Chapter 20. Orbits of Light around the Spinning Black Hole

- *What variety of paths does light follow around a spinning black hole?*
- *Can a spinning black hole reverse the direction of a light beam?*
- *Can a light beam go into orbit around a spinning black hole? If so, how many different orbits are available to it?*
- *What does a distant spinning black hole look like? How can I distinguish it visually from a non-spinning black hole?*

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CHAPTER 20 Orbits of Light around the Spinning Black Hole

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Light does strange things near a spinning black hole; it misleads you about the locations and shapes of things, so you must grope along as if you are in a haunted house. Seeing is definitely not believing!

—The authors

20.1 ■ INTRODUCTION: THE PURPOSES OF LOOKING

Who cares what we see?

New questions about the spinning black hole

Chapter 11 described orbits of light around the non-spinning black hole. That earlier chapter focussed on the question, "What is the visual size of the black hole seen by a raindrop diver falling from a great distance?" The same question about the spinning black hole is of little interest today. Instead, we ask the questions:

- How can we identify a distant spinning black hole in the heavens?
- How can we measure its mass M and spin parameter a?

To answer these questions, the present chapter does use a method similar to that of these earlier chapters:

Stone's mass goes to zero.

- Start with a stone of mass m.
- Let the stone's mass go to zero.

We begin this program with a review of the stone's equations of motion in global Doran coordinates.

*Draft of Second Edition of *Exploring Black Holes: Introduction to General Relativity*

20.2 ■ DORAN GLOBAL EQUATIONS OF MOTION FOR THE STONE

Goal: Get rid of wristwatch time.

Motion of a stone

This chapter analyzes the motion of light in the equatorial plane of the spinning black hole. We derive equations of motion for light by extending equations of motion for a free stone. Begin this process with equations (15), (21), and (20) and (16) for motion of a stone in Section 18.2: . Write them in the form:

$$\frac{dr}{d\tau} = \pm\frac{R}{mr}(E - V_L^+)^{1/2}(E - V_L^-)^{1/2} \qquad \text{(stone)} \qquad (1)$$

$$\frac{d\Phi}{d\tau} = \frac{L}{mR^2} + \frac{\sin^2\alpha}{ma}\left[E - \omega L \pm \frac{1}{\beta}(E - V_L^+)^{1/2}(E - V_L^-)^{1/2}\right] \qquad \text{(stone)} \qquad (2)$$

$$\frac{dT}{d\tau} = \left(\frac{R}{rH}\right)^2 \frac{1}{m}\left[E - \omega L \pm \beta(E - V_L^+)^{1/2}(E - V_L^-)^{1/2}\right] \qquad \text{(stone)} \qquad (3)$$

$$V_L^\pm(r) \equiv \omega L \pm \frac{rH}{R}\left(m^2 + \frac{L^2}{R^2}\right)^{1/2} \qquad \text{(stone)} \qquad (4)$$

Box 1 in Section 18.2 defines α, β, and ω.

Photon does not age.

To apply these equations to light, we must overcome a fundamental problem: The differential aging $d\tau$ of a light flash along its worldline is automatically zero (Section 1.4), so these equations of motion for the stone have no meaning for the light flash. To give them meaning, we eliminate $d\tau$ from these equations of motion, recasting them without $d\tau$. Use the following equations:

$$\frac{dr}{dT} = \left(\frac{dr}{d\tau}\right)\left(\frac{d\tau}{dT}\right) \qquad \text{(stone)} \qquad (5)$$

$$\frac{d\Phi}{dT} = \left(\frac{d\Phi}{d\tau}\right)\left(\frac{d\tau}{dT}\right) \qquad \text{(stone)} \qquad (6)$$

Carry out this combination on equations (1) through (3). *Result:* equations for dr/dT and $d\Phi/dT$. Note that in this process, m disappears from the coefficients—but remains in the expression for $V_L^\pm(r)$ in equation (4).

To convert these equations to describe light, we take the limit of the resulting equation for dr/dT and $d\Phi/dT$ as $m \to 0$. Carry this out first on expressions that contain V_L^+ and V_L^- defined in equation :

$$\left[\left(E - V_L^+\right)\left(E - V_L^-\right)\right]^{1/2} \qquad (7)$$

$$= \left[\left\{E - \omega L - \frac{rH}{R}\left(m^2 + \frac{L^2}{R^2}\right)^{1/2}\right\}\left\{E - \omega L + \frac{rH}{R}\left(m^2 + \frac{L^2}{R^2}\right)^{1/2}\right\}\right]^{1/2}$$

$$= \left[(E - \omega L)^2 - \left(\frac{rH}{R}\right)^2\left(m^2 + \frac{L^2}{R^2}\right)\right]^{1/2} \qquad \text{(stone)} \qquad (8)$$

Let $m \to 0$.

To convert this expression to light, take the limit as $m \to 0$:

$$\lim_{m\to 0}\left[\left(E - V_{\rm L}^{+}\right)\left(E - V_{\rm L}^{-}\right)\right]^{1/2} = \lim_{m\to 0}\left[(E-\omega L)^2 - \left(\frac{rH}{R}\right)^2\left(m^2 + \frac{L^2}{R^2}\right)\right]^{1/2} \quad (9)$$

$$= E\left[\left(1 - \omega\frac{L}{E}\right)^2 - \left(\frac{rH}{R}\right)^2\left(\frac{L}{RE}\right)^2\right]^{1/2} \quad (10)$$

$$= E\left[(1-\omega b)^2 - \frac{b^2}{R^2}\left(\frac{rH}{R}\right)^2\right]^{1/2} \quad (11)$$

$$\equiv E \times F_{\rm spin}(a,b,r) \qquad \text{(light)} \quad (12)$$

Impact parameter b for light

Equation (11) defines a new constant b, while (12) defines a new function $F_{\rm spin}(a,b,r)$:

$$b \equiv \frac{L}{E} \qquad \text{(impact parameter for light)} \quad (13)$$

$$F_{\rm spin}(a,b,r) \equiv \left[(1-\omega b)^2 - \frac{b^2}{R^2}\left(\frac{rH}{R}\right)^2\right]^{1/2} \qquad \text{(light)} \quad (14)$$

QUERY 1. Motion of light around the non-spinning black hole Show that when $a \to 0$, then $F_{\rm spin}(a,b,r) \to F(b,r)$, defined in equation (16), Section 11.3.

Only L/E matters.

The spinning black hole shares an important simplification with the non-spinning black hole (Section 11.2): the motion of light does not depend on L or E separately, but only on their ratio, the impact parameter $b \equiv L/E$.

Comment 1. Both L and b can be positive or negative.
The angular momentum L of a stone can be positive (prograde orbit) or negative (retrograde orbit). Equation (13) shows the same to be true of impact parameter b. For the non-spinning black hole, orbits in the two directions are simply mirror images of one another. In contrast, for the spinning black hole counterclockwise and clockwise orbits of a stone are quite different, as Chapters 18 and 19 show. Prograde and retrograde orbits of light are also different from one another, as Section 20.3 will show.

Equations of motion for light

These results allow us to write down the equations of motion for light in the equatorial plane of the spinning black hole:

$$\frac{dr}{dT} = \pm\left(\frac{rH^2}{R}\right)\frac{F_{\text{spin}}}{1-\omega b \pm \beta F_{\text{spin}}} \qquad \text{(light)} \qquad (15)$$

$$\frac{d\Phi}{dT} = \left(\frac{rH}{R}\right)^2 \frac{\dfrac{b}{R^2} + \dfrac{\sin^2\alpha}{a}\left[1-\omega b \pm \dfrac{1}{\beta}F_{\text{spin}}\right]}{1-\omega b \pm \beta F_{\text{spin}}} \qquad \text{(light)} \qquad (16)$$

$$\frac{dr}{d\Phi} = \pm\frac{R}{r}\frac{F_{\text{spin}}}{\dfrac{b}{R^2} + \dfrac{\sin^2\alpha}{a}\left[1-\omega b \pm \dfrac{1}{\beta}F_{\text{spin}}\right]} \qquad \text{(light)} \qquad (17)$$

QUERY 2. And when $a \to 0$?

In Query 1 you showed that as $a \to 0$, $F_{\text{spin}} \to F$ for the non-spinning black hole. Apply the same limit to expressions in equations(15) through (17). Box 1 in Section 18.2 may be useful.

As $a \to 0$:

A. $R \to r$
B. $H^2 \to (1 - 2M/r)$
C. $\omega \to 0$
D. $\beta \to (2M/r)^{1/2}$
E. $\sin^2\alpha/a \to 0$

With these changes, show that equations (15) through (17) for the motion of light around the spinning black hole reduce to equations (17) through (19) in (Section 11.3) for the non-spinning black hole.

20.3 ■ EFFECTIVE POTENTIAL FOR LIGHT

Orbits at a glance

Derive effective potential for light.

Now we derive the effective potential for light, which allows us to predict at a glance the r-motion of a light flash that moves in the equatorial plane of the spinning black hole. This derivation follows a procedure similar to that for the non-spinning black hole in Section 11.3. Run your finger down that earlier derivation to compare the following derivation for the spinning black hole.

To start, multiply both sides equation (15) by an expression that leaves only $\pm F_{\text{spin}}$ on the right side. Then multiply both sides by M/r and square both sides of the resulting equation. The result has the form:

$$A^2(a,b,r)\left(\frac{dr}{dT}\right)^2 = \left(\frac{M}{b}\right)^2 F^2_{\text{spin}}(a,b,r) \qquad \text{(light)} \qquad (18)$$

$$= \left(\frac{M}{b}\right)^2 - 2M\omega\left(\frac{M}{b}\right) + M^2\omega^2 - \frac{M^2}{R^2}\left(\frac{rH}{R}\right)^2 \qquad (19)$$

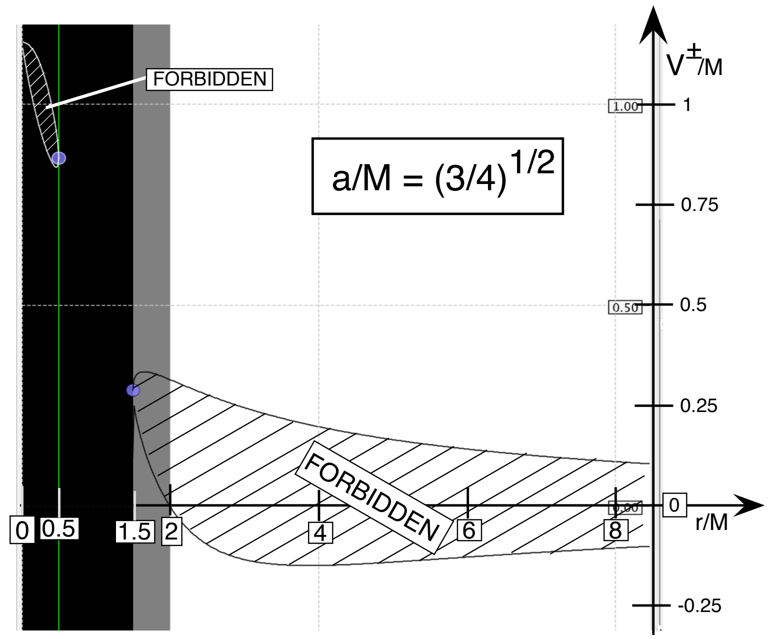

FIGURE 1 Effective potential for light with $a/M = (3/4)^{1/2}$. The gray region extends from the static limit at $r_{\rm S} = 2M$ down to the event horizon at $r_{\rm EH} = 1.5M$. The Cauchy horizon is at $r_{\rm CH} = 0.5M$. There are two forbidden regions for light, one outside the event horizon and one inside the Cauchy horizon. Equation (22) shows that inside a forbidden region dr/dT is imaginary .

where

$$A(a,b,r) \equiv \left(\frac{M}{b}\right)\left(\frac{rH^2}{R}\right)\left[1 - \omega b \pm \beta F_{\rm spin}(a,b,r)\right] \tag{20}$$

The right side of (19) is quadratic in M/b, so factor it using the quadratic equation:

$$\frac{M}{b} = M\omega \pm \left[\frac{M^2}{R^2}\left(\frac{rH}{R}\right)^2\right]^{1/2} \tag{21}$$

Substitute this expression for M/b into the second term on the right side of (19) and collect terms, with the result:

$$\boxed{A^2(a,b,r)\left(\frac{dr}{dT}\right)^2 = \left(\frac{M}{b}\right)^2 - \left[\frac{V^{\pm}(a,r)}{M}\right]^2} \tag{22}$$

where

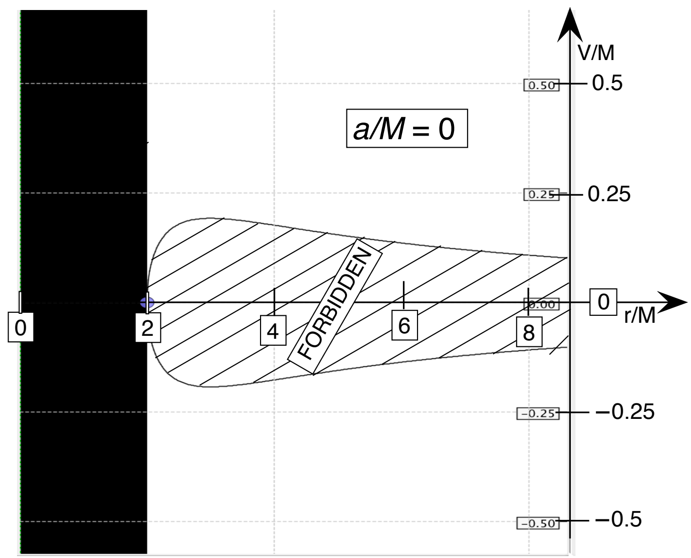

FIGURE 2 For comparison, effective potential for light for the non-spinning black hole ($a = 0$), to the same scale as Figure 1.

$$\left[\frac{V^{\pm}(a,r)}{M}\right]^2 \equiv \frac{M^2}{R^2}\left(\frac{rH}{R}\right)^2 \pm \frac{2M^2\omega}{R}\left(\frac{rH}{R}\right) - M^2\omega^2 \qquad (23)$$

The superscript $\pm$ on the left side of (23) is the same as the $\pm$ on the right side.

Track r-motion of light.

Equation (22) tracks the r-motion of a light flash in the equatorial plane: The first term on the right side is a function of b but not a function of a or r, while the second term—the square of the effective potential—is a function of a and r, but not a function of b. Equation (22) then permits us to plot on the same diagram the effective potential function (23)—Figure 1—and a horizontal line that represents any given value of $(M/b)^2$ in (22).

Forbidden region for light

Equation (22) tells us why the region between V^- and V^+ in Figure 1 is forbidden: If $(M/b)^2$ is less than $(V^+/M)^2$ but greater than $(V^-/M)^2$, then dr/dT is imaginary, which is indeed forbidden. Figure 2 reminds us of the corresponding effective potential for the non-spinning black hole.

QUERY 3. Effective potential for the non-spinning black hole

Show that when $a \rightarrow 0$, equations (22) and (23) reduce to equations (25) and (26) in Section 11.3, shown in Figure 2.

QUERY 4. Mass vs. massless
Compare Figure 1 for light with the corresponding Figure 1 in Section 18.3 for a stone, both for $a/M = (3/4)^{1/2}$. Is the following statement true or false: Outside the event horizon, the forbidden region for light is entirely enclosed in the forbidden region for the stone.

Where light goes

Global radial motion dr/dT must be real. Therefore the right side of (22) must be positive. *Result:* The light flash that moves along a horizontal line at $(M/b)^2$ in Figure 1, for example, cannot enter either forbidden region between the two effective potential curves. A light flash from far away that reaches one of these curves either reverses the sign of its r-motion or holds its r-value at the curve's maximum or minimum.

20.4 ■ EXERCISE

1. Does a spinning back hole appear smaller or larger than a non-spinning black hole?

Think of two black holes. The first is a non-spinning black hole of a particular given mass M. The second is a spinning black hole of the *same* mass M. Can a distant observer tell them apart?

Specifically, (1) How large does a black hole look to an Earth observer? (2) Can the Earth observer determine whether or not the black hole is spinning or non-spinning? (3) Since most black holes in Nature spin, can the Earth observer determine *how fast* the spinning black hole rotates—that is, what is its value of a/M? When you finish this exercise, you should be able to answer these three questions clearly and definitively.

Non-spinning black hole: Review Figures 3 and 12 in Chapter 11. They show that r-coordinate of the knife-edge circular orbit for light around a non-spinning black hole $r = 3M$ corresponds to the maximum of the effective potential curve $V(r)$, and that this knife-edge orbit determines the visual size of the non-spinning black hole through the critical impact parameter b_{critical}: the visual diameter of the non-spinning black hole is $2b_{\text{critical}}$.

Spinning black hole: The visual size of the spinning black hole in the equatorial plane is determined by two critical impact parameters b^+_{critical} and b^-_{critical} that belong to the prograde and retrograde knife-edge orbits of light, respectively, as shown in Figure 3 for the special case $a = (3/4)^{1/2}M$.

A. Figure 1 in this chapter plots the effective potentials $V^{\pm}(r)$ for a light beam that moves near a black hole with $a/M = (3/4)^{1/2}$. From Figure 1, show that

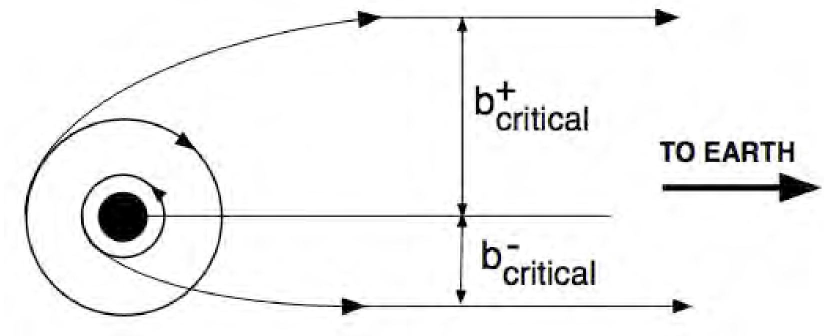

FIGURE 3 Figure for Exercise 1. Schematic diagram showing the visual size of a spinning black hole with $a/M = (3/4)^{1/2} = 0.866$.

$$b^+ = \frac{1}{V^+_{\text{max}}} = \frac{1}{V^+(r^+_{\text{knife edge}})} \quad \text{prograde knife-edge orbit} \tag{24}$$

$$b^- = \frac{1}{V^-_{\text{max}}} = \frac{1}{V^-(r^-_{\text{knife edge}})} \quad \text{retrograde knife-edge orbit} \tag{25}$$

B. Derive the following expressions for $b^{\pm}_{\text{critical}}$ and $r^{\pm}_{\text{knife edge}}$:

$$b^{\pm}_{\text{critical}} = \left(r^{\pm}_{\text{knife edge}} + 3M\right)\left(\frac{r^{\pm}_{\text{knife edge}}}{4M}\right)^{1/2} \tag{26}$$

$$r^{+}_{\text{knife edge}} = 4M\cos^2\Psi^{\pm} \text{ where } \Psi = \frac{1}{3}\arccos\left(\mp\frac{a}{M}\right) \tag{27}$$

C. Evaluate equations (26) and (27) for $a = 0$ and show that both results agree with equation (28) in Chapter 11.

D. Find the values of b^+ and b^- for a spinning black hole with (a) $a/M = (3/4)^{1/2}$ and (b) $a/M = 1$ (maximum-spin black hole).

E. *Optional:* Plot the visual size ($b^+_{\text{critical}} + b^-_{\text{critical}}$) of the spinning black hole as a function of its spin parameter a/M.

F. Answer the question posed in the title of this exercise. Include a sentence that starts, "This depends on . . . .".

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