly omit. *Question:* Would more advanced mathematics—for example tensors—make these derivations simpler? *Answer:* We don't think so, but you can try!

18.2 ■ EQUATIONS OF MOTION FOR A STONE; TWO EFFECTIVE POTENTIALS

Algebra orgies lead to powerful results.

Two equations in three unknowns

Our first task is to find equations of motion for a stone in Doran coordinates. Equation (103) for E/m in Section 17.9 and equation (110) for L/m in Section 17.10 give us two linear equations in the three unknowns $dT/d\tau$, $dr/d\tau$, and $d\Phi/d\tau$. Solve them to find $dT/d\tau$ and $d\Phi/d\tau$ as functions of E/m, L/m and $dr/d\tau$. The result is two equations of motion for the stone, both of them functions of the still-undetermined expression for $dr/d\tau$. Box 1, repeated from Section 17.8, provides expressions for H, ω, β, and R in the following equations:

Box 1. Useful Relations for the Spinning Black Hole

This box repeats Box 1 in Section 17.8.

Static limit from Section 17.3:

$$r_{\mathrm{S}} = 2M \tag{5}$$

Reduced circumference from Section 17.2:

$$R^2 \equiv r^2 + a^2 + \frac{2Ma^2}{r} \tag{6}$$

Horizon function from Section 17.3:

$$H^2 \equiv \frac{1}{r^2}\left(r^2 - 2Mr + a^2\right) \tag{7}$$

$$= \frac{1}{r^2}\left(r - r_{\mathrm{EH}}\right)\left(r - r_{\mathrm{CH}}\right) \tag{8}$$

where r_{EH} and r_{CH} are r-values of the event and Cauchy horizons, respectively, from Section 17.3.

$$\frac{r_{\mathrm{EH}}}{M} \equiv 1 + \left(1 - \frac{a^2}{M^2}\right)^{1/2} \quad \text{(event horizon)} \tag{9}$$

$$\frac{r_{\mathrm{CH}}}{M} \equiv 1 - \left(1 - \frac{a^2}{M^2}\right)^{1/2} \quad \text{(Cauchy horizon)} \tag{10}$$

Ring omega from Section 17.3:

$$\omega \equiv \frac{2Ma}{rR^2} \tag{11}$$

An equivalence from Section 17.3:

$$1 - \frac{2M}{r} + R^2\omega^2 = \left(\frac{rH}{R}\right)^2 \tag{12}$$

Definition of α from Section 17.7:

$$\alpha \equiv \arcsin\left[\left(\frac{2M}{r}\right)^{1/2}\frac{a}{rH}\right] \quad (0 \leq \alpha \leq \pi/2) \tag{13}$$

Definition of β from Section 17.8:

$$\beta \equiv \left(\frac{2M}{r}\right)^{1/2}\left(\frac{r^2 + a^2}{R^2}\right)^{1/2} \tag{14}$$

Box 2 examines the values of some of these expressions at the event and Cauchy horizons.

$$\frac{dT}{d\tau} = \left(\frac{R}{rH}\right)^2\left(\frac{E - \omega L}{m}\right) + \frac{\beta R}{rH^2}\frac{dr}{d\tau} \quad \text{(equations of motion)} \tag{3}$$

$$\frac{d\Phi}{d\tau} = \frac{1}{(rH)^2}\left[\left(1 - \frac{2M}{r}\right)\frac{L}{m} + \frac{2Ma}{r}\frac{E}{m} + a\left(\frac{2Mr}{r^2 + a^2}\right)^{1/2}\frac{dr}{d\tau}\right] \tag{4}$$

Find $dr/d\tau$, the third equation of motion.

To find $dr/d\tau$ on the right sides of these equations, divide both sides of the Doran metric (1) by $d\tau^2$; into the result substitute $dT/d\tau$ and $d\Phi/d\tau$ from equations (3) and (4). Extensive algebra leads to the third equation of motion:

$$\frac{dr}{d\tau} = \pm\frac{R}{r}\left(\frac{E - V_{\mathrm{L}}^{+}}{m}\right)^{1/2}\left(\frac{E - V_{\mathrm{L}}^{-}}{m}\right)^{1/2} \quad \text{(stone)} \tag{15}$$

Here $V_{\mathrm{L}}^{\pm}(r)$ are the **effective potentials** (two of them!) for the spinning black hole:

$$\frac{V_{\mathrm{L}}^{\pm}(r)}{m} \equiv \omega\frac{L}{m} \pm \frac{rH}{R}\left(1 + \frac{L^2}{m^2R^2}\right)^{1/2} \quad \text{(stone, effective potentials)} \tag{16}$$

TWO effective potentials

The $\pm$ sign in (16) chooses between the two effective potentials, while the $\pm$ sign in (15) tells us whether the stone moves to larger or smaller r. Note that the effective potentials are not real-valued (do not exist) at values of r that make the horizon function H imaginary; namely between the event and Cauchy horizons.

QUERY 1. Effective potentials at selected r-values

Show the following:

A. The two effective potential functions become equal, $V_L^+(r) = V_L^-(r)$, at both horizons and at $r = 0$.

B. As $r/M \to \infty$, the two effective potentials become, respectively, $V_L^+(r)/m \to +1$ and $V_L^-(r)/m \to -1$.

Objection 1. *Impossible! Item B in Query 1 says that the spinning black hole has an effective potential that extends outward to infinity. No black hole, spinning or non-spinning, can possibly be that powerful.*

There is no problem with $V_L^+(r)$: Item B in Query 1 simply reaffirms that a stone far from the black hole has $V_L^+(r) \to 1$, the special relativity result in flat spacetime. For the case of $V_L^-(r)$ far from the black hole, read on!

QUERY 2. Map angular momentum of a stone when $a \to 0$

Show that when $a \to 0$ then $d\Phi \to d\phi$ and $R \to r$, so the angular momentum equation (110) in Section 17.10 reduces to the expression for the non-spinning black hole (Section 8.2):

$$\frac{L}{m} = r^2\frac{d\phi}{d\tau} \qquad \text{(non-spinning black hole)} \qquad (17)$$

QUERY 3. Expression for $dr/d\tau$ for the non-spinning black hole

A. Show that when $a \to 0$, equation (15) reduces to equation (19) in Section 8.3 for the non-spinning black hole:

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{E}{m}\right)^2 - \left(\frac{V_L}{m}\right)^2 = \left(\frac{E}{m}\right)^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{m^2r^2}\right) \qquad \text{(non-spinning BH)} \quad (18)$$

B. Show that when $a \to 0$, then $V_L^\pm(r)$ reduces to the single effective potential for a non-spinning black hole in Section 8.4:

Box 2. At the Horizons

What happens to our constants and variables at the event and Cauchy horizons? Here's a summary. (You can derive these expressions as a Query or exercise.)

In the following, the subscript H stands for the value of that quantity at either the event horizon or the Cauchy horizon.

$$R(r) \to R_{\rm H} = 2M \quad \text{(Fig. 1, Section 17.2.)} \qquad (22)$$

$$H(r) \to H_{\rm H} = 0 \qquad (23)$$

$$\omega \to \omega_{\rm H} = \frac{a}{2Mr_{\rm H}} \qquad (24)$$

$$\frac{r^2+a^2}{2Mr} \to 1 \qquad (25)$$

$$\left(\frac{E-V_{\rm L}^+}{m}\right)^{1/2}\left(\frac{E-V_{\rm L}^-}{m}\right)^{1/2} \to \frac{E-\omega_{\rm H}L}{m} \qquad (26)$$

$$\beta = \left(\frac{2M}{r}\right)^{1/2}\left(\frac{r^2+a^2}{R^2}\right)^{1/2} \to \beta_{\rm H} = 1 \qquad (27)$$

$$\frac{V_{\rm L}(r)}{m} \equiv \left(1-\frac{2M}{r}\right)^{1/2}\left(1+\frac{L^2}{m^2r^2}\right)^{1/2} \qquad \text{(non-spinning black hole)} \qquad (19)$$

Equations of motion $dT/d\tau$ and $d\Phi/d\tau$

Use expressions (15) and (16) for $dr/d\tau$ to complete the equations of motion begun with (3) and (4), and rearrange the results to give the following expressions. These extensive derivations use several expressions in Box 1.

$$\frac{dT}{d\tau} = \left(\frac{R}{rH}\right)^2\left[\frac{E-\omega L}{m} \pm \beta\left(\frac{E-V_{\rm L}^+}{m}\right)^{1/2}\left(\frac{E-V_{\rm L}^-}{m}\right)^{1/2}\right] \qquad (20)$$

$$\frac{d\Phi}{d\tau} = \frac{L}{mR^2} + \frac{\sin^2\alpha}{a}\left[\frac{E-\omega L}{m} \pm \frac{1}{\beta}\left(\frac{E-V_{\rm L}^+}{m}\right)^{1/2}\left(\frac{E-V_{\rm L}^-}{m}\right)^{1/2}\right] \qquad (21)$$

In these equations, the plus sign in front of β or $1/\beta$ corresponds to an increasing r-value and the minus sign to a decreasing r-value.

18.3 ■ USING EFFECTIVE POTENTIALS

Where to go, where to stop, where to bounce, where to stay

Every equation of motion—(15), (20), and (21)—contains the following expression, which must be real if the stone can move, or even exist, with that map energy E:

$$\left(\frac{E-V_{\rm L}^+}{m}\right)^{1/2}\left(\frac{E-V_{\rm L}^-}{m}\right)^{1/2} \qquad \text{must be real.} \qquad (28)$$

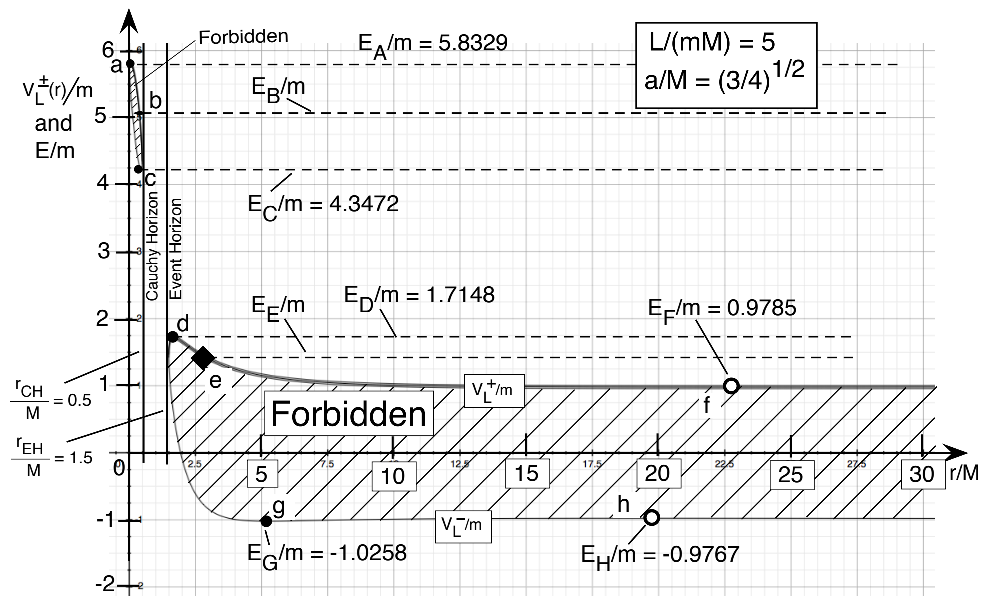

FIGURE 1 Effective potentials $V_L^+(r)$ and $V_L^-(r)$ for a stone with $L/m = 5M$ orbiting a spinning black hole with spin parameter $a/M = (3/4)^{1/2}$. Turning points (Definition 2) lie on the effective potential curves: a little filled circle at the r-value of an unstable circular orbit; a little open circle at the r-value of a stable circular orbit; a rotated little black square at a bounce point. Figure 2 shows a magnified view of effective potentials inside the Cauchy horizon.

Equations of motion must be real.

From (16), $V_L^+(r) > V_L^-(r)$ at every r-value where effective potentials exist. Expression (28) is real at these r-values when either $E > V_L^+(r)$ or $E < V_L^-(r)$. In contrast, expression (28) is imaginary in regions where map energy lies between the effective potentials, that is where $V_L^+(r) > E > V_L^-(r)$. The stone cannot move, or even exist, with map energy E in that region. We say that this is a *forbidden map energy region* (Definition 1).

Figures 1 and 2 plot the two effective potentials from (16) for given values $a/M = (3/4)^{1/2}$ and $L/(mM) = 5$, along with several values of the stone's map energy. These figures illustrate forbidden map energy regions, which we now define.

Definition: **Forbidden energy region**

DEFINITION 1. Forbidden map energy region

A **forbidden map energy region** (which we often call simply a *forbidden region*) is a region between the $V_L^-(r)$ and $V_L^+(r)$ effective potential curves on the $V_L^{\pm}(r)/m$ vs r/M plot. Why forbidden? Because if the map energy E/m of the stone did lie in this region, its equations of motion would be imaginary or complex.

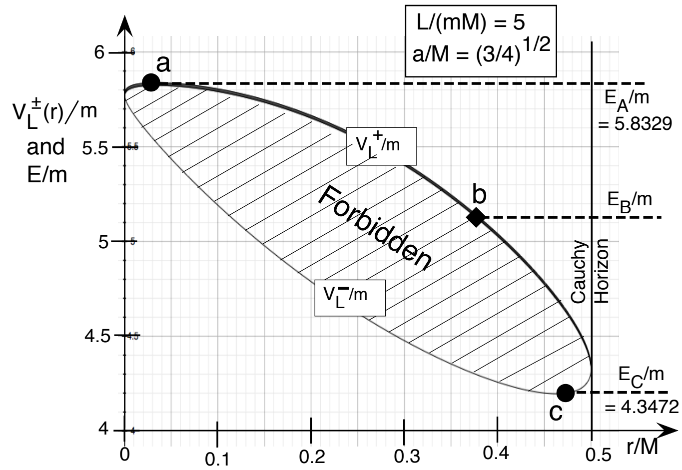

FIGURE 2 Magnified view of the pair of effective potentials in Figure 1 inside the Cauchy horizon. Little filled circles at points a and c show r-values for unstable circular orbits; the rotated filled square symbol locates a bounce point.

Figures 1 and 2 exhibit not only forbidden map energy regions but also what we call *turning points*, which we subdivide into *circle points* and *bounce points*. (Recall similar definitions in Section 8.4 for the non-spinning black hole.)

DEFINITION 2. Turning point, circle point, and bounce point

Definition: **Turning point**

A **turning point** is a point on the $V_{\rm L}^{\pm}(r)/m$ vs r/M curve for which either $E = V_{\rm L}^{+}$ or $E = V_{\rm L}^{-}$. At a turning point $dr/d\tau = 0$—equation (15). Examples: points labeled a through h in Figure 1. We distinguish two kinds of turning points: circle point and bounce point.

Definition: **Circle point**

A **circle point** is a turning point at a maximum or minimum of the effective potential. At a circle point $dr/d\tau = 0$ and remains zero, at least temporarily, so a stone at a circle point is in an unstable or stable circular orbit. We plot a circle point as a little filled circle (at an unstable circular orbit) or a little open circle (at a stable circular orbit). See Definition 3. Examples: points labeled a, c, d, f, g, and h in Figure 1.

Definition: **Bounce point**

A **bounce point** is a turning point that is *not* at a maximum or minimum of the effective potential. At a bounce point, $dr/d\tau = 0$ for an instant but then reverses sign. We plot a bounce point as a little filled rotated square (a diamond). Examples: points b, and e in Figure 1 and point b in Figure 2.

Return to the circle point. There are two different kinds of circular orbits: *stable* and *unstable*.

DEFINITION 3. Stable and unstable circular orbits

Definition: **Stable circular orbit**

A stone occupies a **stable circular orbit** when it lies at a circle point in the $V_{\rm L}^{\pm}(r)/m$ *vs.* r/M diagram at which displacement either right or left, while keeping E/m constant puts it inside a forbidden map energy region. We plot a stable circular orbit location as a little open circle. Examples: points f and h in Figure 1.

Definition: **Unstable circular orbit**

The stone occupies an **unstable circular orbit** when it lies at a circle point in the $V_{\rm L}^{\pm}(r)/m$ *vs.* r/M diagram at which displacement either right or left, while keeping E/m constant does *not* put it inside a forbidden map energy region in that diagram. We often call an unstable circular orbit a **knife-edge** orbit to emphasize its instability. We plot an unstable circular orbit location as a little filled circle. Examples: points a, c, d, and g in Figure 1.

Table 18.1 expresses these definitions analytically. Table 18.2 lists details for turning points in Figures 1 and 2.

?

Objection 2. *Stop! Figure 1 shows circular orbits g and h at negative map energies; negative-energy orbits cannot exist. Everyone knows that energy must be a positive quantity. Circular orbits at points g and h in Figure 1 cannot exist!*

!

Beware of phrases such as "everyone knows." First, even in Newton's mechanics we can choose the zero of gravitational energy at any height in a gravitational field; then the potential energy of a stationary stone at any lower height becomes negative. Second, in general relativity the map energy is typically not measurable; it's a constant of motion that can be negative without physical consequence. Chapter 19 gives formulas for the energy of a free stone measured in a local inertial frame, which yields a positive frame energy even for a negative map energy.

?

Objection 3. *Phooey! Your whole analysis is a fantasy! Even Figures 1 and 2 describe structures inside the event horizon that no observer can possibly see or measure. Physical theory has to be "falsifiable:" it must be vulnerable to disproof by observation.*

!

In principle (or possibly in the future) we can observe and measure these results: Someone who rides a free stone inward across the event horizon can make measurements to verify results of this theory. Let an astronaut initially outside the event horizon have positive map energy above the forbidden map energy region. Chapter 21 describes a set of maneuvers inside the event horizon that brings this astronaut back out through the event horizon with negative map energy. Then she can report on her measurements during her earlier descent. More generally, a scientific theory often predicts what we will observe when new conditions or improved equipment become available.

TABLE 18.1 Classification of Circular Orbits using $V_L^{\pm}$

When $E = V_L^+$ and	When $E = V_L^-$ and
$dV_L^+/dr = 0$, then the orbit is	$dV_L^-/dr = 0$, then the orbit is
STABLE if $d^2V_L^+/dr^2 > 0$, but	**STABLE** if $d^2V_L^-/dr^2 < 0$, but
UNSTABLE if $d^2V_L^+/dr^2 < 0$.	**UNSTABLE** if $d^2V_L^-/dr^2 > 0$.

TABLE 18.2 Map Energies of Circular Orbits with $L/(mM) = 5$ and $a/M = (3/4)^{1/2}$ (Figures 1 and 2). Circle orbit Type numbers from equations (31)–(38).

Circular orbit letter: r/M-value	Type: E/m-value, unstable or stable
Point a: $r/M = 0.0341$	Type 1: $E_A/m = 5.8329$, unstable
Point c: $r/M = 0.4660$	Type 2: $E_C/m = 4.3472$, unstable
Point d: $r/M = 1.6963$	Type 1: $E_D/m = 1.7148$, unstable
Point f: $r/M = 22.744$	Type 1: $E_F/m = 0.9785$, STABLE
Point g: $r/M = 5.2469$	Type 3: $E_G/m = -1.0258$, unstable
Point h: $r/M = 19.7855$	Type 3: $E_H/m = -0.9767$, STABLE

QUERY 4. Application of Table 18.1
Which entries in Table 18.1 apply to circular orbits around the *non-spinning* black hole?

Comment 2. Two non-communicating regions
What goes on below the forbidden map energy region in Figure 1? This figure implies, and equations show, that this forbidden map energy region extends as far as $r \to \infty$. Apparently both stable and unstable circular orbits exist below the forbidden map energy region. We have verified that no stone can exist in the forbidden map energy region, and Chapter 20 demonstrates that light is similarly forbidden to travel directly between an upper and lower region. *Result:* two regions that cannot communicate directly with one another.

Map energy is negative below the forbidden map energy region, but that need not worry us: nobody observes or measures map energy. You can show that almost every (but not every) local inertial frame (defined in Chapter 17) that exists above the forbidden region can exist below the forbidden region. Indeed, for almost every (but not every) event that occurs at T, r, Φ above the forbidden map energy region an event can occur at T, r, Φ below this region.

Where are events that occur below the forbidden map energy region? Is there an entire separate Universe there, a Universe we cannot see from ours? Can we get to that Universe? Can we come back? Answers in Chapter 21!

18.4 ■ FOUR TYPES OF CIRCULAR ORBITS

How many circular orbits, and of what types?

Multiple circular orbits at the same r

The spinning black hole has (many!) more surprises for us. One of these is the existence of *multiple distinct circular orbits at the same r-value.* Figure 3 shows

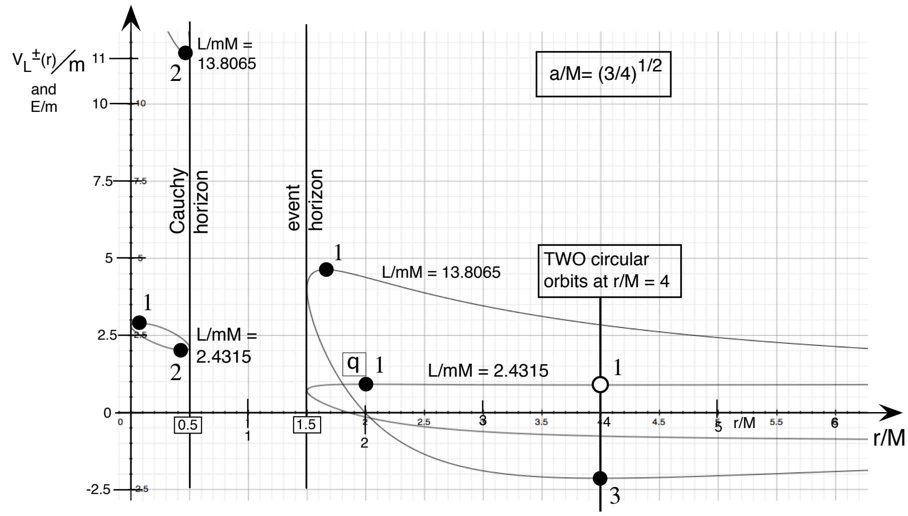

FIGURE 3 Two different effective potentials for a spinning black hole, each of which leads to a circular orbit at $r/M = 4$, one stable and the other unstable. Numbers 1 through 3 indicate circular orbit Types from equations (31) through (38). Figure 4 shows the possibility of *four* circular orbits at $r/M = 4$. The label q refers to the same orbit in Figure 13. In order to display all turning points clearly, we do not shade forbidden map energy regions in this plot.

two different effective potentials for a spinning black hole with $a/M = (3/4)^{1/2}$ that lead to two different circular orbits at $r/M = 4$. Note that these occur for two different (positive) values of the map angular momentum $L/(mM)$. Even more astonishing, Figure 4 shows a total of four circular orbits at $r/M = 4$, two for the pair of positive values of $L/(mM)$ in Figure 3 plus two more for the corresponding *negative* values of these map angular momenta.

QUERY 5. Number of circular orbits at given r: Newton and the non-spinning black hole

Both the non-spinning black hole and the spherically symmetric center of attraction of Newton's mechanics are spherically symmetric, which allows an unlimited number of differently oriented $[r, \Phi]$ slices through the centers of these objects on which circular orbits can exist. On a single one of these slices,

A. Newton: For what values of r do circular orbits exist?
B. Newton: How many distinct circular orbits exist at that r?
C. Newton: If your answer to Item B predicts more than one circular orbit, what determines the difference between circular orbits at that r-value?
D. Repeat Items A through C for the non-spinning black hole.

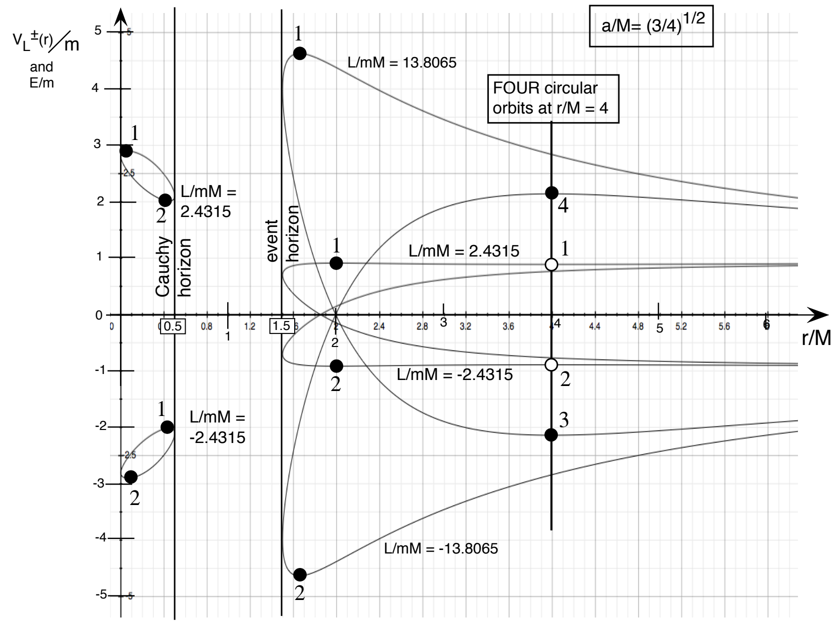

FIGURE 4 Four different effective potentials for a spinning black hole with $a/M = (3/4)^{1/2}$, all of which have circular orbits at $r/M = 4$, two stable and two unstable. This figure adds to Figure 3 effective potential curves for negative values of the stone's angular momentum. Effective potentials for $L/(mM) = \pm 13.8065$ inside the Cauchy horizon lie beyond the vertical range of this plot. The number on each circular orbit symbol gives its Type. We do not shade forbidden map energy regions, in order to display all turning points clearly.

No circular orbits between event horizon and Cauchy horizon

We want to derive general expressions for map energies and map angular momenta of circular orbits around a spinning black hole. Definition 2 tells us that a circular orbit occurs at r-values for which either $E = V_L^+(r)$ and $dV_L^+(r)/dr = 0$ or $E = V_L^-(r)$ and $dV_L^-(r)/dr = 0$. Between the event horizon and the Cauchy horizon the third equation of motion (15) is imaginary, so carries no physical meaning there. In addition, circular orbits near the horizons lie separated in r-value from the horizons, illustrated in Figures 1 and 2 (Query 6). Now we turn these qualitative observations into analytical and numerical results.

QUERY 6. Circular orbits avoid horizons and the singularity.

In this Query you show that the circular orbits do not exist at the singularity or at the two horizons.

A. Show that the slope of each effective potential function increases without limit ($dV_L^{\pm}/dr \rightarrow \infty$) at both horizons and at $r = 0$.

B. The slope of the effective potential is zero at the r-value of every circular orbit. Item A tells us that this slope is vertical at three r-values: both horizons and the singularity. The effective potentials are continuous at these three r-values and at the nearest circle points. Circular orbits are impossible at each horizon and at the singularity.

Generating equation for circular orbits

To find all r-values of circular orbits, set the derivatives of the two functions $V_{\mathrm{L}}^{+}(r)$ and $V_{\mathrm{L}}^{-}(r)$ equal to zero and from them derive an equation that contains all terms containing L. *Result:* an expression for the value of L for a circular orbit (if any) at that r-value. Equations (29) and (30) are the generating equations for circular orbits.

$$\pm AL = B\left(L^2 + m^2R^2\right)^{1/2} + \frac{C}{\left(L^2 + m^2R^2\right)^{1/2}} \tag{29}$$

where the $\pm$ symbol matches that in the superscript of $V_{\mathrm{L}}^{\pm}$, and symbols A, B, C stand for the following functions of a and r:

$$A \equiv -\frac{d\omega}{dr}, \qquad B \equiv \frac{d}{dr}\left(\frac{rH}{R^2}\right), \qquad C \equiv m^2\left(r - \frac{Ma^2}{r^2}\right)\frac{rH}{R^2} \tag{30}$$

QUERY 7. *Optional:* Derive the generating equation for circular orbits.
Carry out the derivation of equations (29) and (30).

QUERY 8. Pairs of solutions

A. Show that when $L = +L_1$ is a solution of (29) with $E = V_{\mathrm{L}}^{+}(r)$, then $L = -L_1$ is also a solution at the same r with $E = V_{\mathrm{L}}^{-}(r)$. Conclusion: *Circular orbits come in pairs.*
B. Identify all such pairs in Figure 4.
C. Show also that the orbits in a pair are either both stable or both unstable. Hint: Use a symmetry argument.

Four circular orbit types

Solve equation (29) for L/m as a function of r. Lots of algebra yields two solutions for L/m. For each of these solutions set $E = V_{\mathrm{L}}^{+}$ or $E = V_{\mathrm{L}}^{-}$ at this value of r. Result: four types of circular orbits described by the following equations. (Section 18.5 defines the labels on the right sides of these equations.)

TYPE 1 for $E = V_{\rm L}^{+}(r)$

$$\left(\frac{L}{m}\right)_{\rm Type\,1} = \left(\frac{M}{r}\right)^{1/2} \frac{r^2 + a^2 - 2a(Mr)^{1/2}}{\left[r^2 - 3Mr + 2a(Mr)^{1/2}\right]^{1/2}} \quad \text{(forward, prograde)} \quad (31)$$

$$\left(\frac{E}{m}\right)_{\rm Type\,1} = \frac{V_{\rm L}^{+}(r)}{m} = \frac{r^2 - 2Mr + a(Mr)^{1/2}}{r\left[r^2 - 3Mr + 2a(Mr)^{1/2}\right]^{1/2}} \quad \text{(forward, prograde)} \quad (32)$$

TYPE 2 for $E = V_{\rm L}^{-}(r)$

$$\left(\frac{L}{m}\right)_{\rm Type\,2} = -\left(\frac{L}{m}\right)_{\rm Type\,1} \quad \text{(backward, prograde)} \quad (33)$$

$$\left(\frac{E}{m}\right)_{\rm Type\,2} = \frac{V_{\rm L}^{-}(r)}{m} = -\left(\frac{E}{m}\right)_{\rm Type\,1} \quad \text{(backward, prograde)} \quad (34)$$

TYPE 3 for $E = V_{\rm L}^{-}(r)$

$$\left(\frac{L}{m}\right)_{\rm Type\,3} = \left(\frac{M}{r}\right)^{1/2} \frac{r^2 + a^2 + 2a(Mr)^{1/2}}{\left[r^2 - 3Mr - 2a(Mr)^{1/2}\right]^{1/2}} \quad \text{(backward, retrograde)} \quad (35)$$

$$\left(\frac{E}{m}\right)_{\rm Type\,3} = \frac{V_{\rm L}^{-}(r)}{m} = -\frac{r^2 - 2Mr - a(Mr)^{1/2}}{r\left[r^2 - 3Mr - 2a(Mr)^{1/2}\right]^{1/2}} \quad \text{(backward, retrograde)} \quad (36)$$

TYPE 4 for $E = V_{\rm L}^{+}(r)$

$$\left(\frac{L}{m}\right)_{\rm Type\,4} = -\left(\frac{L}{m}\right)_{\rm Type\,3} \quad \text{(forward, retrograde)} \quad (37)$$

$$\left(\frac{E}{m}\right)_{\rm Type\,4} = \frac{V_{\rm L}^{+}(r)}{m} = -\left(\frac{E}{m}\right)_{\rm Type\,3} \quad \text{(forward, retrograde)} \quad (38)$$

QUERY 9. Pairs of map energies and map angular momenta
Show that Figure 4 illustrates the results of Query 8. As a result, show that Type 1 implies the existence of Type 2 and also that Type 3 implies the existence of Type 4.

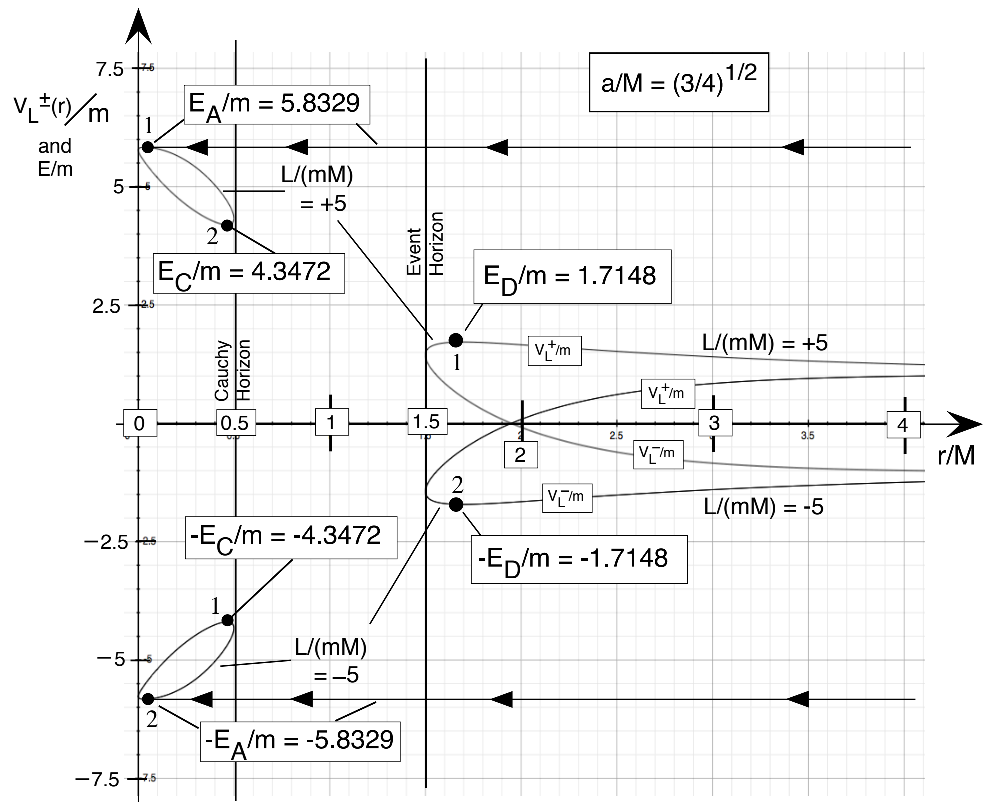

FIGURE 5 Extension of Figure 1 to positive and negative values of angular momentum: $L/(mM) = \pm 5$ to show the relation between Types 1 and 2 circular orbits. Reverse the sign of L to reverse the sign of E at the same r-value (Query 8). A stone of map energy $E_{\rm A}$ and $L/(mM) = +5$ (horizontal line at the top of the plot) goes into a Type 1 circular orbit, which is distinct from the Type 2 circular orbit with $E = -E_{\rm A}$ at the same r (bottom of the plot). Similarly for other circular orbits at the same r-values but of different types.

QUERY 10. Other pairs of solutions

A. Show that when we change ω to $-\omega$ in (16), then $V_{\rm L}^{+}(r)$ becomes $-V_{\rm L}^{-}(r)$ and $V_{\rm L}^{-}(r)$ becomes $-V_{\rm L}^{+}(r)$.

B. From Item A and equation (11), show that when we change a to $-a$ in (31) and (32)—that is, when the black hole spins in the opposite sense—then a circular orbit of Type 1 becomes a circular orbit of Type 3 at the same r-value.

C. Likewise, show that when we change a to $-a$ in (33) and (34), then a Type 2 circular orbit becomes a Type 4 circular orbit at the same r-value.

Comment 3. Convenient to define four types of circular orbits
Queries 8 through 10 show that reversing the sign of the orbital angular momentum of a stone and/or the spin parameter of the black hole yield new circular orbits. Result: We can derive from Type 1 the other three types of circular orbits for a given absolute value of the black hole spin parameter $|a/M|$. It is informative, however, to consider each of the four types separately.

How many circular orbits at a given r?

How many circular orbits exist at r for the spinning black hole with a given value of a/M? To answer this question, look at equations (31) through (38). Map energy and map angular momentum of the stone must be real, so orbits exist only at r-values where functions inside the square roots in the denominators of these equations are positive:

$$r^2 - 3Mr + 2a(Mr)^{1/2} > 0 \quad \text{(where orbits exist for Types 1 and 2)} \quad (39)$$

$$r^2 - 3Mr - 2a(Mr)^{1/2} > 0 \quad \text{(where orbits exist for Types 3 and 4)} \quad (40)$$

How many circular orbits at various values of r?

From these inequalities we can sort out the r-locations at which different circular orbit types exist. As $r \to \infty$, both inequalities (39) and (40) are satisfied, so all four types of circular orbits exist far from the black hole. At some intermediate values of r (but outside the event horizon) inequality (39) is satisfied, but inequality (40) is not satisfied, so only prograde orbits exist at those r-values. Only prograde orbits exist inside the Cauchy horizon, as in Figure 2 (Table 1). Finally, a region exists in which even $r^2 - 3Mr + 2a(Mr)^{1/2} < 0$, so no circular orbits can exist in that region. Each of these conditions depends on the value of the black hole's spin parameter a/M. Figure 6 plots these results for different values of a/M.

Comment 4. Orbits of light
The r-values where equations (39) and (40) become equalities are places where the denominators vanish in equations (31) through (38). Multiply both sides of each of these equations through by m, the mass of the orbiting stone. Then circular orbits can exist with the corresponding values of E and L if, and only if, $m \to 0$. Therefore, these are r-values for circular orbits of *light* (Figure 6). Chapter 20 explores orbits of light in greater generality.

Which of these circular orbits are stable? Figures such as 2 and 4 preview the result that *all* circular orbits inside the Cauchy horizon are unstable. Sections 18.6 through 18.8 pursue the stability question after we investigate further the differences among Types 1 through 4 circular orbits outside the event horizon. In this process we will finally define *prograde vs. retrograde* circular orbits and *forward vs. backward* circular orbits.

18.5 ■ MAP $dT/d\tau$ AND MAP $d\Phi/dT$ FOR CIRCULAR ORBITS

Add Doran Φ and T to the specification of circular orbits.

$dT/d\tau$ and $d\Phi/d\tau$ for circular orbits

Look again at equations of motion (20) and (21). The final term on the right side of each of these equations equals zero for the special case of a circular orbit, for which either $E = V_{\rm L}^{+}$ or $E = V_{\rm L}^{-}$:

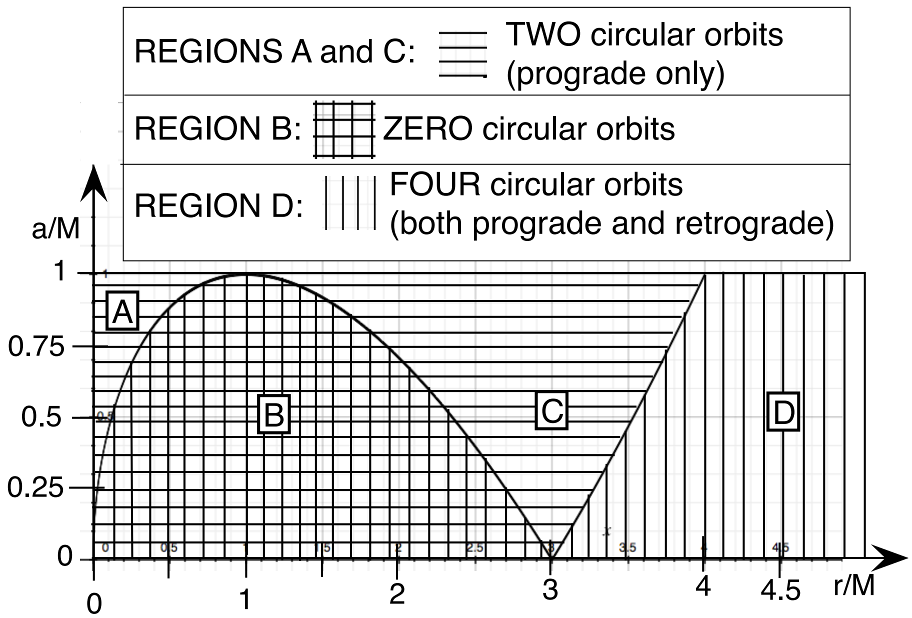

FIGURE 6 This figure uses inequalities (39) and (40) to answer the question, "How many circular orbits of a stone exist at a given r for different values of the spin parameter a/M?" In Region B, zero circular orbits exist. In Regions A and C, only Type 1 and Type 2 (prograde) circular orbits exist. In Region D, all four types of circular orbits exist. Circular orbits along the curves that divide regions are photon orbits (Comment 4).

$$\frac{dT}{d\tau} = \left(\frac{R}{rH}\right)^2 \left(\frac{E-\omega L}{m}\right) \qquad \text{(circular orbit)} \qquad (41)$$

$$\frac{d\Phi}{d\tau} = \frac{L}{mR^2} + \frac{\sin^2\alpha}{a}\left(\frac{E-\omega L}{m}\right) \qquad \text{(circular orbit)} \qquad (42)$$

Four types of $dT/d\tau$ and $d\Phi/d\tau$

Now plug values of L/m and E/m from (31) through (38) into equations (41) and (42). This leads to expressions for $dT/d\tau$ and $d\Phi/d\tau$ for the four types of circular orbits in in Section 18.4:

TYPE 1 for $E = V_L^+$

$$\left(\frac{dT}{d\tau}\right)_{\text{Type 1}} = \frac{r + a\,(M/r)^{1/2}}{\left[r^2 - 3Mr + 2a(Mr)^{1/2}\right]^{1/2}} \qquad \text{(forward, prograde)} \qquad (43)$$

$$\left(\frac{d\Phi}{dT}\right)_{\text{Type 1}} = \frac{(Mr)^{1/2}}{r^2 + a\,(Mr)^{1/2}} \qquad \text{(forward, prograde)} \qquad (44)$$

TYPE 2 for $E = V_L^-$

$$\left(\frac{dT}{d\tau}\right)_{\text{Type 2}} = -\left(\frac{dT}{d\tau}\right)_{\text{Type 1}} \qquad \text{(backward, prograde)} \qquad (45)$$

$$\left(\frac{d\Phi}{dT}\right)_{\text{Type 2}} = +\left(\frac{d\Phi}{dT}\right)_{\text{Type 1}} \qquad \text{(backward, prograde)} \qquad (46)$$

TYPE 3 for $E = V_L^-$

$$\left(\frac{dT}{d\tau}\right)_{\text{Type 3}} = \frac{-r + a(M/r)^{1/2}}{\left[r^2 - 3Mr - 2a(Mr)^{1/2}\right]^{1/2}} \qquad \text{(backward, retrograde)} \qquad (47)$$

$$\left(\frac{d\Phi}{dT}\right)_{\text{Type 3}} = \frac{(Mr)^{1/2}}{-r^2 + a\,(Mr)^{1/2}} \qquad \text{(backward, retrograde)} \qquad (48)$$

TYPE 4 for $E = V_L^+$

$$\left(\frac{dT}{d\tau}\right)_{\text{Type 4}} = -\left(\frac{dT}{d\tau}\right)_{\text{Type 3}} \qquad \text{(forward, retrograde)} \qquad (49)$$

$$\left(\frac{d\Phi}{dT}\right)_{\text{Type 4}} = +\left(\frac{d\Phi}{dT}\right)_{\text{Type 3}} \qquad \text{(forward, retrograde)} \qquad (50)$$

Note: Equations for $d\Phi/dT$, with dT in the denominator, are not typographical errors: We choose to solve for $d\Phi/dT$, not for $d\Phi/d\tau$, for two reasons: *Minor reason*: Equations for $d\Phi/dT$ are simpler than equations for $d\Phi/d\tau$. *Major reason*: This choice simplifies the categories. Type 1 and 2 circular orbits (labeled prograde) always have $d\Phi/dT > 0$, while Type 3 and 4 circular orbits (labeled retrograde) always have $d\Phi/dT < 0$.

QUERY 11. Plus or minus? Signs of important expressions

A. From the requirement that $r^2 - 3Mr - 2a(Mr)^{1/2} > 0$ for Types 3 and 4 circular orbits, show that $-r^2 + a(Mr)^{1/2} < 0$.

B. As a result, show that for Types 3 and 4 circular orbits, we have $d\Phi/dT < 0$ for all values of r.

C. Show that $(dT/d\tau)_3 < 0$ for all values of r, so $(dT/d\tau)_4 > 0$ for all values of r.

This analysis leads to definitions of *prograde* and *retrograde* orbits.

Prograde and **retrograde** orbits

DEFINITION 4. Prograde and retrograde orbits

We divide circular orbits into two classes, **prograde** and **retrograde**. In a prograde orbit the stone "revolves in the direction that the black hole rotates" in global Doran coordinates so that $d\Phi/dT > 0$, while in a retrograde orbit the stone revolves in the opposite direction, $d\Phi/dT < 0$. Note that the condition $d\Phi/dT = 0$ for the raindrop worldline (Section 17.7) marks the separation between prograde and retrograde orbits. As shown in Figure 6, retrograde orbits exist only outside the event horizon, while prograde orbits exist inside the Cauchy horizon as well as outside the event horizon.

?

Objection 4. *Your definitions of prograde and retrograde orbits are nothing but manipulations of Doran map coordinates Φ and T. You keep saying that we cannot observe map coordinates directly. Worse: Except for wristwatch time τ, this chapter uses **only** map coordinates. Your messy results tell us nothing about what we can **see** and **measure** as we move near a spinning black hole. Stop wasting our time!*

!

Nice objection! We use global constants of motion to discover possible motions of a stone. For example, we now know how many circular orbits—zero, two, or four—can exist at each r-value around a black hole with given spin parameter a. This significant achievement says nothing whatsoever about what you will *see* as you ride an unpowered rocket ship in any circular orbit. Such predictions require analysis of orbits of light near the spinning black hole. Hang on: Visual results arrive in future chapters!

Forward or backward orbits from sign of $dT/d\tau$

The other pair of labels attached to circular orbits, *forward* or *backward*, derive from the sign of $dT/d\tau$. We have chosen the stone's wristwatch time τ to increase—to make $d\tau$ positive—as the stone proceeds along its worldline (Comment 7, Section 1.11). So the sign of dT determines the sign of $dT/d\tau$. If $dT/d\tau > 0$, then T also runs forward along the worldline of that stone. In contrast if $dT/d\tau < 0$ then T runs backward along that worldline. This leads to definitions of forward and backward orbits.

Definition: **forward** and **backward** orbits

DEFINITION 5. Forward and backward orbits

Along a **forward orbit**, $dT/d\tau > 0$, so both T and τ increase as the stone proceeds along its worldline. Along a **backward orbit**, $dT/d\tau < 0$, so τ increases and T decreases as the stone proceeds along its worldline.

The concept of a global rain T (or global Schwarzschild t for a non-spinning black hole) that runs backward along a stone's worldline is

TABLE 18.3 Signs of circular orbit quantities

Type	$E =$	E/m	L/m	$dT/d\tau$	$d\Phi/dT$	$d\Phi/d\tau$
1	V_L^+	$\pm$	$\pm$	$+$	$+$	$+$
2	V_L^-	$\mp$	$\mp$	$-$	$+$	$-$
3	V_L^-	$-$	$+$	$-$	$-$	$+$
4	V_L^+	$+$	$-$	$+$	$-$	$-$

Types 1 and 2 for L/m and E/m: Upper sign for orbits outside the event horizon, either sign for orbits inside the Cauchy horizon. Type 3 and 4 orbits exist only outside the event horizon.

nothing new. Figure 8 in Section 3.7 displayed the worldline of Stone B inside the event horizon along whose worldline Schwarzschild global t runs backward. No contradiction results; nobody measures these global coordinate differences.

Global T can run either forward or backward along a worldline.

For the spinning black hole there are two new results: *First new result:* The orbits that run forward and backward in T come in pairs: if one exists, the other exists at the same r, with opposite signs of E/m and L/m—equations (32) through (37). *Second new result:* For a spinning black hole, global T can run backward along a stone's worldline even outside the event horizon, indeed, all the way out: $r/M \to \infty$.

Comment 5. ALWAYS forward? ALWAYS backward?
Are orbits with $E = V_L^+(r)$ *always* forward? Are orbits with $E = V_L^-(r)$ *always* backward? Yes to both questions—at least for circular orbits. These results follow from (31) through (38) and (43) through (50). Can you fill in the argument?

QUERY 12. Orbit pairs

A. Show that the signs of $dT/d\tau$ and $d\Phi/dT$ in Table 18.3 agree with Definitions 4 and 5.
B. Show that for each pair of circular orbits in Query 8, one orbit is forward, the other is backward.
C. Show that for eqch pair of circular orbits in Query 10, one orbit is prograde, the other is retrograde.

QUERY 13. More signs of important expressions

A. Use Table 18.3 and the signs of L/m and E/m to verify the assignment of Types to the 6 circular orbits listed in Table 18.2.
B. Verify the signs of L/m and E/m in Table 18.3. Hint: To show that both signs are possible for Types 1 and 2, examine Point c in Table 18.2.
C. Verify the signs of $dT/d\tau$ and $d\Phi/d\tau$ in Table 3 using equations (43) to (50) and Query 11.

QUERY 14. Elapsed ΔT and $\Delta\tau$ for one circular orbit

A. Define one complete circular orbit to have $\Delta\Phi = 2\pi$. Use equations (44), (46), (48), and (50) to find the following expression for ΔT, the advance of Doran global T-coordinate, during one circular orbit:

$$\Delta T(\text{one orbit}) = \pm 2\pi \left[\frac{\pm r^2 + a(Mr)^{1/2}}{(Mr)^{1/2}}\right] = \pm 2\pi M \left[\pm\left(\frac{r}{M}\right)^{3/2} + \frac{a}{M}\right] \qquad (51)$$

The $\pm$ sign outside the square brackets comes from $\pm\Delta T$ for forward and backward orbits and the $\pm$ sign inside the square bracket for prograde and retrograde orbits.

B. Next, define one complete circular orbit to have $\Delta\Phi = +2\pi$ if $d\Phi/d\tau > 0$ but $\Delta\Phi = -2\pi$ if $d\Phi/d\tau < 0$. Then use all eight equations (43) through (50) to find the elapsed wristwatch time $\Delta\tau$:

$$\Delta\tau(\text{one orbit}) = 2\pi r \left[\frac{r^2 - 3Mr \pm 2a(Mr)^{1/2}}{Mr}\right]^{1/2} = 2\pi r \left[\frac{r}{M} - 3 \pm \frac{2a}{(Mr)^{1/2}}\right]^{1/2} \qquad (52)$$

with the plus for prograde and the minus for retrograde orbits.

C. Show that for the non-spinning black hole, equation (52) reduces to equation (37) in Section 8.5. What happens to the $\pm$ sign in (52) in this reduction?

D. Answer one of the initial questions on the first page of this chapter: *Does black hole spin make orbits go faster? slower?* Pay special attention to the meaning(s?) of the word "go" in that question.

In the following three sections we examine which circular orbits are stable and which are unstable: Section 18.6 for Newton's circular orbits; Section 18.7 for circular orbits around the non-spinning black hole; Section 18.8 for circular orbits around the spinning black hole.

Why do we care about *stable* circular orbits? Why are they important? Stable circular orbits are important to us for two primary reasons:

WHY ARE STABLE CIRCULAR ORBITS IMPORTANT?

1. A stone perched at the peak of the effective potential does not stay there long, so you do not observe unstable circular orbits in Nature. In contrast, the accretion disk around the spinning black hole (Section 18.9) consists of a series of nested stable circular orbits which a stone occupies in sequence as it radiates away its loss of orbital energy.
2. When we carry out an exploration program of the spinning black hole (Chapter 19), we can temporarily perch our unpowered spaceship in an unstable circular orbit on our way to somewhere else. "Somewhere else" is often a stable circular orbit, from which we can make relaxed observations without worry about falling off the effective potential maximum.

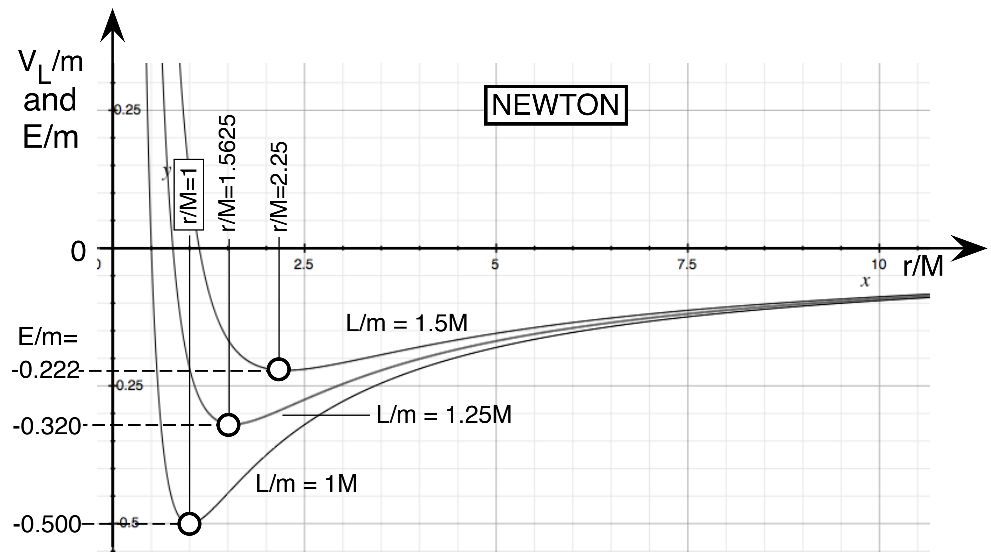

FIGURE 7 Examples of effective potentials and the radii of Newton's stable circular orbits around a point mass. A stable orbit (little open circle) exists at the minimum of each effective potential curve. The area under each effective potential is a forbidden map energy region for the stone with that angular momentum.

18.6 ■ STABILITY OF NEWTON'S CIRCULAR ORBITS

Angular momentum makes the world go 'round.

Newton: Total energy

Begin the analysis of Newton's circular orbits with his expression for the total energy (kinetic plus potential) of a stone in a central gravitational field:

$$E = \frac{1}{2}mv^2 - \frac{mM}{r} \qquad \text{(Newton, conservation of energy)} \qquad (53)$$

Newton's force law $F = ma$ demands that in a circular orbit the inward gravitational force $-mM/r^2$ equals mass m times the inward acceleration $-v^2/r$:

$$-\frac{mM}{r^2} = -\frac{mv^2}{r} \quad \text{so} \quad r^2v^2 = Mr \quad \text{(Newton force law, circular orbit)} \quad (54)$$

Newton: Angular momentum of a circular orbit

Newton defines the angular momentum of a stone in a circular orbit as its radius r times its tangential linear momentum mv:

$$L \equiv mrv \qquad \text{(Newton, circular orbit)} \qquad (55)$$

so that from (54):

$$L = m(Mr)^{1/2} \qquad \text{(Newton, circular orbit)} \qquad (56)$$

Figure 7 suggests that total orbital energy decreases with decreasing radius of the stable circular orbit. To check this, find an expression for v^2 from (55) and substitute the result into (53), thereby defining the effective potential:

$$V_L(r) \equiv \frac{L^2}{2mr^2} - \frac{mM}{r} \qquad \text{(Newton, effective potential)} \qquad (57)$$

Newton: Total energy of a circular orbit

Now substitute for L from (56) and rearrange the result to yield the energy of a stone in a circular orbit as a function of the radius of that orbit:

$$E = V_L(r) = \frac{1}{2}\frac{mM}{r} - \frac{mM}{r} = -\frac{1}{2}\frac{mM}{r} \qquad \text{(Newton, circular orbit energy)} \qquad (58)$$

Newton's conclusion: Every circular orbit is stable, all the way down to $r = 0$.

Figure 7 and our accompanying algebraic analysis tell us that Newton's effective potential has only one zero-slope point, and that one point is at a minimum. Definition 3 then tells us that *in Newton's mechanics EVERY circular orbit is stable.* More: Newton's circular orbits are stable all the way down to $r = 0$, or until the stone strikes the surface of a spherically symmetric center of attraction.

Add a little friction.

Now suppose that a stone in a circular orbit encounters a little friction—perhaps from dust or a rarified atmosphere. This friction converts some orbital energy into heat, electromagnetic radiation, or other forms of energy. Where does this converted energy come from? For Newton the only source is the orbital energy of the stone. We analyze the result with a simple model: Assume that this loss of energy per orbit is minuscule, so the stone's orbit remains circular, but its radius changes slightly. How can we track changes in energy, angular momentum, and radius of the orbit during this process? Begin to answer these questions by differentiating both sides of (58):

$$\frac{dE}{dr} = +\frac{1}{2}\frac{mM}{r^2} \qquad \text{(sequence of Newton's circular orbits)} \qquad (59)$$

Similarly, differentiate both sides of (56):

$$\frac{dL}{dr} = +\frac{m}{2}\left(\frac{M}{r}\right)^{1/2} \qquad \text{(sequence of Newton's circular orbits)} \qquad (60)$$

Circular orbit E, L, and r all decrease.

Figure 7 shows what equations (59) and (60) tell us, namely that when the energy of the circular orbit decreases, the angular momentum also decreases, as does the radius of the orbit.

Equations (59) and (60) imply that energy and angular momentum can *change*. How can this be?

External force: friction

The stone's energy and angular momentum are constant for free-fall motion, but they change if an external force is applied to the stone, whether this force arises from a rocket or from friction in an accretion disk. For a circular orbit, r, E, and L are all related. As E and L change, the radius of the circular orbit changes. To see how, think of an incremental change ΔE in energy. Equation (59) then implies that r changes by the amount

$$\Delta r \approx \left(\frac{dE}{dr}\right)^{-1} \Delta E \qquad \text{(Newton AND Einstein circular orbits)} \qquad (61)$$

We can adapt (60) to express the same change in radius between stable orbits of different angular momentum:

$$\Delta r \approx \left(\frac{dL}{dr}\right)^{-1} \Delta L \qquad \text{(Newton AND Einstein circular orbits)} \qquad (62)$$

To summarize: For Newton's circular orbits, a small amount of friction decreases the energy E and the angular momentum L of the orbiting stone and causes it to move to smaller radii through a sequence of *stable* circular orbits. Why stable? Because *all* Newton's circular orbits are stable; every circular orbit nests at a minimum of an effective potential (Figure 7).

?

Objection 5. *Whoa! In this section you use the terms "radius," "energy," and "angular momentum" without modifiers. But you keep saying that these terms have no measurable meaning. Instead, you force us to use modifiers such as "map energy," "map angular momentum," "shell frame energy," and so forth. Why did you use single-word terms that you label forbidden? Follow your own rules!*

!

These distinctions—important in general relativity—do not exist in Newton's mechanics. When carefully used, everyday terms are perfectly accurate for Newton. So we have just enjoyed a short vacation from our terminology rules for general relativity. Sorry, our little vacation is now over!

Comment 6. Wide application of Definitions 1 and 3

Our analysis of Newton's circular orbits uses Definition 1 (forbidden region) and Definition 3 (stable and unstable orbits). The energy region under each of Newton's effective potential curves in Figure 7 is forbidden to the stone (Definition 1), because in that region the stone's kinetic energy would be negative. The stable circular orbit (Definitition 3) nestles at the minimum of the effective potential. These same definitions have wide usefulness: They apply to circular orbits around the non-spinning black hole (Section 18.7) and around the spinning black hole (Section 18.8).

QUERY 15. Time for one orbit according to Newton

A. From Newton's equation for orbit speed in (54) and the circumference of a circle $= 2\pi r$ in flat spacetime, show that for Newton the elapsed time for one circular orbit is:

$$\frac{\Delta t}{M}(\text{one orbit}) = \frac{\Delta \tau}{M}(\text{one orbit}) = 2\pi\left(\frac{r}{M}\right)^{3/2} \qquad \text{(Newton)} \qquad (63)$$

B. Show that equations (51) and (52) both reduce to Newton's result (63) when $r/M \to \infty$.

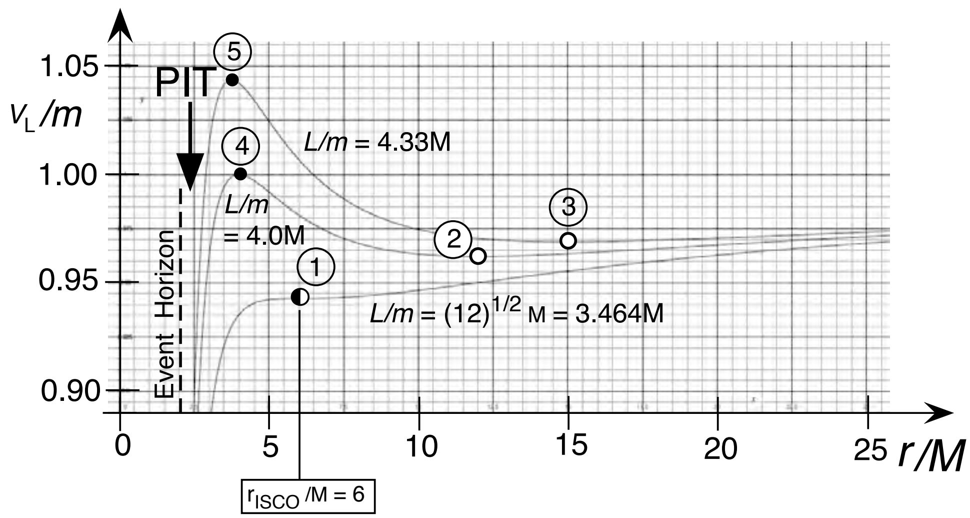

FIGURE 8 Effective potentials for the non-spinning black hole (repeat of Figure 4 in Section 8.4). The area under each curve is the forbidden map energy region for a stone with that value of map angular momentum. Little filled circles locate unstable circular orbits, little open circles locate stable circular orbits, and the little half-filled circle locates a "half-stable" $r_{\rm ISCO}$ circular orbit, one that is "stable to the right and unstable to the left." A small amount of friction moves stable orbits downward and to the left along the sequence of circled numbers $3 \rightarrow 2 \rightarrow 1$ until $r = r_{\rm ISCO}$, after which the stone spirals inward across the event horizon.

18.7 ■ STABILITY OF CIRCULAR ORBITS: NON-SPINNING BLACK HOLE

Add unstable circular orbits to stable circular orbits.

Non-spinning black hole: Stable circular orbits exist for $r > 6M$.

Next analyze the stability of circular orbits around the non-spinning black hole. Figure 8 replots the effective potential for several values of L from Figure 4 in Section 8.4. In Newton's case, Figure 7, all curves have one minimum, the location of a stable circular orbit. But for the spinning black hole, Figure 8, the effective potential to the left of each minimum is radically different. In particular, Figure 8 exhibits the famous PIT in the potential of the non-spinning black hole. Unstable orbits exist at maxima of the effective potential between this pit and the stable-orbit r-values, provided that $L/(mM) > (12)^{1/2}$. Points 4 and 5 are examples of this maximum. Unstable circular orbits are the new contribution of the non-spinning black hole.

To analyze circular orbits for the non-spinning black hole, let $a/M \rightarrow 0$ in equations (31) for E/m and (32) for L/m. Results:

$$\frac{E}{m} = \frac{r^2 - 2Mr}{r\left(r^2 - 3Mr\right)^{1/2}} \qquad \text{(circular orbits, non-spinning black hole)} \qquad (64)$$

$$\frac{L}{m} = \left(\frac{M}{r}\right)^{1/2} \frac{r^2}{\left(r^2 - 3Mr\right)^{1/2}} \text{(circular orbits, non-spinning black hole)}(65)$$

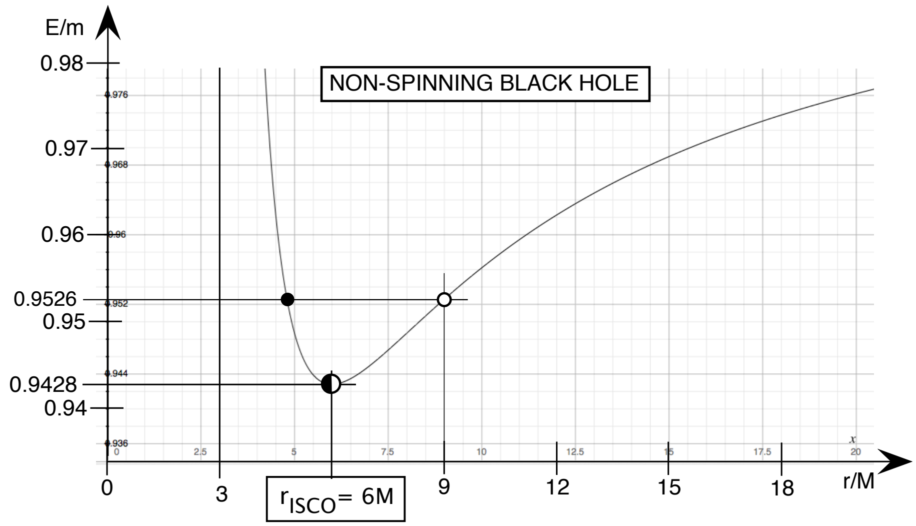

FIGURE 9 Plot of equation (64) for circular orbits around the non-spinning black hole. *Every point* on this curve represents the map energy of a circular orbit. The curve has a minimum $(E/m)_{\rm min} = (8/9)^{1/2} = 0.9428$ at $r_{\rm ISCO} = 6M$ (little half-filled circle). A horizontal line above this minimum at, say, $E/m = 0.9526$ fixes the r-value of an unstable circular orbit (little filled circle) and also the r-value of a stable circular orbit (little open circle).

These correspond to equations (58) and (56) in Newton's case.

QUERY 16. Circular orbits in Newton's limit

Check (64) and (65) in Newton's limit $r/M \to \infty$, that is $M/r \to 0$.

A. Does (65) reduce to (56)?

B. Does (64) reduce to (58). Before doing the algebra, guess the answer by comparing the vertical scales of Figures 7 and 8 and the number that E/m approaches as $r/M \to \infty$.

C. Interpret the physical difference between Newton's circular orbit energy (58) and the Newtonian limit of circular orbit energy (64).

We want to trace the result of a little friction on these orbits. To follow an analysis similar to that for Newton's circular orbits in Section 18.6, take derivatives of both sides of (64) and (65) in Query 17.

QUERY 17. Non-spinning black hole: dE/dr and dL/dr for a sequence of circular orbits.

A. Differentiate (64) and (65) to obtain, for a non-spinning black hole:

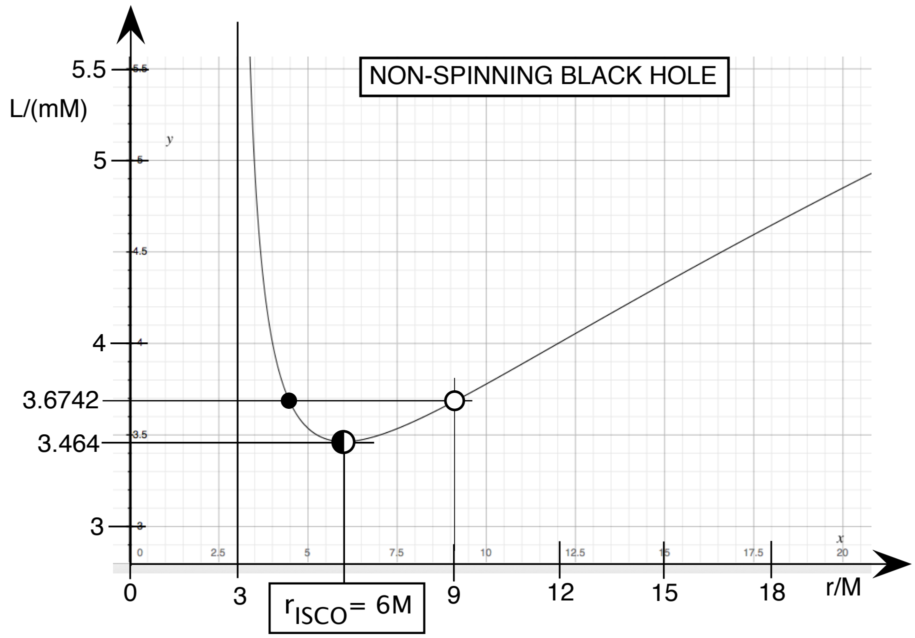

FIGURE 10 Plot of equation (65) for circular orbits around the non-spinning black hole. *Every point* on this curve represents the map angular momentum of a circular orbit. This curve has a minimum $[L/(mM)]_{\text{min}} = (12)^{1/2} = 3.464$ at $r_{\text{ISCO}} = 6M$ (little half-filled circle). A horizontal line above this minimum at, say, $L/(mM) = 3.6742$ fixes the r-value of an unstable circular orbit (little filled circle) and a stable circular orbit (little open circle).

$$\frac{dE}{dr} = \frac{mM(r - 6M)}{2r^3(r - 3M)^{3/2}} \qquad \text{(sequence of circular orbits)} \qquad (66)$$

$$\frac{dL}{dr} = \frac{mM^{1/2}(r - 6M)}{2(r - 3M)^{3/2}} \qquad \text{(sequence of circular orbits)} \qquad (67)$$

B. Show that when $r \gg M$, these reduce to Newton's results (59) and (60).

C. Show how Figure 8 reflects the result that the right sides of both equations (66) and (67) reverse sign at $r = 6M$.

Figure 9 plots E/m vs r/M from equation (64), while Figure 10 plots L/m vs r/M from equation (65). These figures show what equations (66) and (67) describe: E and L have minima at $r = 6M$ for circular orbits around a non-spinning black hole and both have positive slopes, $dL/dr > 0$ and $dE/dr > 0$, for $r > 6M$. From Figure 8, orbits in this range of r-values are stable because r-displacement in either direction at constant E moves the circle point into a forbidden map energy region (Definition 2).

Comment 7. Not another kind of effective potential
Figure 9 looks like an effective potential for the non-spinning black hole, but it is not. Instead, it tells us the r-values of circular orbits for *all possible* values of E/m.

Add friction: Shrinking orbits for non-spinning black hole unstable for $3M < r \leq 6M$.

Now trace the consequences of a little friction for circular orbits around the non-spinning black hole. Start with a stone in a circular orbit at $r > 6M$ in Figures 9 and 10. Friction causes the orbit to lose both angular momentum and energy. Because $dL/dr > 0$ and $dE/dr > 0$ for $r > 6M$, therefore both L and E decrease when r decreases: the orbit shrinks, as confirmed by equations (61) and (62).

What happens after the orbit r-value reaches $r = 6M$, where $L/(mM) = (12)^{1/2} = 3.4641$ and $E/m = (8/9)^{1/2} = 0.9428$? Answer: Friction continues to drain angular momentum and energy. But $dL/dr = 0$ and $dE/dr = 0$ for circular orbits at $r = 6M$, so the stone can no longer change L and E by changing its orbital r-value: *No circular orbits exist for* $L/(mM) < (12)^{1/2}$ *and* $E/m < (8/9)^{1/2}$. Equations (61) and (62) bear this out: Δr is undefined at $r = 6M$.

To determine what happens next, see circled number 1 in Figure 8: Displacement to the left does *not* move the circle point into a forbidden map energy region. Instead, it leads to a continual decrease of r. Result: The stone spirals inward across the event horizon.

For stable circular orbit: $dE/dr > 0$ and $dL/dr > 0$

As long as $dE/dr > 0$ and $dL/dr > 0$ along a sequence of circular orbits, the orbits are stable. Query 17 shows that dE/dr and dL/dr both change sign at $r = 6M$, which marks the transition to unstable circular orbits. Comparing Figures 8 through 10, we see that circular orbits are unstable at r-values where $dE/dr < 0$ and $dL/dr < 0$.

The smallest r-value of a stable circular orbit is called r_{ISCO}. The subscript ISCO stands for **Innermost Stable Circular Orbit,** defined in Section 8.5.

Recall that the ISCO is both stable and unstable: Increasing the r-value at the same energy puts the stone into a forbidden map energy region, but decreasing the r-value does not; the orbit is stable to increasing r, but unstable to decreasing r. We can call the r_{ISCO} orbit a **half-stable circular orbit.**

?

Objection 6. *Wrong again! You tell us that "Map quantities L and E are not measured quantities." So how can you say that friction causes them to decrease? Only physical quantities like velocity and energy in a local frame have a measurable meaning. So you talk nonsense when you say that friction causes (unmeasurable!) map quantities L and E to decrease!*

!

Guilty as charged! Values of L and E are not directly measurable. However, we can use global coordinates to predict, for example, the energy E_{shell} of the stone measured in a local inertial shell frame. For a non-spinning black hole, the result shows that when the tangential velocity measured, for example, in the local shell frame decreases, then L decreases, and *vice versa*. And when local E_{shell} decreases, then map E also decreases, and *vice versa*. Map angular momentum and map energy serve as "proxies" for measurable quantities and both do decrease as claimed. Chapter 19 carries out this analysis for a spinning black hole using the ring frame.

18.8 ■ STABILITY OF CIRCULAR ORBITS: SPINNING BLACK HOLE

Find four types of stable and unstable circular orbits.

Four types of circular orbits for spinning black hole.

How many stable and unstable circular orbits exist around the spinning black hole? We follow an analysis similar to the one for the non-spinning black hole (Section 18.7). But there is a complication: The spinning black hole has four types of circular orbits, introduced in Section 18.4. The symmetry among these four types allows us to concentrate on the two types with positive map energy outside the event horizon, Type 1 and Type 4. (The other two types are related to these by sign changes, described in Query 9.) Figure 11 plots effective potentials that show locations of two Type 1 circular orbits. Compare this plot with Figure 8 for the non-spinning black hole. Figure 12 plots effective potentials that show locations of two Type 4 circular orbits.

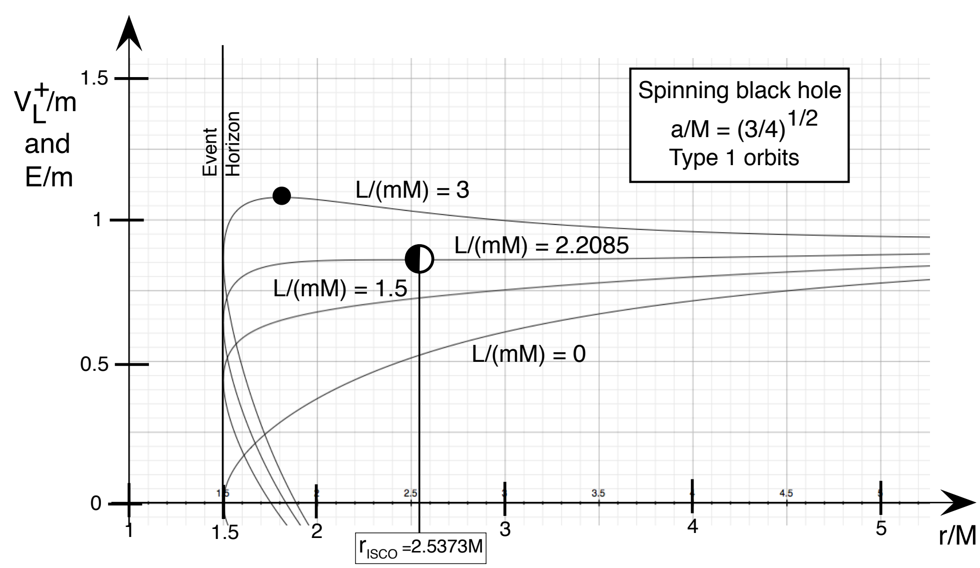

FIGURE 11 Magnified view of the effective potential $V_{\text{L}}^{+}(r)$ near the event horizon for several values of $L/(mM)$, showing r-values of two Type 1 (prograde) circular orbits from (32). Compare with Figure 8. In this plot the forbidden map energy region exists below and to the right of each curve for every value of map angular momentum, including zero. The horizontal axis begins at $r/M = 1$ to hide the distraction of unstable circular orbits inside the Cauchy horizon.

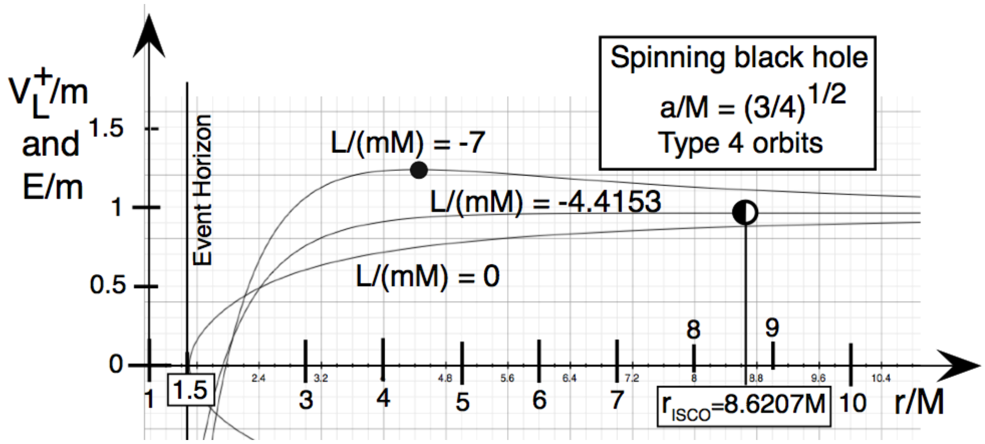

FIGURE 12 Magnified view of the effective potential $V_L^+(r)$ near the event horizon for several values of $L/(mM)$, showing r-values of two Type 4 (retrograde) circular orbits from (37).

Comment 8. Always a forbidden map energy region for spinning black hole
Figures 11 and 12 show that for $a/M = (3/4)^{1/2}$ the forbidden map energy region exists for every value of the stone's angular momentum, including zero. This result is general: For *every* spinning black hole and for *every* value of the stone's angular momentum in orbit around it, *every pair* of effective potentials $V_L^-(r)$ and $V_L^+(r)$ embrace a forbidden map energy region.

Figure 13 plots E/m vs. r/M from Type 1 (prograde) and Type 4 (retrograde) orbits for $a/M = (3/4)^{1/2}$ from equations (32) and (38). Figure 14 shows corresponding plots of $L/(mM)$ vs. r/M from equations (31) and (37). Sample horizontal lines show pairs of unstable and stable orbits at the same map energy or map angular momentum.

Adding friction shrinks stable orbits for spinning black hole

To see where and why circular orbits become unstable, start with the stone in a stable prograde (Type 1) circular orbit at large r. Now introduce a little friction that decreases the stone's energy E. Figures 13 and 14 show positive derivatives $dL/dr > 0$ and $dE/dr > 0$ for stable Type 1 orbits at large r. Then equations (61) and (62) tell us that the r-value of the orbit shrinks.

Orbits stable down to r_{ISCO}

The condition for stability of Type 1 orbits is $dE/dr > 0$ and $dL/dr > 0$ from equations (31) and (32), or equivalently $dV_L^+/dr = 0$ and $d^2V_L^+/dr^2 > 0$ (Table 18.1). Either way gives, after lots of algebra, the inequality:

$$r^2 - 6Mr + 8a(Mr)^{1/2} - 3a^2 > 0 \qquad \text{(stable orbits, Types 1 and 2)} \qquad (68)$$

Although we derived equation (68) for Type 1, it is also valid for Type 2 ($E = V_L^-$ and, outside the event horizon, $L < 0$). You can see this from Figure 4 and Query 8. Both stable and unstable circular orbits come in pairs.

The left hand side of equation (69) vanishes at only one r-value and is negative for smaller r-values. The r-value of the innermost stable circular orbit is therefore given by the solution of this equation:

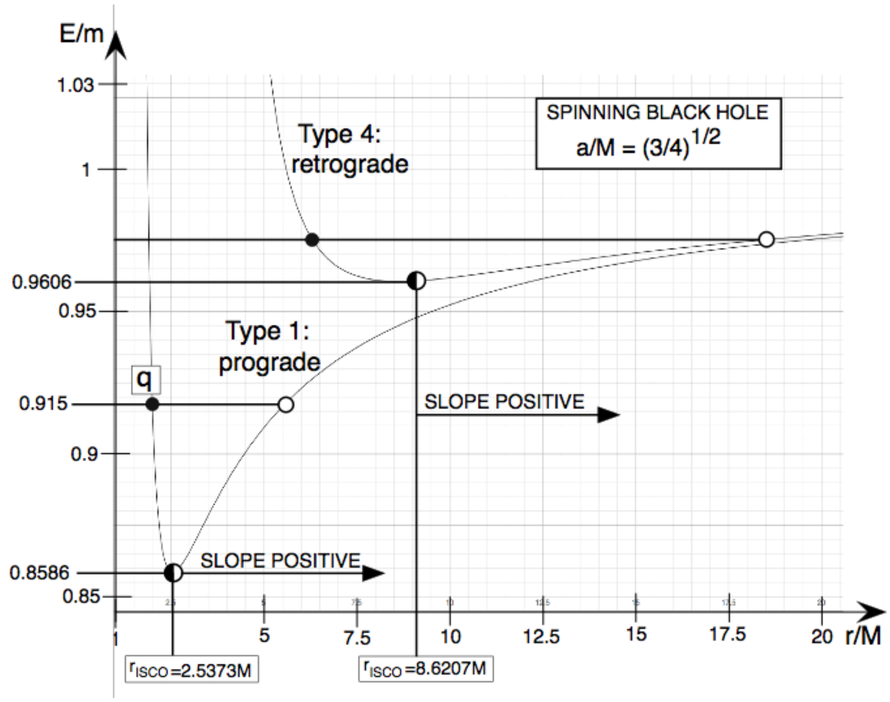

FIGURE 13 Map energy vs r for circular orbits outside the event horizon of the spinning black hole with $a/M = (3/4)^{1/2}$, from equation (32) for Type 1 and equation (38) for Type 4, showing $r_{\rm ISCO}$ at the minima and one example of unstable and stable orbits for each type. The prograde circular orbit labeled q at $r = 1.95$ and energy $E = 0.915M$ is the orbit labeled q in Figure 3; the figure above proves that orbit q in Figure 3 is unstable.

$$r_{\rm ISCO}^2 - 6Mr_{\rm ISCO} + 8a\left(Mr_{\rm ISCO}\right)^{1/2} - 3a^2 = 0 \qquad \text{(prograde orbits)} \qquad (69)$$

Stable circular orbits exist only for $r > r_{\rm ISCO}$.

Minimum $|L|$ and $|E|$ at $r_{\rm ISCO}$

For a stone in a Type 1 or 2 (prograde) circular orbit at $r_{\rm ISCO}$, further decrease of $|L|$ or $|E|$ can no longer result in a circular orbit, because $|L|$ and $|E|$ have already reached their minimum values for circular orbits, shown in Figures 13 and 14. To determine what happens next, look at the little half-filled circle in Figure 11: Displacement of the stone to the left does *not* move it into a forbidden map energy region. Instead, it leads to a continual decrease of r. Result: The stone spirals inward, then crosses the event horizon!

$r_{\rm ISCO}$ for retrograde orbits.

Next turn attention to retrograde orbits, Types 3 and 4. It is simplest to start with Type 4, $E = V_{\rm L}^{+} > 0$ and $L < 0$ (Table 18.3). Then stability for Type 3 follows as a “mirror image,” as was the case for prograde circular orbits. At large r for Type 4, $dE/dr > 0$ (Figure 13), while $dL/dr < 0$ (Figure

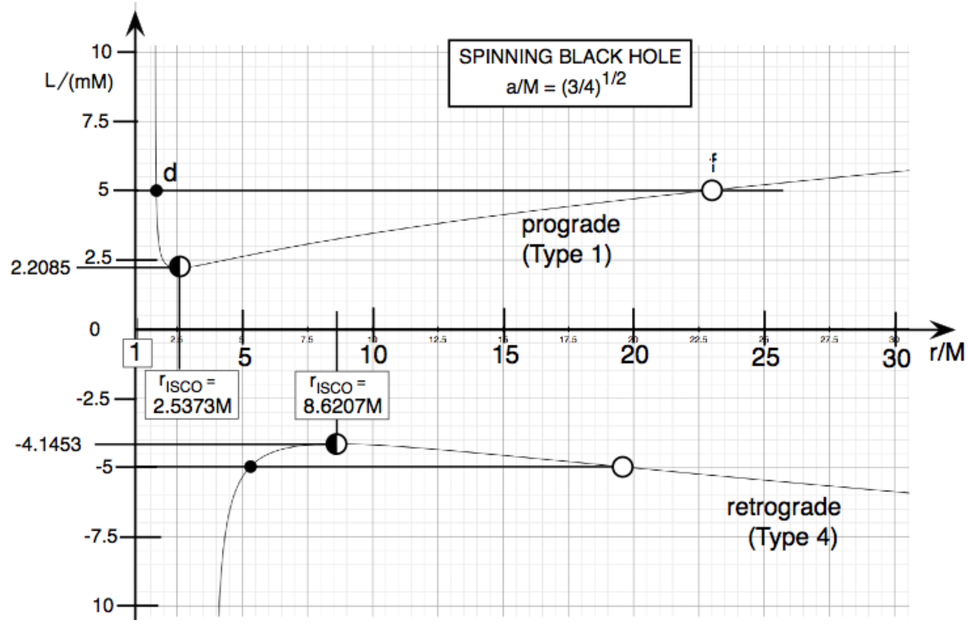

FIGURE 14 Map angular momentum vs r for circular orbits outside the event horizon of the spinning black hole with $a/M = (3/4)^{1/2}$, from equation (31) for Type 1 and equation (37) for Type 4, showing $r_{\rm ISCO}$ at the minima and one example of unstable and stable orbits for each type. Points d and f along the horizontal line at $L/(mM) = +5$ have the same labels in Figure 1 and Table 2.

14). Whether L is positive or negative, a little friction decreases $|L|$. Thus the condition for stability is that there exists a circular orbit of slightly smaller r and slightly smaller $|L|$; this condition requires that $d|L|/dr > 0$ and therefore $dL/dr < 0$ when $L < 0$ in Figure 14.

Comment 9. Signs of dE/dr and dL/dr for stable orbits
When $E < 0$, as in Type 3, the condition on E for stability becomes $d|E|/dr > 0$. For both signs of E, the stability condition is $d|E|/dr > 0$, similar to the condition $d|L|/dr > 0$ for stability. The reason for this is that a little friction decreases both $|L|$ and $|E|$ regardless of the signs of L and E, and for orbits to exist with smaller $|L|$ and $|E|$, the graphs of $|L(r)|$ and $|E(r)|$ must have positive slope with respect to r.

The stability condition for Type 4 circular orbits is $dE/dr > 0$ and $dL/dr < 0$ from equations (37) and (38), or equivalently $dV_{\rm L}^{+}/dr = 0$ and $d^2V_{\rm L}^{+}/dr^2 > 0$ (Table 18.1). Either way yields, after lots of algebra, the inequality:

$$r^2 - 6Mr - 8a(Mr)^{1/2} - 3a^2 > 0 \qquad \text{(stable orbits, Types 3 and 4)} \qquad (70)$$

Although we derived equation (70) for Type 4 orbits, it is also valid for Type 3. At $r = r_{\rm ISCO}$, equation (70) becomes an equality.

$$r_{\rm ISCO}^2 - 6Mr_{\rm ISCO} - 8a\left(Mr_{\rm ISCO}\right)^{1/2} - 3a^2 = 0 \qquad \text{(retrograde orbits)} \qquad (71)$$

As in Query 10, this result follows from the prograde case merely by changing the sign of a. We can solve the two equations (69) and (71) to find two expressions for $a(r_{\rm ISCO})$.

$$a(r_{\rm ISCO}) = \pm\frac{1}{3}\left(Mr_{\rm ISCO}\right)^{1/2}\left[4 - \left(3\frac{r_{\rm ISCO}}{M} - 2\right)^{1/2}\right] \qquad (72)$$

The plus sign in this equation describes prograde orbits and the minus sign describes retrograde orbits.

Limits on value of $r_{\rm ISCO}$

Black holes exist only for $0 \leq a/M \leq 1$. Equation (72) then limits prograde and retrograde orbits to to the following values of $r_{\rm ISCO}$:

$$M \leq r_{\rm ISCO} \leq 6M \qquad (0 \leq a/M \leq 1,\ \text{prograde}) \qquad (73)$$

$$6M \leq r_{\rm ISCO} \leq 9M \qquad (0 \leq a/M \leq 1,\ \text{retrograde}) \qquad (74)$$

Values of $r_{\rm ISCO}$

The curves in Figure 15 plot a as a function of $r_{\rm ISCO}$ from equation (72). Bardeen, Press, and Teukolsky solved (72) to give $r_{\rm ISCO}$ as a function of a, a combination of three equations (see the references):

$$\frac{r_{\rm ISCO}}{M} = 3 + Z_2 \mp \left[(3 - Z_1)(3 + Z_1 + 2Z_2)\right]^{1/2} \qquad (75)$$

minus sign for prograde, plus sign for retrograde, and

$$Z_2 \equiv \left(3a^2/M^2 + Z_1^2\right)^{1/2} \qquad (76)$$

$$Z_1 \equiv 1 + \left(1 - a^2/M^2\right)^{1/3}\left[(1 + a/M)^{1/3} + (1 - a/M)^{1/3}\right] \qquad (77)$$

QUERY 18. Values of $r_{\rm ISCO}$ for $a/M = (3/4)^{1/2}$. (*Optional*)

Use equations (75) through (77) to verify the following values of $r_{\rm ISCO}$ for a spinning black hole with $a/M = (3/4)^{1/2}$:

$$r_{\rm ISCO}/M = 2.537331951 \qquad \text{for prograde orbit} \qquad (78)$$

$$r_{\rm ISCO}/M = 8.620665097 \qquad \text{for retrograde orbit}$$

Summary: For circular orbits around a spinning black hole, a small amount of friction decreases the absolute values of map energy and map

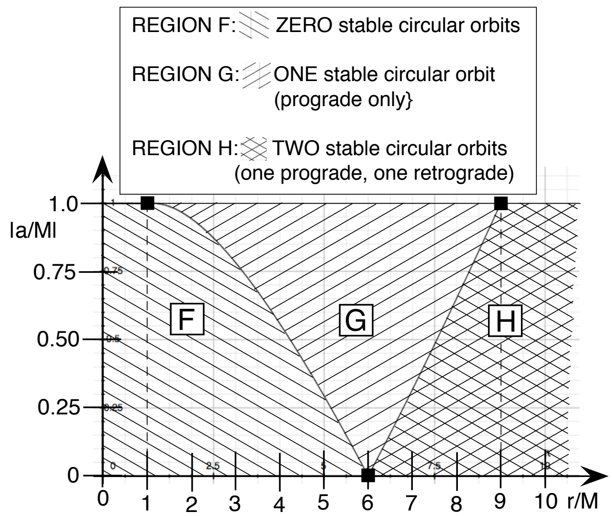

FIGURE 15 How many *stable* circular orbits exist at a given r for different values of the spin parameter a/M? This figure uses inequalities (68) and (70) to answer that question. The regions are separated by curves for r_{ISCO} from equations (69) and (71). In Region F there are *zero* stable circular orbits; in Region G there is *one* stable prograde circular orbit; in Region H there are *two* stable circular orbits, one prograde and one retrograde. Compare this figure with Figure 6 for *all* circular orbits.

Summary: Sequence of stable circular orbits

angular momentum, $|E|$ and $|L|$, which causes the stone to occupy a sequence of stable circular orbits with decreasing r—until both $|E|$ and $|L|$ reach their minima at $r = r_{\text{ISCO}}$. Increasing black hole spin moves the ISCO inward from $r_{\text{ISCO}} = 6M$ to $r_{\text{ISCO}} = M$ for prograde orbits and outward from $r_{\text{ISCO}} = 6M$ to $r_{\text{ISCO}} = 9M$ for retrograde orbits (Figure 15). These results have profound consequences for the accretion disk around the spinning black hole, which we explore in Section 18.10.

18.9 ■ TIMING CIRCULAR ORBITS FROM A LARGE r

On whose watch?

We are (thank goodness!) far from a spinning black hole. *Surprise:* We can nevertheless hold a stopwatch on each circular orbit in the sequence of circular orbits as a stone works its way inward through the accretion disk (Section 18.10). In practice we might observe a blob of incandescent matter as it moves in each circular orbit. This section provides the background for such an observation.

Flash-emitting orbiter

Replace the circulating blob with an astronaut in a circular orbit who emits a flash of light as she completes each orbit. Equation (52) tells us the lapse $\Delta\tau_{\text{emit}}$ on her wristwatch for one orbit, where we have added the

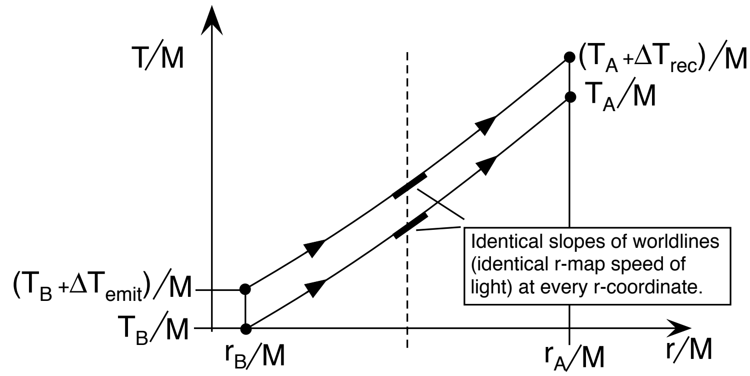

FIGURE 16 Schematic plot in Doran global coordinates of worldlines of two flashes emitted by the Below emitter at the beginning and end of one circular orbit and received by a distant Above observer. The lapse ΔT_{rec} between receptions is equal to the lapse ΔT_{emit} between emissions. Similar plot for the Global Positioning System: Figure 2 in Section 4.2.

subscript "emit" for clarity in what follows. How does the orbiter *know* that she has completed one orbit? The pattern of stars she sees overhead repeats as she returns to the same r, Φ. We have not yet predicted this star pattern, which depends on the observer's orbit and the worldline of light from each distant star to the observer. Still, we know that this visual pattern repeats, so the observer can emit a flash at each repetition.

Equation (51) tells us the Doran coordinate lapse ΔT_{emit} between flash emissions by the orbiter. A distant observer at rest in Doran coordinates $(dr/dT = d\Phi/dT = 0)$ receives two sequential flashes emitted by the orbiter and records his wristwatch time lapse $\Delta\tau_{\mathrm{rec}}$ between these two receptions.

Timing these flashes from far away

At the location of this stationary distant observer the Doran metric reduces to $d\tau^2 = dT^2$. Therefore, the distant observer measures a time lapse $\Delta\tau_{\mathrm{rec}} = \Delta T_{\mathrm{rec}}$ between flashes, where ΔT_{rec} is the Doran coordinate lapse between the receptions of sequential light flashes.

How is ΔT_{rec} related to ΔT_{emit}? Light rays travel along curves $r(T)$ in global coordinates. Let one light flash be emitted at $r = r_{\mathrm{A}}$ and $T = T_{\mathrm{A}}$ and a second one from the same r-value at $T = T_{\mathrm{A}} + \Delta T_{\mathrm{emit}}$ (Figure 16). When are these two flashes received by a distant observer stationary in Doran coordinates?

We cannot answer this question without integrating the equation of motion of light, but we can answer a simpler question: What is the difference between two global T-values of reception by a distant observer? That is, how is ΔT_{rec} related to ΔT_{emit}?

Figure 16 shows that at every value of r the curves $r(T)$—or equivalently $T(r)$—have the same slope for two sequential light pulses emitted from the same global location. Therefore these curves are vertically displaced by the same offset in Doran T at every r-value. As a result, $\Delta T_{\mathrm{rec}} = \Delta T_{\mathrm{emit}}$.

This analysis leads to the prediction that the wristwatch time $\Delta\tau_{\mathrm{far}}$ for one orbit measured by a distant observer at rest in Doran coordinates is equal to the lapse ΔT for one orbit given by equation (51). This answers the question, "What is the wristwatch time lapse $\Delta\tau_{\mathrm{far}}$ for one circular orbit measured by a distant observer?"

QUERY 19. Careful with wristwatch times!

Show that the wristwatch time $\Delta\tau_{\mathrm{far}}$ between reception of flashes for the distant observer is NOT equal to the wristwatch time $\Delta\tau_{\mathrm{emit}}$ between emission of flashes for the orbiter.

Pulse emitter: black hole GR1915

Figure 17 shows X-ray pulses emitted by the spinning black hole labeled GR1915, with about 14 times the mass of the Sun located near the plane of the galaxy about 40 light-years from us. A companion star feeds a pulse of material to the accretion disk of GR1915. This pulse of matter heats to high temperature and emits radiation whose pressure temporarily prevents more matter from entering the accretion disk from the companion. After the accreted material drops into the black hole, a new blob enters the accretion disk from the companion. The resulting "heartbeat" of X-rays are about 50 seconds apart.

18.10 ■ THE ACCRETION DISK

Circling toward doom

Section 8.6 constructed the toy model of an accretion disk around a non-spinning black hole, but we have not observed a non-spinning black hole, much less one with an accretion disk. We do observe energetic radiation from quasars, each of which appears to be a spinning black hole surrounded by an accretion disk that emits this radiation. What creates this radiation? Interactions within the accretion disk are complex and defy simple analysis, but here is the basic idea: The accretion disk consists of dust and particles in orbit. This material changes energy as it moves inward through a sequence of circular orbits. The change in energy heats up the accretion disk, with consequent emission of radiation.

QUASAR: Emission as material circles inward through accretion disk.

Assume that material in the accretion disk passes in sequence through a series of circular orbits. Initial circular orbits are at large r-values; their final circular orbit is at r_{ISCO}, after which the material spirals inward through the event horizon. We cannot see radiation emitted after stones and dust pass through the event horizon. Now for some details.

Stone in distant circular orbit has $E/m = 1$ and $L/(mM) = 0$.

Start with a stone far from the black hole, a stone that moves so slowly in its circular orbit that it is effectively at rest in Doran global coordinates, with initial map energy $E/m = 1$ and initial map angular momentum $L/(mM) = 0$. Consider this stone to be in a forward, prograde Type 1 circular orbit.

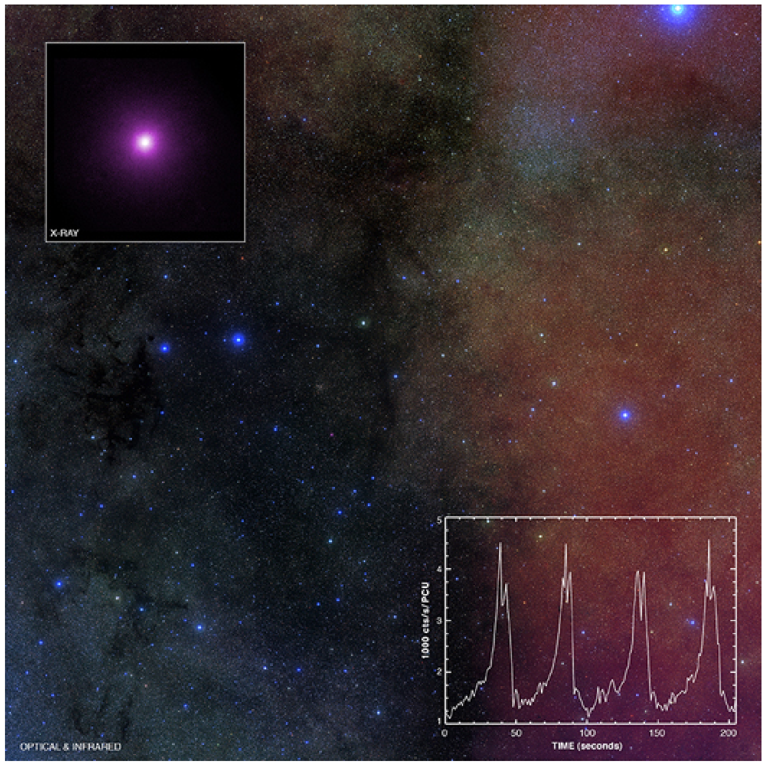

FIGURE 17 Upper left corner: the spinning black hole GR1915-105 fed by material from a companion star (not visible). Lower right corner: the "heartbeat" of emitted X-rays.

For very large r, $R \to r$ and the Doran metric (2) becomes:

$$d\tau^2 \to dT^2 - dr^2 - r^2 d\Phi^2 \qquad \text{(for } r \to \infty) \qquad (79)$$

This is the metric of flat spacetime in which we can define local shell coordinates: $\Delta t_{\text{shell}} \equiv \Delta T$, $\Delta y_{\text{shell}} \equiv \Delta r$, and $\Delta x_{\text{shell}} \equiv \bar{r}\Delta\Phi$. A stone at rest in this local frame must have $(E/m)_{\text{shell}} = 1 = E/m$, where E/m is the map energy. *Summary: Far from the black hole the directly measurable shell energy* $(E/m)_{\text{shell}}$ *of a stone is equal to its Doran map energy* E/m.

Next the stone loses map energy as it passes gradually inward through a series of circular orbits of decreasing r until it reaches the innermost stable circular orbit at r_{ISCO}. How much map energy does the stone lose during this process? Assume the material emits its change in map energy in the form of radiation. What total radiated energy do we detect far from the black hole?

What is the map energy of the stone in the ISCO orbit just before it drops across the event horizon?

When $a/M = (3/4)^{1/2}$, equations (75) through (76) tell us that $r_{\rm ISCO} = 2.5373M$ so that from equation (32) $E/m = 0.8586$. Hence the radiated energy is $\Delta E = (1 - 0.8586)m = 0.1414m$.

In contrast, when $a/M = 1$, then equations (75) through (76) tell us that $r_{\rm ISCO} = M$ so that from equation (32) $E/m = 0$. Hence the radiated energy is $\Delta E = m$. The entire rest energy of the stone is emitted as radiation. No wonder the quasar shines so brightly!

QUERY 20. More typical emission of radiation

A more typical upper value of a/M for a spinning black hole is 0.85. Use Figure 15 to estimate numerical value of $r_{\rm ISCO}$ for $a/M = 0.85$. Optional: Use equations (75) through (77) to calculate the numerical value of $r_{\rm ISCO}$ to four decimal digits in this case.

QUERY 21. Power output of a quasar

A distant quasar swallows $m = 10M_{\rm Sun}$ = ten times the mass of our Sun every Earth-year. Recall that watts equals joules/second and, from special relativity,
$\Delta E[\text{joules}] = \Delta m[\text{kilograms}]c^2[\text{meters}^2/\text{second}^2]$. Assume that this quasar has $a/M = 0.85$.

A. How many watts of radiation does this quasar emit, according to our model?

B. Our Sun emits radiation at the rate of approximately 4×10^{26} watts. The quasar is how many times as bright as our Sun?

C. Compare your answer in Item B to the total radiation output of a galaxy of approximately 10^{11} Sun-like stars.

QUERY 22. How long does a quasar shine?

We see most quasars with large red shifts of their light, which means they were formed not long after the Big Bang, about 14×10^9 years ago. A typical quasar is powered by a black hole of mass less than 10^9 solar masses. Explain, from the results of Query 21, what this says about the lifetime during which the typical quasar shines.

18.11 ■ CHAPTER SUMMARY

Key ideas of the chapter

Two effective potentials

The spinning black hole has not one but *two* effective potentials, which depend on the stone's angular momentum and the spin parameter of the black hole. Circular orbits of a stone are possible at maxima and minima of these effective potentials, which (for different values of the stone's map angular momentum) can occur at most r-values outside the event horizon and inside the Cauchy horizon.

Forbidden energy region

Each pair of effective potentials encloses a **forbidden map energy region**. A stone cannot have its map energy in a forbidden map energy region.

Prograde and **retrograde** orbits

We divide circular orbits into two classes, **prograde** and **retrograde**. In a prograde orbit the stone "revolves in the direction that the black hole rotates" in global Doran coordinates, $d\Phi/dT > 0$, while in a retrograde orbit the stone revolves in the opposite direction, $d\Phi/dT < 0$.

Stable circular orbits and the *innermost* stable circular orbit

Most circular orbits around the spinning black hole are unstable; a few are stable. To analyze orbital stability, we trace the effects of a little friction, which slowly decreases orbital r (leaving the orbit effectively circular), while it also decreases values of $|L|$ and $|E|$. The r-value of the **innermost stable circular orbit**, labeled r_{ISCO}, occurs when values of $|L|$ and $|E|$ for a circular orbit reach their minima. When the circular orbit of a stone reaches r_{ISCO}, further loss of energy to friction leads the stone to spiral inward through the event horizon.

Accretion disk

In Nature a spinning black hole is surrounded by an *accretion disk* that consists of material circulating in stable prograde circular orbits in the equatorial plane. (Why prograde? Because a stone circulating in a prograde ISCO has a much smaller map energy than a stone in a retrograde ISCO; see Figure 13.) Orbiting dust and particles emit energy in the form of electromagnetic radiation as they descend gradually through circular orbits of decreasing r. A distant stationary observer measures this emitted radiation to have energy equal to the map energy E/m. We continue to observe this radiation as material spirals down from the minimum stable ISCO orbit, but not after the material crosses the event horizon.

18.12 ■ EXERCISES

0. SOLVED EXERCISE. Add angular momentum to a maximum-spin black hole?

Suppose that the spinning black hole has maximum spin: $a/M = 1$. Can you increase this (maximum!) spin by sending into the black hole a stone with positive angular momentum? Try a specific example:

Figure 18 plots the effective potential for a black hole with maximal spin $a/M = 1$ and incoming stones with angular momentum $L/(mM) = 5$ and three different map energies, including $E_{\text{C}}/M = 6$, above the energy of the forbidden map energy regions. When it falls into the black hole, can this highest-energy stone increase the black hole spin beyond its maximum value? Answer this question using the following steps.

$$\frac{E}{m} = 6 \qquad \text{and} \qquad \frac{L}{mM} = 5 \tag{80}$$

A. When this stone enters the black hole, it changes the black hole's mass according to equation (28) in Section 6.5 and increases the black hole's

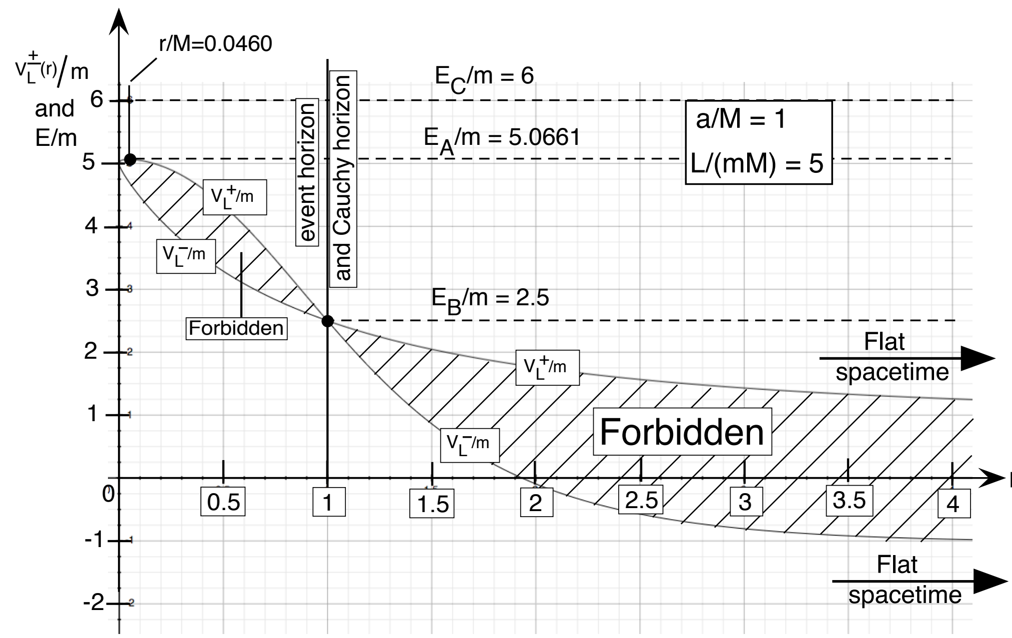

FIGURE 18 Effective potentials $V_{\mathrm{L}}^{+}(r)$ and $V_{\mathrm{L}}^{-}(r)$ for a stone with $L/m = 5M$ in orbit around a spinning black hole with maximum spin parameter $a/M = 1$. There are two stable circular orbits at larger r/M than the maximum in this diagram, one prograde, one retrograde. Two of the dashed lines show map energies E_{A}/m and E_{B}/m of two stones that take up unstable circular orbits. Can a third stone, with $E_{\mathrm{C}}/m = 6$ and angular momentum $L/(mM) = 6$ fall into this black hole and increase its angular momentum above the maximum?

angular momentum beyond the old maximum in equation (2) in Section 17.1:

$$M_{\mathrm{new}} = M + E_{\mathrm{stone}} \qquad \text{and} \qquad J_{\mathrm{new}} = M^2 + L_{\mathrm{stone}} \tag{81}$$

B. Then equation (1) in Section 17.1 tells us that

$$\frac{a_{\mathrm{new}}}{M_{\mathrm{new}}} = \frac{J_{\mathrm{new}}}{M_{\mathrm{new}}^2} = \frac{M^2 + L_{\mathrm{stone}}}{(M + E_{\mathrm{stone}})^2} = \frac{1 + L_{\mathrm{stone}}/M^2}{(1 + E_{\mathrm{stone}}/M)^2} \tag{82}$$

C. Now L_{stone} and E_{stone} are properties of the incoming stone, which has mass $m \ll M$, therefore $L_{\mathrm{stone}} \ll M^2$ and $E_{\mathrm{stone}} \ll M$, so we can approximate (82) with the formula inside the front cover:

$$\frac{a_{\text{new}}}{M_{\text{new}}} \approx (1 + L_{\text{stone}}/M^2)(1 - 2E_{\text{stone}}/M) \tag{83}$$

$$\approx 1 + \frac{L_{\text{stone}}}{M^2} - \frac{2E_{\text{stone}}}{M} \tag{84}$$

$$\approx 1 + \left(\frac{L_{\text{stone}}}{mM}\right)\left(\frac{m}{M}\right) - 2\left(\frac{E_{\text{stone}}}{m}\right)\left(\frac{m}{M}\right)$$

$$\approx 1 + \frac{m}{M}(5 - 2 \times 6) = 1 - 7\frac{m}{M} \tag{85}$$

The step from (83) to (84) neglects the product of two small quantities. The final expression (85) is (slightly) smaller than the initial (maximum) spin parameter $a/M = 1$.

For this example, the incoming stone does not increase the spin parameter of the black hole. Why not? Because it increases the mass of the black hole, which changes the value of its maximum spin.

1. *Optional:* Repeat exercise 0 with GRorbits

Use interactive GRorbits software to plot the analysis of Exercise 0

A. Plot the case described in Exercise 0 with your choice of numerical values for $m \ll M$ and $M = 10M_{\text{Sun}}$.

B. Repeat Item A for $M = 10^7 M_{\text{Sun}}$. Describe how your results differ from those in Item A?

C. Report what you have learned in this exercise that supplements or reinforces results in Exercise 0.

2. Fast orbits!

Write a computer program to fill in Tables 18.4 and 18.5 for a spinning black hole with $a/M = (3/4)^{1/2}$. Write "None" in entries for which circular orbits do not exist. Section 18.10 shows that a distant observer records a wristwatch time equal to map ΔT for one circular orbit. In the table, "progr." means "prograde" and "retrogr." means "retrograde".

A. For a "small" black hole with mass $M = 10M_{\text{Sun}}$ fill in entries in Table 18.4.

B. For a "large" black hole with mass $M = 4 \times 10^6 M_{\text{Sun}}$ (the approximate mass of the spinning black hole at the center of our galaxy), fill in the entries in Table 18.5.

TABLE 18.4 "Small" black hole: "TIMES" for one orbit, in SECONDS

$M = 10M_{\mathrm{Sun}}$	$r/M =$ 0.2	$r/M =$ 2	$r/M =$ 6	$r/M =$ 10	$r/M =$ 20
Newton time					
Nonspin $\Delta\tau$					
Spin progr. $\Delta\tau$					
Spin retrogr. $\Delta\tau$					
Nonspin ΔT					
Spin progr. ΔT					
Spin retrogr. ΔT					

NOTE: Spinning black hole has $a/M = (3/4)^{1/2}$. Equation (52) for τ and (51) for T.

TABLE 18.5 "Large" black hole: "TIMES" for one orbit, in DAYS

$M = 4 \times 10^6 M_{\mathrm{Sun}}$	$r/M =$ 0.2	$r/M =$ 2	$r/M =$ 6	$r/M =$ 10	$r/M =$ 20
Newton time					
Nonspin $\Delta\tau$					
Spin progr. $\Delta\tau$					
Spin retrogr. $\Delta\tau$					
Nonspin ΔT					
Spin progr. ΔT					
Spin retrogr. ΔT					

NOTE: Spinning black hole has $a/M = (3/4)^{1/2}$. Equation (52) for τ and (51) for T.

3. Can a stone exist in a region where the effective potential is not real-valued?

In Section 18.2 we found from equation (16) that the effective potentials are not real-valued (do not exist) at r-values for which the horizon function H is imaginary, namely between r_{C} and r_{E}. This seems to imply that the equation of motion (15) for $dr/d\tau$ is complex-valued, so the stone cannot move or even exist between the horizons. Demonstrate conclusively that the stone can exist and move between the two horizons.

4. Forbidden map energy region for non-spinning black hole?

Review the effective potential diagrams for the *non-spinning* black hole in Chapter 8 Circular Orbits and answer the following questions without doing any calculation.

A. Show that a forbidden map energy region exists for the non-spinning black hole.

B. Does this forbidden map energy region extend all the way to flat spacetime, $r \to \infty$?

C. What is the experimental (observational) consequence—if any—of the forbidden map energy region near the non-spinning black hole for an observer far away where spacetime is flat?

D. *Optional:* Take the limit of equation (16) as $a/M \to 0$ and $L/(mM) \to 0$. Plot the resulting effective potential curve for a stone moving radially near a non-spinning black hole.

5. Forward time travel using a knife edge circular orbit of a spinning black hole.

Review Exercise 7 in Chapter 8. The Space Administration is now accepting proposals for forward time travel that use a forward prograde knife-edge circular orbit around a spinning black hole with $a/M = (3/2)^{1/2}$. They consider a satellite with a non-relativistic velocity far from the black hole so that $E/m \approx 1$. While still far from the black hole, the spaceship captain uses small rocket thrusts to achieve the value of map angular momentum L required so that $V_L^+/m = E/m = 1$ on the peak of the $V_L^+(r)/m$ curve.

A. Substitute the condition that $V_L^+/m = 1$ at the peak of the $V_L^+(r)/m$ curve into equation (32). Solve the resulting equation for r.

B. Substitute the solution of Item A into (31) to find the factor $d\tau/dT$ for the spaceship in this knife-edge orbit. What speed in flat spacetime gives the same time-stretch ratio?

C. Compare $d\tau/dT$ in Item B with the time-stretch ratio for the non-spinning black hole (Exercise 7, Item B in Chapter 8).

6. Effect of friction when starting from an unstable circular orbit

Section 18.7 analyzes the motion of a stone that starts from a *stable* circular orbit at $r > 6M$ around a non-spinning black hole, and loses map energy and angular momentum through friction (see Figures 9 and 10). Use Figures 9 and 10 to answer the following question: What happens if a stone is in an *unstable* circular orbit at $r < 6M$, then loses map energy and map angular momentum in small steps through friction?

7. How many stable circular orbits are there for the non-spinning black hole?

Figure 15 shows that at $a/M = 0$ regions F, G and H meet in a single point at $r/M = 6$. Are there ZERO, ONE or TWO stable circular orbits there?

8. Circular orbits inside the Cauchy horizon

Figures 11 through 14 all plot the horizontal r-axes for $r/M > 1$ in order to avoid complications with the spacetime region between the singularity and the Cauchy horizon. Yet Figure 15 plots the horizontal axis all the way down to the singularity at $r/M = 0$. Use Figures 1 and 2 to explain why the region $0 < r/M < 1$ in Figure 15 is correct.

9. Stone map energy and map angular momentum at the ISCO for $a = M$

Equation (69) shows that for the maximum-spin black hole, $r_{\rm ISCO} = M$. For these values of a and $r_{\rm ISCO}$, equations (31) and (32) give indeterminate values $(L/(mM))_{\rm Type1} = 0/0$ and $(E/m)_{\rm Type1} = 0/0$.

To find the numerical values of L/m and E/m for this orbit, we need to set $r/M = 1 + \epsilon$ and take the limit of equations (31) and (32) as $\epsilon \rightarrow 0$. The answer, to one significant digit, is $L/(mM) = 1.2$ and $E/m = 0.6$.

A. Find numerical values for $L/(mM)$ and E/m to three significant digits. [Warning: our familiar approximation inside the front cover does not work everywhere in this case. Under the square root in the denominator of the right side of (31) and (32) you need to include the second (quadratic) term in the expansion, so that: $(r/M)^{1/2} = (1+\epsilon)^{1/2} \approx 1 - \epsilon/2 - \epsilon^2/8$

B. *Optional*. Plot $V_{\rm L}^{+}(r)/m$ vs. r/M for the value of $L/(mM)$ you calculated in Item A. Check that the minimum of the effective potential occurs at $r/M = 1$ at the value of E/m you obtained in Item A.

10. Two light cone diagrams for the maximally spinning black hole ($a = M$)

A. Review Sections 3.6 through 3.9 in Chapter 3 for the meaning of *spacetime slice, light cone diagram*, and *embedding diagram*. Use the technique outlined there to construct a light cone diagram, similar to Figure 8 of Chapter 3, on the $[r, T]$ slice of a spinning black hole with $a/M = 1$.

B. Construct a light cone diagram on the $[\Phi, T]$ slice of a spinning black hole with $a/M = 1$.

C. Answer the following questions for both light cones in Items A and B: Why cannot a stone or spaceship remain static in Doran coordinates for $r < 2M$? How can a stone or spaceship still escape to infinity from $r = 2M$? Does the rotation of the black hole drag a stone or spaceship at $r = 2M$ inevitably along the direction in which the black hole spins? Is your answer to this third question coordinate-free?

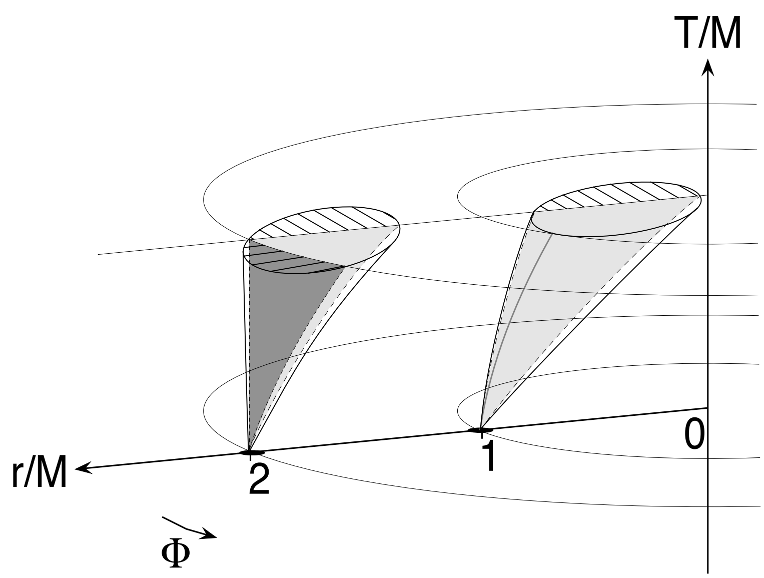

FIGURE 19 A three-dimensional Doran coordinate r, Φ, T plot of two light cones near the maximally-spinning black hole $a/M = 1$.

11. ***Difficult!*** Three-dimensional light cone diagram for the maximally-spin black hole

Figure 19 shows a three-dimensional Doran coordinate plot of two light cones for the maximally-spinning black hole. Discuss the following characteristics of these light cone plots/ plot.

A. Both light cones start on the r/M axis. Why are they both deflected inward in the r direction? Are they deflected in the Φ direction? Why or why not?

B. Why is the light cone that starts at $r/M = 1$ deflected more in the r direction than the light cone that starts at $r/M = 2$?

C. What is the physical difference between that part of the area at the top of the $r/M = 2$ light cone whose lines lie in the r direction and the part of that area whose lines lie in the Φ direction? Why is there no corresponding area of the $r/M = 1$ light cone lined in the r direction?

D. Does either light cone tell you that a circular orbit of a stone is possible at that value of r/M? If not, why not? If so, what does it say about that circular orbit?

E. Answer Item C in exercise 10 for the two lightcones of Figure 19.

12. Light cone diagrams for a spinning black hole with $a/M = (3/2)^{1/2}$

Refer to your answers for Items A through C of exercise 10. The present exercise asks to you apply a similar analysis to a black hole with $a/M = (3/2)^{1/2}$.

A. Repeat Item A of exercise 10 for $a/M = (3/2)^{1/2}$.

B. Ditto for Item B of exercise 10.

C. In Section 17.8 we found from equations (77) through (79) the surprising result that local ring frames can exist between the Cauchy horizon and the singularity. Use the 3D light cone diagram of Item C to show how once a stone crosses the Cauchy horizon, in principle—*that is, without any mathematical analysis of particular orbits*—the stone is *not necessarily* dragged further towards smaller r-values and into the singularity, but can remain in circular orbits.

D. Knowing what you know from the present chapter, how many different circular orbits can there be for a free stone inside the Cauchy horizon? Why is your answer to this Item D different from your answer to Item C?

13. Limiting values of constants and variables at the horizons

Derive expressions (22) through (27) in Box 2.

14. Stable circular orbits at $r/M = 9$ for maximum-spin black hole

Equations (68) and (70) tell us that stable orbits come in pairs (prograde Types 1 and 2 always occur together, and retrograde Types 3 and 4 also always occur together). Figure 15 shows that for a maximum-spin black hole, $r/M = 9$ is on the boundary between region G (where one prograde pair of stable circular orbits exist) and region H (where two pairs of stable circular orbits exist—one prograde, one retrograde).

This argument implies that $r/M = 9$ is the innermost stable circular orbit (ISCO) for retrograde (Types 3 and 4) orbits, but just an ordinary stable circular orbit for prograde (Types 1 and 2) orbits.

Use equations (31) through (38) for L/m and E/m and equation (16) for $V_{\mathrm{L}}^{\pm}/m$ to verify the conclusion in the preceding paragraph.

15. Orbiting in the direction of rotation of the black hole

Out of the four types of circular orbits discussed in this chapter, in which type(s) does the stone actually orbit in the direction that the black hole rotates? Does this question have a coordinate-free meaning?

16. Circle points for the maximum-spin black hole

Table 2 shows the r/M and E/m values of circular orbits for a black hole with $a/M = (3/2)^{1/2}$ and a stone with a map angular momentum $L/(mM) = 5$. How were these numerical values calculated? Construct a similar table for stone moving with the same map angular momentum around a spinning black hole with $a/M = 1$. Display the effective potentials $V_{\mathrm{L}}^{\pm}(r)$ for this case in a plot similar to Figure 1.

17. Possible orbits and their orbit parameters for a given a/M and r/M

Use equations (31) through (38) and equations (43) through (50) to find all possible types of circular orbits and their values of $L/(mM)$, E/m, $dT/d\tau$, and $d\Phi/dT$, for black hole spin $a/M = (3/2)^{1/2}$ at the following r-coordinates.

(a) $r/M = 22.76$. Check your result in Figure 1.
(b) $r/M = 19.87$. Check your result in Figure 1.
(c) $r/M = 4$. Check your result in Figure 3.
(d) $r/M = 0.4475$. Check your result in Figure 3.

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