PROJECT F

The Spinning Black Hole

Black holes are macroscopic objects with masses varying from a few solar masses to millions of solar masses. To the extent they may be considered as stationary and isolated, to that extent, they are all, every single one of them, described exactly by the Kerr solution. This is the only instance we have of an exact description of a macroscopic object. Macroscopic objects, as we see them all around us, are governed by a variety of forces, derived from a variety of approximations to a variety of physical theories. In contrast, the only elements in the construction of black holes are our basic concepts of space and time. They are, thus, almost by definition, the most perfect macroscopic objects there are in the universe. And since the general theory of relativity provides a single unique two-parameter family of solutions for their description, they are the simplest objects as well.

—S. Chandrasekhar

1 Introduction

In this project we explore some of the properties of spacetime near a spinning black hole. Analogous properties describe spacetime external to the surface of the spinning Earth, Sun, or other spinning uncharged heavenly body. For a black hole these properties are truly remarkable. Near enough to a spinning black hole—even outside its horizon—you cannot resist being swept along tangentially in the direction of rotation. You can have a negative total energy. From outside the horizon you can, in principle, harness the rotational energy of the black hole.

Do spinning black holes exist? The primary question is: Do black holes exist? If the answer is yes, then spinning black holes are inevitable, since astronomical bodies most often rotate. As evidence, consider the most compact stellar object short of a black hole, the neutron star. Detection of radio and X-ray pulses from some spinning neutron stars (called **pulsars**) tells us that many neutron stars rotate, some of them very rapidly. These are impressive structures, with more mass than our Sun, some of them spinning once every few milliseconds. Conclusion: If black holes exist, then spinning black holes exist.

General relativity predicts that when an isolated spinning star collapses to a black hole, gravitational radiation quickly (in a few seconds of far-away time!) smooths any irregularities in rotation. Thereafter the metric exterior to the horizon of the spinning black hole will be the Kerr metric used in this project.

However, not all spinning black holes are isolated; many are surrounded by other matter attracted to them. The inward-swirling mass of a resulting **accretion disk** can affect spacetime in its vicinity, distorting the metric away from that of the isolated spinning black hole that we analyze here.

2 Angular Momentum of the Black Hole

An isolated spinning uncharged black hole is completely specified by just two quantities: its mass M and its angular momentum. In Chapter 4 (page 4-3) we defined the angular momentum per unit mass for a particle orbiting a nonspinning black hole as $L/m = r^2 d\phi / d\tau$. In this expression, the angle ϕ has no units and proper time τ has the unit meter. Therefore L/m has the unit meter. To avoid confusion, the angular momentum of a spinning black hole of mass M is given the symbol J and its angular momentum per unit mass is written J/M. The ratio J/M appears so often in the analysis that it is given its own symbol: $a = J/M$. We call the constant "a" the **angular momentum parameter**. Just as the angular momentum L/m of a stone orbiting a non-rotating black hole has the unit meter, so does the angular momentum parameter $a = J/M$ have the unit meter. In what follows it is usually sufficient to treat the angular momentum parameter a as a positive scalar quantity.

Newman and others found the metric for a spinning black hole with net electric charge (see references in Section 14 and also equation [51] for the metric of a charged *nonspinning* black hole). The most general steady-state black hole has mass, angular momentum, and electric charge. However, we have no evidence that astronomical bodies carry sufficient net electric charge (which would ordinarily be rapidly neutralized) to affect the metric. If actual black holes are uncharged, then the Kerr metric describes the most general stable isolated black hole likely to exist in Nature.

3 The Kerr Metric in the Equatorial Plane

For simplicity we are going to study spacetime and particle motion in the **equatorial plane** of a symmetric spinning black hole of angular momentum J and mass M. The equatorial plane is the plane through the center of the spinning black hole and perpendicular to the spin axis.

Here is the **Kerr metric** in the equatorial plane, expressed in what are called **Boyer-Lindquist coordinates**. The angular momentum parameter $a = J/M$ appears in a few unaccustomed places.

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 + \frac{4Ma}{r}dt d\phi - \frac{dr^2}{1 - \frac{2M}{r} + \frac{a^2}{r^2}} - \left(1 + \frac{a^2}{r^2} + \frac{2Ma^2}{r^3}\right)r^2 d\phi^2 \quad [1]$$

For the *nonrotating* black hole examined in Chapters 2 through 5, the Schwarzschild metric describing spacetime on a plane is the same for *any* plane that cuts through the center of the black hole, since the Schwarzschild black hole is spherically symmetric. The situation is quite different for the spinning Kerr black hole; the metric [1] is correct *only* for neighboring events that occur in the plane passing through the center of the black hole and perpendicular to its axis of rotation. We choose the equatorial plane because it leads to the simplest and most interesting results.

The time t in equation [1] is the "far-away time" registered on clocks far from the center of attraction, just as for the Schwarzschild metric. In contrast, for $a > 0$ the Boyer-Lindquist r-coordinate does *not* have the simple geometrical meaning that it had for the Schwarzschild metric. More on the meaning of r in Sections 4 and 9. The metric [1] provides a *complete* description of spacetime in the equatorial plane outside the horizon of a spinning uncharged black hole. No additional information is needed to answer every possible question about its (nonquantum) properties and (with the Principle of Extremal Aging) about orbits of free particles and light pulses in the equatorial plane.

You say that the Kerr metric provides a complete nonquantum *description of the spinning black hole. Why this reservation? What more do we need to know to apply general relativity to quantum phenomena?*

In answer, listen to Stephen Hawking as he discusses the "singularity" of spacetime at the beginning of the Universe. A similar comment applies to the singularity inside any black hole.

The general theory of relativity is what is called a classical theory. That is, it does not take into account the fact that particles do not have precisely defined positions and velocities but are "smeared out" over a small region by the uncertainty principle of quantum mechanics that does not allow us to measure simultaneously both the position and the velocity. This does not matter in normal situations, because the radius of curvature of space-time is very large compared to the uncertainty in the position of a particle. However, the singularity theorems indicate that space-time will be highly distorted, with a small radius of curvature at the beginning of the present expansion phase of the universe [or at the center of a black hole]. In this situation, the uncertainty principle will be very important. Thus, general relativity brings about its own downfall by predicting singularities. In order to discuss the beginning of the universe [or the center of a black hole], we need a theory that combines general relativity with quantum mechanics.

—Stephen Hawking

Suggestion: As you go along, check the units of all equations, the equations in the project and also your own derived equations. An equation can be wrong if the units are right, but the equation cannot be right if the units are wrong!

Do Spinning Black Holes Power Quasars?

In contrast to dead solitary black holes, the most powerful steady source of energy we know or conceive or see in all the universe may be powered by a spinning black hole of many millions of solar masses, gulping down enormous amounts of matter swirling around it. Maarten Schmidt, working at the Palomar Mountain Observatory in 1956, was the first to uncover evidence for these **quasi-stellar objects**, or **quasars**, starlike sources of light located not billions of kilometers but billions of light-years away. Despite being far smaller than any galaxy, the typical quasar manages to put out more than a hundred times as much energy as our entire Milky Way with its hundred billion stars. Quasars—unsurpassed in brilliance and remoteness—can justly be called lighthouses of the heavens.

Observation and theory have come together to explain in broad outline how a quasar operates. A spinning black hole of some hundreds of millions of solar masses, itself perhaps built by accretion, accretes more mass from its surroundings. The incoming gas, and stars converted to gas, does not fall in directly, any more than the water rushes directly down the bathtub drain when the plug is pulled. This gas, as it goes round and round, slowly makes its way inward to regions of ever-stronger gravity. In the process it is compressed and heated and finally breaks up into positive ions and electrons, which emit copious amounts of radiation at many wavelengths. The in-falling matter brings with it some weak magnetic fields, which are also compressed and powerfully strengthened. These magnetic fields link the swirling electrons and ions into a gigantic accretion disk. Matter little by little makes its way to the inner boundary of this accretion disk and then, in a great swoop, falls across the horizon into the black hole. During that last swoop, hold on the particle is relinquished. Therefore, the chance is lost to extract as energy the full 100 percent of the mass of each in-falling bit of matter. However, magnetic fields do hold on to the ions effectively enough and long enough to extract, as radiant energy, several percent of the mass. In contrast, neither nuclear fission nor nuclear fusion is able to obtain a conversion efficiency of more than a fraction of 1 percent. No one has ever seen evidence for a more effective process to convert bulk matter into energy than accretion into a spinning black hole, and no one has ever been able to come up with a more feasible scheme to explain the action of quasars.
See Section 11 for more details.

QUERY 1 **Equatorial-plane Kerr metric in the limit of zero angular momentum.** Show that for zero angular momentum ($a = J/M = 0$), the Kerr metric, equation [1], reduces to the Schwarzschild metric (equation [A] in Selected Formulas at the end of this book).

QUERY 2 **Motion stays in plane.** Make an argument from symmetry that a free object that begins to orbit a spinning black hole in the equatorial plane will stay in the equatorial plane.

The Kerr metric has four central new features that distinguish it from the Schwarzschild metric.

The first new feature of the Kerr metric is a new *r*-value for the horizon. In the Schwarzschild metric, the coefficient of dr^2 is $1/(1 - 2M/r)$. This coefficient increases without limit at the Schwarzschild horizon, $r_H = 2M$. For the Kerr metric, in contrast, the horizon—the point of no return—has an r-value that depends on the value of the angular momentum parameter a. *(Note:* A true proof that a horizon exists requires the demonstration that worldlines can run through it only in the inward direction, not outward. For the corresponding proof for the non-spinning black hole, see Project B, pages B-14–15. Our choice here of the horizon at the place where the coefficient of dr^2 blows up is an intuitive, but yet correct, selection.)

QUERY 3 **Radial coordinate of the horizon.** Show that for the spinning black hole, the coefficient of dr^2 increases without limit at the r-value:

$$r_H = M \pm (M^2 - a^2)^{1/2} \qquad [2]$$

Look first at the case with the plus sign. What value does r_H have when $a = 0$? For a spinning black hole, is the value of r_H greater or less than the corresponding r-value for the Schwarzschild horizon?

Unless stated otherwise, when we say "the horizon" we refer to equation [2] with the plus sign.

Research note: Choosing the minus sign in equation [2] leads to a second horizon that is *inside* the outer, plus-sign horizon. This inner horizon is called the **Cauchy horizon**. Theoretical research shows that spacetime is stable (correctly described by the Kerr metric) immediately inside the outer horizon and most of the way down to the inner (Cauchy) horizon. However, near the Cauchy horizon, spacetime becomes unstable and therefore is *not* described by the Kerr metric. At the Cauchy horizon is located the so-called *mass-inflation singularity* described in the box on page B-5. The presence of the mass-inflation singularity at the Cauchy horizon bodes ill for a diver wishing to experience in person the region between the outer horizon and the center of a rotating black hole. It is delightful to read in a serious theoretical research paper a sentence such as the following: "Such . . . results strongly suggest (though they do not prove) that inside a black hole formed in a generic collapse, an observer falling toward the inner [Cauchy] horizon should be engulfed in a wall of (classically) infinite density immediately after seeing the entire future history of the outer universe pass before his eyes in a flash." (Poisson and Israel)

4 The Kerr Metric for Extreme Angular Momentum

In this project we want to uncover the central features of the spinning black hole with minimum formalism. The equations become simpler for the case of a black hole that is spinning at the maximum possible rate.

QUERY 4 **Maximum value of the angular momentum.** How "live" can a black hole be? That is, how large is it possible to make its angular momentum parameter $a = J/M$? Show that the largest value of the angular momentum parameter, a, consistent with a real value of r_H is $a = M$. This maximum value of the angular momentum parameter a is equivalent to angular momentum $J = M^2$. What happens to the inner (Cauchy) horizon in this case?

A black hole spinning at the maximum rate derived in Query 4 is called an **extreme Kerr black hole.** How fast are existing black holes likely to spin; how "live" are they likely to be? Listen to Misner, Thorne, and Wheeler (page 885): "Most objects (massive stars; galactic nuclei; . . .) that can collapse to form black holes have so much angular momentum that the holes they produce should be 'very live' (the angular momentum parameter $a = J/M$ nearly equal to M; J nearly equal to M^2)."

> QUERY 5 **Maximum angular momentum of Sun?** A recent estimate of the angular momentum of Sun is 1.91×10^{41} kilogram meters^2 per second (see the references). What is the value of the angular momentum parameter $a = J/M$ for Sun, in meters? (*Hint:* Divide the numerical value above by M_{kg}, the mass of Sun in kilograms, to obtain an intermediate result in units of meter^2/second. What conversion factor do you then use to obtain the result in meters?) What fraction a/M is this of the maximum possible value permitted by the Kerr metric?

The metric for the equatorial plane of the extreme-spin black hole results if we set $a = M$ in equation [1], which then becomes

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 + \frac{4M^2}{r}dtd\phi - \frac{dr^2}{\left(1 - \frac{M}{r}\right)^2} - R^2 d\phi^2 \qquad \text{[3. extreme Kerr]}$$

Note how the denominator of the dr^2 term in the Kerr metric differs in two ways from the dr^2 term in the Schwarzschild metric: here the denominator is squared and also contains M/r instead of $2M/r$.

Equation [3] has been simplified by defining

$$R^2 \equiv r^2 + M^2 + \frac{2M^3}{r} \qquad \text{[4. extreme Kerr]}$$

The form $R^2 d\phi^2$ of the last term on the right side of equation [3] tells us that R is the **reduced circumference** for extreme Kerr spacetime. That is, the value of R is determined by measuring the circumference of a stationary ring in the equatorial plane concentric to the black hole and dividing this circumference by 2π. This means that r is *not* the reduced circumference but has a value derived from equation [4]. Finding an explicit expression for r in terms of R requires us to solve an equation in the third power of r, which leads to an algebraic mess. Rather than solving such an equation, we carry along expressions containing both R and r. Note from equation [4] that R is not equal to r even for large values of r, although the *percentage* difference between R and r does decrease as r increases.

> QUERY 6 **Limiting values of *R*.** What is r_H, the value of r at the horizon of an extreme spinning black hole? What is R_H, the value of R at the horizon? Find the *approximate* range of r-values for which the value of R differs from the value of r by less than one part in a million.

QUERY 7	**More general R_a.** Consider the more general case of arbitrary angular momentum parameter *a* given in equation [1]. What is the expression for R^2 (call it R_a^2) in this case? What is the value of R_a in the limiting case of the nonspinning black hole?

Now move beyond the new *r*-value for the horizon—the first new feature of the Kerr metric—to the **second new feature of the Kerr metric,** which is the presence of the product $dtd\phi$ of two different spacetime coordinates, called a **cross product**. The cross product implies that coordinates ϕ and t are intimately related. In the following section we show that the Kerr metric predicts **frame dragging**. What does "frame dragging" mean? Near any center of attraction, radial rocket thrust is required to keep a stationary observer at a fixed radius. Near a spinning black hole an additional *tangential* rocket thrust is required during *initial* placement of an object in a stationary position, a position from which the fixed stars do not appear to move overhead. (See box page F-20.) One might say that spacetime is swept around by the rotating black hole: spacetime itself on the move!

Unless otherwise noted, everything that follows applies to the equatorial plane around an extreme Kerr black hole.

5 The Static Limit

The third new feature of the Kerr metric is the presence of a so-called **static limit**. The horizon of a rotating black hole lies at an *r*-value *less* than $2M$ (equation [2] with the plus sign). The horizon is where the metric coefficient of dr^2 blows up. In contrast, for the equatorial plane, the coefficient of dt^2, namely, $(1 - 2M/r)$, goes to zero at $r = 2M$, just as it does in the Schwarzschild metric for a nonrotating black hole. The *r*-value $r = 2M$ in the equatorial plane at which the coefficient of the dt^2 term goes to zero is called the **static limit**. An examination of equations [3] and [1] shows that the expression for the static limit in the equatorial plane is the same whatever the value of the angular momentum parameter *a*, namely

$$r_S = 2M \qquad [5]$$

The static limit gets its name from the prediction that for radii smaller than r_S (but greater than that of the horizon r_H) an observer cannot remain at rest, cannot stay *static*. The space between the static limit and the horizon is called the **ergosphere**. Inside the ergosphere you are inexorably dragged along in the direction of rotation of the black hole. Not even a tangential rocket allows you to stand at one fixed angle ϕ. For you the fixed stars cannot remain at rest overhead. In principle, a small amount of frame dragging is detectable near any spinning astronomical object. An experimental Earth satellite (Gravity Probe B), now under construction at Stanford University, will measure the extremely small frame-dragging effects predicted near the spinning Earth. Inside the static limit of a rotat-

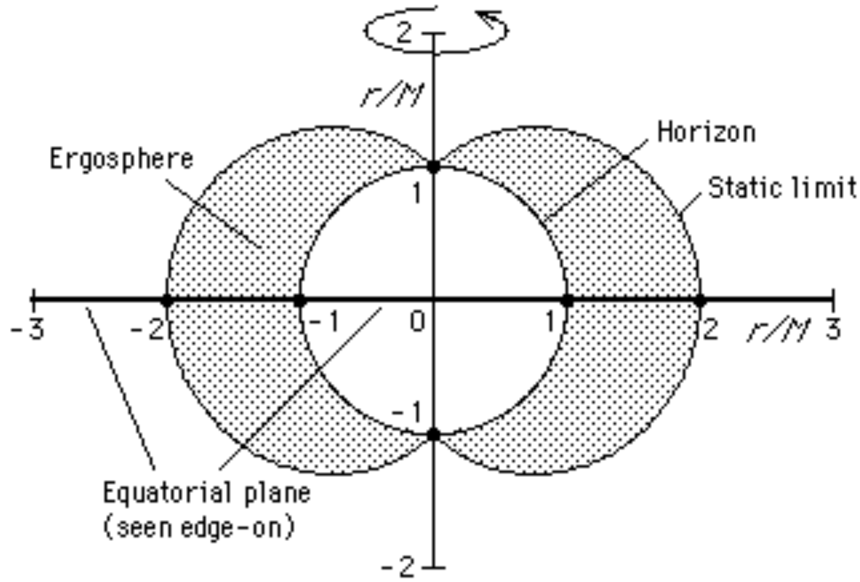


Figure 1** Computer plot of the cross-section of an extreme black hole showing the static limit and horizon using the Kerr bookkeeper (Boyer-Lindquist) coordinate r (not R). From inside the horizon no object can escape, even one traveling at the speed of light. Between the horizon and the static limit lies the* ***ergosphere, *shaded in the figure. Within this ergosphere everything—even light—is swept along by the rotation of the black hole. Inside the ergosphere, too, a stone can have a negative total energy (Section 10).*

ing black hole, in contrast, the frame dragging is irresistible, as will be described on the following page.

The Kerr metric for three space dimensions—not discussed in this book—reveals that the horizon has a constant *r*-value in all directions (is a sphere) while the static limit has cusps at the poles. Figure 1 shows this result. This figure is drawn using the Kerr bookkeeper (Boyer-Lindquist) *r*-coordinate, which shows only one possible way to view these structures. When Figure 1 is plotted in terms of the reduced circumference R/M instead of r/M, then the radius of the horizon is greater in the equatorial plane than along the axis of rotation, giving the horizon the approximate shape of a hamburger bun.

QUERY 8 **Reduced circumference of the static limit.** For the extreme black hole, find an expression for R_S, the reduced circumference of the static limit, in the equatorial plane.

QUERY 9 **Displaying the spinning black hole from above.** Draw a cross-section of the extreme black hole in the equatorial plane. That is, display the static limit and horizon in bookkeeper coordinates on a plane cut through the horizontal axis of Figure 1, as if viewing that figure downward along the vertical axis from above. Label the static limit, horizon, and ergosphere and put in expressions for their radii.

Now look more closely at the nature of the static limit in the equatorial plane. Examine the Kerr metric for the case of a light flash moving initially in the ϕ direction ($dr = 0$). (Only the *initial* motion in the equatorial plane will be tangential; later the flash may be deflected radially away from the tangential direction.) Because this is light, the proper time is zero between adjacent events on its path: $d\tau = 0$. Make these substitutions in the metric [3], divide through by dt^2, and rearrange to obtain

$$R^2\left(\frac{d\phi}{dt}\right)^2 - \frac{4M^2}{r}\left(\frac{d\phi}{dt}\right) - \left(1 - \frac{2M}{r}\right) = 0 \qquad \text{[6. light, } dr = 0\text{]}$$

Equation [6] is quadratic in the angular velocity $d\phi/dt$.

> QUERY 10 **Tangential motion of light.** Solve equation [6] for $d\phi/dt$. Show that the result has two possible values (simplified in equation [11], page F-10):
>
> $$\frac{d\phi}{dt} = \frac{2M^2}{rR^2} \pm \frac{2M^2}{rR^2}\left[1 + \frac{r^2R^2}{4M^4}\left(1 - \frac{2M}{r}\right)\right]^{1/2} \qquad \text{[7. light, } dr = 0\text{]}$$

Look closely at this expression at the static limit, namely, where $r = 2M$ and $R^2 = 6M^2$. The two solutions are

$$\frac{d\phi}{dt} = 0 \qquad \text{and} \qquad \frac{d\phi}{dt} = \frac{4M^2}{rR^2} = \frac{1}{3M} \qquad \text{[8. light, } dr = 0\text{]}$$

To paraphrase Schutz (see references), the second solution in [8] represents light sent off in the same direction as the hole is rotating. The first solution says that the other light flash—the one sent “backward”—does not move at all as recorded by the far-away bookkeeper. The dragging of orbits has become so strong that this light cannot move in the direction opposite to the rotation! Clearly, any material particle, which must move slower than light, will therefore have to rotate with the hole, even if it has an angular momentum arbitrarily large in the sense opposite to that of hole rotation.

> QUERY 11 **Light dragging in the ergosphere.** Show that *inside* the ergosphere (r such that $r_H < r < r_S$), light launched in *either* tangential direction in the equatorial plane moves in the direction of rotation of the black hole as recorded by the far-away bookkeeper. That is, show that the initial tangential angular velocity $d\phi/dt$ is always positive.

The static limit creates a difficulty of principle in measuring the reduced circumference R, defined by equation [4] on page F-6. According to that definition, one measures R by laying off the total distance—the circumference—around a *stationary* ring in the equatorial plane concentric to the black hole, then dividing that circumference by 2π to find the value of R. But inside the static limit no such ring can remain stationary; it is inevitably swept along in a tangential direction, even if we fire powerful rockets tangentially trying to keep it stationary. Thus, for the present, we have no practical definition for R inside the static limit. We will overcome this difficulty in principle in Section 9.

To anticipate a later result, we mention here the fourth new feature of the Kerr metric, which is analyzed further in Sections 10 and 11.

The fourth new feature of the Kerr metric is available energy. No net energy can be extracted from a nonspinning black hole (except for the quantum "Hawking radiation," page 2-4, which is entirely negligible for star-mass black holes). For this reason, the nonspinning black hole carries the name *dead*. In contrast, energy of rotation is available from a spinning black hole, which therefore deserves its name *live*. See Section 12.

6 Radial and Tangential Motion of Light

QUERY 12 **Radial motion of light.** For light ($d\tau = 0$) moving in the radial direction ($d\phi = 0$), show from the metric that

$$\frac{dr}{dt} = \pm\left(1 - \frac{M}{r}\right)\left(1 - \frac{2M}{r}\right)^{1/2} \qquad \text{[9. light, } d\phi = 0\text{]}$$

Show that this radial speed goes to zero at the static limit and is imaginary (therefore unreal) inside the ergosphere. *Meaning:* No purely radial motion is possible inside the ergosphere. See Figure 2.

For light ($d\tau = 0$) moving in the tangential direction ($dr = 0$), we call the tangential velocity $Rd\phi/dt$ as recorded by the Kerr bookkeeper. From equation [7], this tangential velocity is given by

$$R\frac{d\phi}{dt} = \frac{2M^2}{rR} \pm \frac{2M^2}{rR}\left[1 + \frac{r^2R^2}{4M^4}\left(1 - \frac{2M}{r}\right)\right]^{1/2} \qquad \text{[10. light, } dr = 0\text{]}$$

The second term on the right side of [10] can be simplified by substituting for R^2 in the numerator from equation [4]. (Trust us or work it out for yourself!) Equation [10] becomes

$$R\frac{d\phi}{dt} = \frac{2M^2}{rR} \pm \frac{r - M}{R} \qquad \text{[11. light, } dr = 0\text{]}$$

QUERY 13 **Light dragging at the horizon.** What happens to the light dragging at the horizon (r_H given by equation [2] with the plus sign and $a = M$, and R_H derived in Query 6)? Show that at the horizon the initial tangential rotation $d\phi/dt$ for light has a single value whichever way the pulse is launched. Show that the bookkeeper initial tangential velocity $Rd\phi/dt$ for this light at the horizon has the value shown in Figure 2.

The radial and tangential velocities of light in equations [9] and [11] are *bookkeeper velocities*, reckoned by the Kerr bookkeeper using the coordinates r and ϕ and the far-away time t. Nobody measures the Kerr bookkeeper velocities directly, just as nobody measured directly bookkeeper velocities near a non-spinning black hole (Chapters 3 through 5).

Figure 2 shows the radial and tangential bookkeeper velocities of light for the extreme Kerr metric. Note again that these plots show the *initial* velocity of a light flash launched in the various directions. After launch, a radially moving light flash may be dragged sideways, or a tangentially moving flash may be deflected inward.

QUERY 14 **Locked-in motion?** (Optional) Kip Thorne says, "I guarantee that, if you send a robot probe down near the horizon of a spinning hole, blast as it may it will never be able to move forward or backward [in either tangential direction] at any speed other than the hole's own spin speed. . . ." What evidence do equation [11] and Figure 2 give for this conclusion? What is "the hole's own spin speed"? (See Kip S. Thorne, *Black Holes and Time Warps*, W. W. Norton & Co., New York, 1994, page 57.)

7 Wholesale Results, Extreme Kerr Black Hole

Now suppose that you have never heard of the Kerr metric and someone presents you with the "anonymous" metric [3] (which we know to be the metric for the extreme Kerr black hole) plus the definition of R:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 + \frac{4M^2}{r}dtd\phi - \frac{dr^2}{\left(1 - \frac{M}{r}\right)^2} - R^2 d\phi^2 \qquad [3]$$

$$R^2 \equiv r^2 + M^2 + \frac{2M^3}{r} \qquad [4]$$

You say to yourself, "This equation is just a crazy kind of mixed-up Schwarzschild-like metric, with a nutty denominator for the dr^2 term, a cross-term in $dtd\phi$, and R^2 instead of r^2 as a coefficient for $d\phi^2$. Still, it's a metric. So let's try deriving expressions for angular momentum, energy, and so forth for a particle moving in a region described by this metric in analogy to similar derivations for the Schwarzschild metric." So saying,

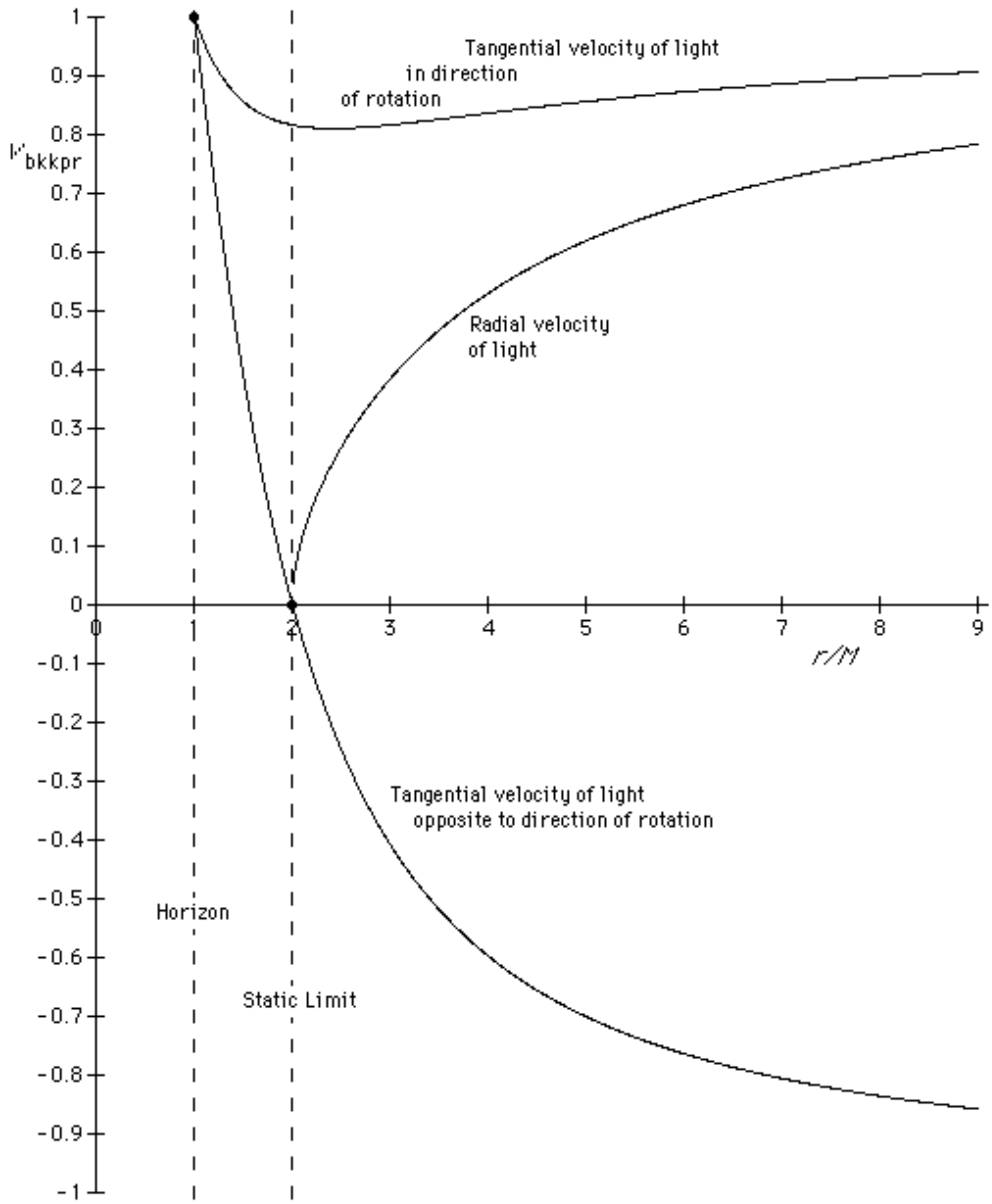


Figure 2 *Computer plot of bookkeeper radial and tangential velocities of light near an extreme Kerr black hole (a = J/M = M). Note that as r/M becomes large, the different bookkeeper velocities all approach plus or minus unity. Note also that purely radial motion of light is not possible inside the static limit. Important: These are* initial *velocities of light just after launch in the given direction. After launch, the light will generally change direction. For the case of a nonrotating black hole, see Figures 6 and 7, pages B-18–19.*

Table 1 Comparison of results of nonspinning and extreme-spin black holes

Quantity	Nonspinning Schwarzschild black hole	Extreme-spin Kerr black hole ("shell" = stationary ring outside static limit)
Define r and R	Reduced circumference = $r \equiv \frac{\text{(circumference of shell)}}{2\pi}$ [12]	Reduced circumference R given by: $R^2 \equiv r^2 + M^2 + \frac{2M^3}{r}$ [13]
Shell time vs. far-away time: (gravitational red shift)	$dt_{\text{shell}} = \left(1 - \frac{2M}{r}\right)^{1/2} dt$ [14]	$dt_{\text{shell}} = \left(1 - \frac{2M}{r}\right)^{1/2} dt$ [15. stationary]
dr_{shell} vs. dr	$dr_{\text{shell}} = \left(1 - \frac{2M}{r}\right)^{-1/2} dr$ [16]	$dr_{\text{shell}} = \left(1 - \frac{M}{r}\right)^{-1} dr$ [17. stationary]
Energy (constant of the motion)	$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$ [18]	$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} + \frac{2M^2}{r}\frac{d\phi}{d\tau}$ [19]
Angular momentum (constant of the motion)	$\frac{L}{m} = r^2\frac{d\phi}{d\tau}$ [20]	$\frac{L}{m} = R^2\frac{d\phi}{d\tau} - \frac{2M^2}{r}\frac{dt}{d\tau}$ [21]

you use the Principle of Extremal Aging and other methods of Chapters 2 through 5 to derive expressions similar to results in those chapters and enter them in the right hand column of Table 1.

Notes: (1) We limit ourselves to the equatorial plane. (2) Outside the static limit we can still set up stationary spherical shells (which we have limited to stationary *rings* in the equatorial plane). However, equation [21] with $d\phi/d\tau = 0$ tells us that a *stationary* ring has *negative* angular momentum. So during construction we need to provide an *initial tangential* rocket blast to give negative angular momentum to the ring structure in order to make it stationary. (See box page F-20.)

QUERY 15 **Energy and angular momentum as constants of the motion.** Derive entries [19] and [21] in Table 1 for energy and angular momentum of a free object moving in the equatorial plane of an extreme Kerr black hole.

8 Plunging: The "Straight-In Spiral"

Near the nonrotating black hole, the simplest motion was radial plunge (Chapter 3). What is the simplest motion near a spinning black hole? By analogy, examine the motion of a stone dropped from rest at a great distance which thereafter falls inward, maintaining zero angular momentum.

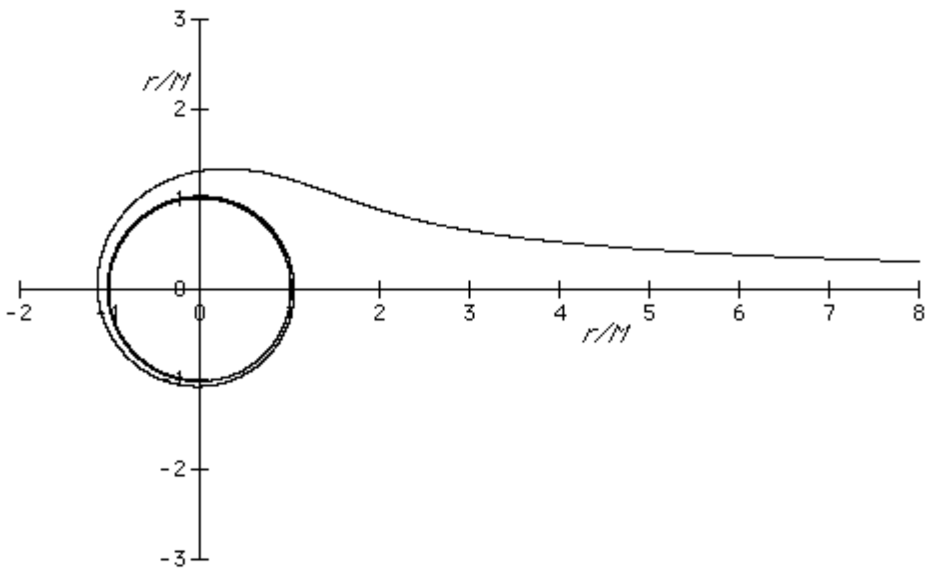


Figure 3 *Computer plot: Kerr map (Kerr bookkeeper plot) of the trajectory in space of a stone dropped from rest far from a black hole (therefore with zero angular momentum). According to the far-away bookkeeper, the stone spirals in to the horizon at r = M and circulates there forever.*

QUERY 16 **No angular momentum. But angular motion!** Set angular momentum [21] equal to zero and verify the following equation:

$$\frac{d\phi}{dt} = \frac{2M^2}{rR^2} \qquad [22.\ L = 0]$$

Equation [22] gives the remarkable result that a particle with zero angular momentum nevertheless circulates around the black hole! This result is evidence for our interpretation that the black hole drags nearby spacetime around with it. Figure 3 shows the trajectory of an inward plunger with zero angular momentum, as calculated in what follows.

Let's see if we can set up the equations to describe a stone that starts at rest far from a rotating black hole and moves inward with zero angular momentum. At remote distance, in flat spacetime, the stone has energy $E/m = 1$. It keeps the same energy as it falls inward. From equation [19] in Table 1,

$$\frac{E}{m} = 1 = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} + \frac{2M^2}{r}\frac{d\phi}{d\tau} \qquad [23]$$

Equations [22] and [23] are two equations in the four unknowns dr, dt, $d\tau$, and $d\phi$. A third equation is the metric [3] for the extreme-spin black hole. With these three independent equations, we can eliminate three of the four unknowns to find a relation between any two remaining differentials. We choose to solve for the quantities dr and $d\phi$, because we want to draw the trajectory, the Kerr map. Don't bother doing the algebra—it is a mess. After substituting equation [4] for R^2 into the result, one obtains the relation between dr and $d\phi$:

$$\begin{aligned} dr &= \frac{r-M}{r}\left[\frac{r^5}{2M^3} - \frac{r^4}{M^2} + \frac{r^3}{M} - r^2 + \frac{Mr}{2}\right]^{1/2} d\phi \\ &= \frac{(r-M)^2}{r}\left[\frac{r^3}{2M^3} + \frac{r}{2M}\right]^{1/2} d\phi \end{aligned} \qquad [24.\ L = 0]$$

The computer has no difficulty integrating and plotting this equation, as shown in Figure 3. Since we used the Kerr bookkeeper angular velocity [22], the resulting picture is that of the Kerr bookkeeper. For her, the zero-angular-momentum stone spirals around the black hole and settles down in a tight circular path at $r = M$, there to circle forever.

QUERY 17	**Final circle according to the bookkeeper.** Verify that dr goes to zero (that is, r does not change) once this stone reaches the horizon.

Remember that for the nonspinning black hole an object plunging inward slows down as it approaches the horizon, according to the records of the Schwarzschild bookkeeper. For both spinning and nonspinning black holes, the in-falling stone with $L = 0$ never crosses the horizon when clocked in far-away time.

QUERY 18	**Bookkeeper speed in the "final circle."** *Guess:* At the horizon, what is the value of the tangential speed $Rd\phi/dt$ of the stone dropped from rest at infinity, as measured by the Kerr bookkeeper? Now derive a formula that gives you this numerical value. Was your guess correct?

The observer who has fallen from rest at infinity has quite a different perception of the trip inward! For her there is no pause at the horizon; she has a quick, smooth trip to the center (assuming that the Kerr metric holds all the way to the center!). An algebra orgy similar to the previous one gives a relation between dr and $d\tau$, where $d\tau$ is the wristwatch time increment of the in-faller:

$$\begin{aligned} \left(\frac{dr}{d\tau}\right)^2 &= \frac{2Mr^3 - 4M^2r^2 + 4M^3r - 4M^4 + \dfrac{2M^5}{r}}{r^2(r-M)^2} \\ &= \frac{2M}{r}\left(1 + \frac{M^2}{r^2}\right) \end{aligned} \qquad [25.\ L = 0]$$

Figure 4 compares the magnitude of the square root of this expression with the magnitude of the velocity of the stone dropped from rest at a great distance in the Schwarzschild case (equation [32], page 3-22):

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \qquad [26.\ L = 0 \text{ Schwarzschild}]$$

Both equations [25] and [26] show bookkeeper radial components of speed greater than unity in the region of small radius. The resulting speed is even more impressive when one adds the tangential component of motion forced on the diver descending into the spinning black hole (Figure 3). Does such motion violate the "cosmic speed limit" of unity for light? A similar question is debated for the Schwarzschild black hole in Section 3 of Project B, Inside the Black Hole, pages B-6–12.

Research note: When applied inside the horizon, equation [25] assumes that the Kerr metric correctly describes spacetime all the way to the center of the extreme Kerr black hole. This may not be the case. See the box Egg-beater Spacetime? on page B-5.

9 Ring Riders

Equation [22] on page F-14 describes the angular rotation rate ω of an in-falling stone that has zero angular momentum:

$$\frac{d\phi}{dt} \equiv \omega = \frac{2M^2}{rR^2} \qquad [27.\ L = 0]$$

In some way, ω in this equation describes the angular rate at which space is "swept along" by the nearby spinning black hole. What happens if we "go with the flow," moving tangentially at angular rate ω given by this equation? How do we guarantee that our rotation is at the correct rate to yield zero angular momentum? What happens to us at the static limit?

To pursue these ideas, we envision a set of nested rings in the equatorial plane and concentric to the black hole (Figure 5). Each of these rings revolves at an angular rate given by equation [27] as reckoned by the Kerr bookkeeper. Rings at different values of *r* rotate at different angular rates.

The result of this construction is a set of observers in the equatorial plane whom we call **ring riders**. A ring rider is an observer who stands at rest on one of the rotating rings with zero angular momentum. In times past, ring riders were known as **locally nonrotating observers**, but now the customary name is **zero angular momentum observers** or **ZAMOs**. Each ring rider, like each shell observer in Schwarzschild geometry, is subject to a gravitational acceleration directed toward the center of the black hole. In both cases the radially inward gravitational acceleration becomes infinite at the horizon, destroying any possible circumferential ring structures at or inside the horizon. According to ring rider measurements, light has speed unity, the same speed in both tangential directions, as we shall see.

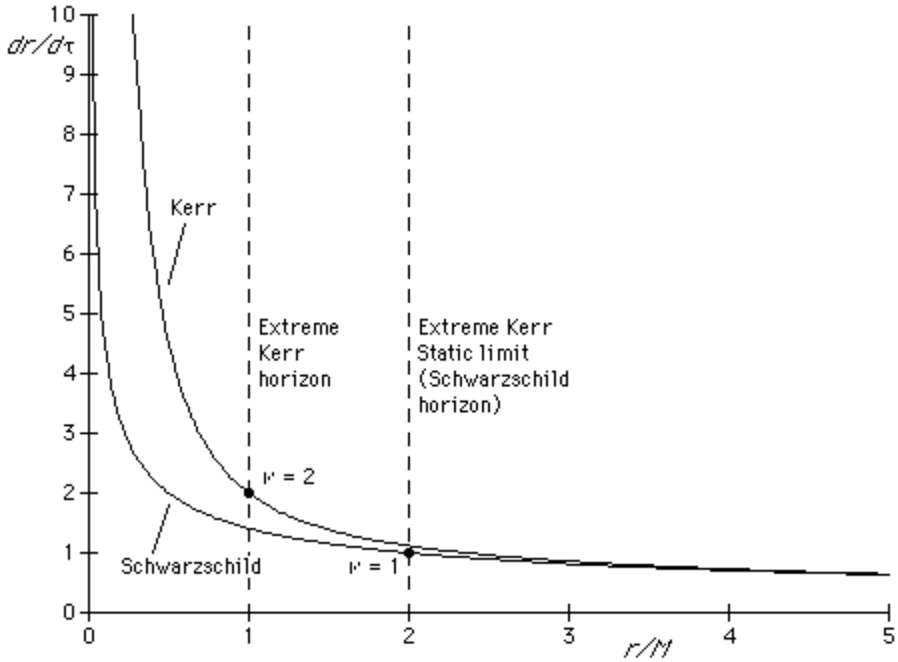


Figure 4 *Computer plot: comparison of radial components of plunge velocities experienced by different in-fallers who drop from rest (so with L = 0) at a great distance from Schwarzschild and extreme Kerr black holes.*

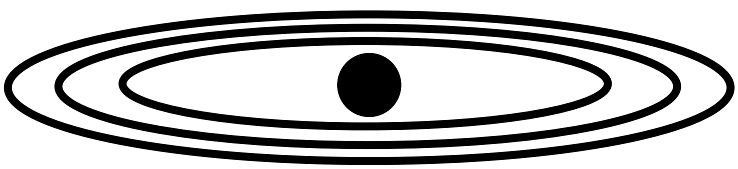


Figure 5 *Kerr map (perspective plot) of rings surrounding a spinning black hole. The rings rotate in the same direction as the black hole but at angular rates that differ from ring to ring, as given by equation [27], page F-16.*

QUERY 19 **Ring slippage.** Will the inner rings rotate with larger or smaller angular velocity than the rings farther out? Justify your choice.

QUERY 20 **Ring speed according to the bookkeeper.** What are the units of ω in equation [27]? What is the numerical value of the bookkeeper speed $R\omega$ for each of the rings $r = 100M$, $r = 10M$, $r = 2M$, and $r = M$? Express each answer as a fraction of the speed of light.

QUERY 21 **Does rain fall vertically?** Present an argument that a stone dropped from rest starting at a great radial distance falls vertically past the rider on every zero angular momentum ring. *Guess:* Is the same true if the stone is *flung* radially inward from a great distance? *Guess:* What about light?

Can we write a simplified metric for the rider on the zero angular momentum ring? Probably not for events separated radially because of shearing, the slippage between adjacent rings. So limit attention to events separated tangentially along the ring. According to the remote observer, each ring revolves with an angular velocity ω given by equation [27]. Define an azimuthal angle increment $d\phi_{\text{ring}}$ measured along the ring with respect to some zero mark on the ring. Let an object move uniformly along the ring. Then, as recorded by the Kerr bookkeeper, the object's total angular velocity $d\phi/dt$ is the angular velocity $d\phi_{\text{ring}}/dt$ with respect to the ring added to the bookkeeper angular velocity ω of the ring, or

$$\frac{d\phi}{dt} = \frac{d\phi_{\text{ring}}}{dt} + \omega \qquad [28]$$

The positive direction of both $d\phi$ and $d\phi_{\text{ring}}$ is in the direction of rotation of the black hole.

Now think of two events separated by the angle $d\phi_{\text{ring}}$ along the ring and at far-away time separation dt. Then the angular separation $d\phi$ between these two events for the far-away observer is, from [27] and [28],

$$d\phi = d\phi_{\text{ring}} + \frac{2M^2}{rR^2}dt \qquad [29.\ dr = 0]$$

The metric [3] with the same limitation to motion along the ring ($dr = 0$) is

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 + \frac{4M^2}{r}dtd\phi - R^2 d\phi^2 \qquad [30.\ dr = 0]$$

QUERY 22 **New metric for the ring.** Substitute equation [29] into [30]. Show first that the coefficient of the cross-term in $dtd\phi_{\text{ring}}$ is equal to zero. Second, collect terms in dt^2 and $d\phi_{\text{ring}}{}^2$ to show that the resulting metric is given by equation [31] for motion along the ring. *Hint:* Group over a common denominator r^2R^2, then substitute in the numerator for R^2 (equation [4]):

$$d\tau^2 = \frac{r^2}{R^2}\left(1 - \frac{M}{r}\right)^2 dt^2 - R^2 d\phi_{\text{ring}}^2 \qquad [31.\ dr = 0]$$

QUERY 23 **Time on the ring rider clock.** A ring rider is at rest on a (zero angular momentum) ring. Show that the time dt_{ring} between ticks on his clock and the time dt between ticks on the far-away clock are related by the equation

$$dt_{\text{ring}} = \frac{r - M}{R}dt \qquad [32.\ dr = d\phi_{\text{ring}} = 0]$$

Show that, with this substitution, the metric for $dr = 0$ becomes

$$d\tau^2 = dt_{\text{ring}}^2 - R^2 d\phi_{\text{ring}}^2 \qquad [33.\ dr = 0]$$

In brief, for nearby events along the ring the metric [33] looks like that of flat spacetime. But spacetime is *not* flat near a spinning black hole. Equation [33] describes a *local frame* useful only in analyzing events that are limited in space and time and for which the "local gravitational force" in the radial direction can be neglected. However, this equation is useful for analyzing events that occur near to one another along the same ring.

Now (finally!) we can define the reduced circumference R everywhere external to the horizon, even inside the static limit. A ring rider measures the circumference of his ring and then divides this circumference by 2π.

$$\begin{pmatrix}\text{circumference of}\\ \text{freely rotating ring}\end{pmatrix} \equiv 2\pi R \qquad [34]$$

The result is a formal definition of the **reduced circumference** R for this zero angular momentum ring. The value of R, along with the value of r from equation [4], is then stamped on each rotating ring for all to see and everyone to use. The same values of R and r can also be stamped on each stationary ring that coincides with an already measured rotating ring. (Of course, nonrotating rings can exist only outside the static limit.)

This set of zero angular momentum rotating rings can extend from the horizon to infinite radius. For a pair of events near one another along a given ring, the proper distance $d\sigma$ between them is given by the equation

$$d\sigma = R d\phi_{\text{ring}} \qquad [35.\ dr = dt = 0]$$

QUERY 24 **Speed of light along the ring is unity for ring riders.** From the metric [33], show that the ring rider measures the speed of light along the ring to have the magnitude unity. Is this value the same for motion of the light in both directions along the ring (Figure 6)?

QUERY 25 **Is motion along ring free or locked?** *Hard thought question; optional.* Equation [33] says that the ring rider on every ring can use special relativity in analyzing motion along the ring. So he must be able to move freely back and forth along the ring, even on a ring near the horizon. In contrast, Query 14 asserts that the tangential motion near the horizon is rigidly locked to the rotation of the black hole. Locked or free? What's going on?

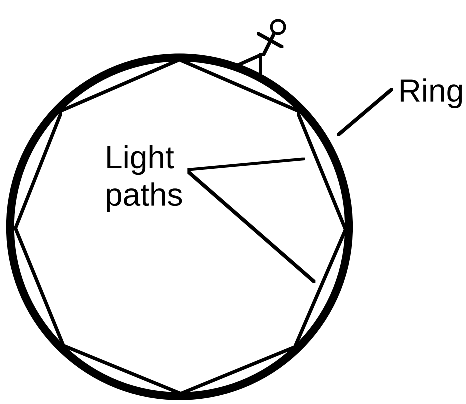


Figure 6 *Silvered inner surface of rotating zero angular momentum ring allows signaling along ring with light flashes. Light-path segments shown as straight will be curved. We assume that each segment is arbitrarily short so that light skims along close to the ring. Equal time for light transmission in opposite directions around the ring verifies that the ring has zero angular momentum (equation [27] and Query 24). Then light signals at locally-measured speed* $v_{\text{ring}} = 1$ *allow synchronization of clocks around the ring.*

Tornado Without a Wind?

In what sense does spacetime near a spinning black hole "circulate like a tornado"? In the vicinity of a spinning black hole can we *feel* this rotation of spacetime?

For comparison, think about trying to stand still on the surface of Earth as the circulating wind of a tornado passes over you. You must lean into the wind or hang onto something fixed and solid in order to keep from being swept along in the tangential direction in which the wind moves around the center of the tornado. While standing still you *feel* a force in the direction in which the wind blows.

Now suppose you stand still, at rest on a stationary ring concentric to a spinning black hole. "Stationary" and "at rest" mean that for you the remote stars do not move overhead. (Such a stationary ring can be constructed only outside the static limit.) You experience the same kind of radially-inward "gravitational force" you felt while resisting the tornado on Earth. But is there an additional tangential "tornado force" due to swirling spacetime, a "force" pushing you in the direction of rotation of the black hole? We said so in early printings of this book, but we made an error. Standing at rest on a stationary ring concentric to the center of a black hole, you experience NO sideways tangential force.

We were misled by the analogy to a tornado. To begin to understand the difference between an Earth-tornado and spacetime near a spinning black hole, look at the graph below. This graph plots the angular momentum of an object AT REST outside an extreme spinning black hole as a function of radius (equation at the end of this box). Note that for an object at rest this angular momentum is *negative*. Key idea: An object at rest *already has* the angular momentum appropriate for that radius and does not need a tangential force to maintain this angular momentum. The situation is similar to that of a stone in a circular orbit around Earth; the stone feels no force in the tangential direction and does not need such a force in order to continue in its orbit with constant angular momentum. The essential difference between the Earth-orbiting stone and the person standing on a stationary ring outside a spinning black hole is that the person has (negative) angular momentum *while standing still*.

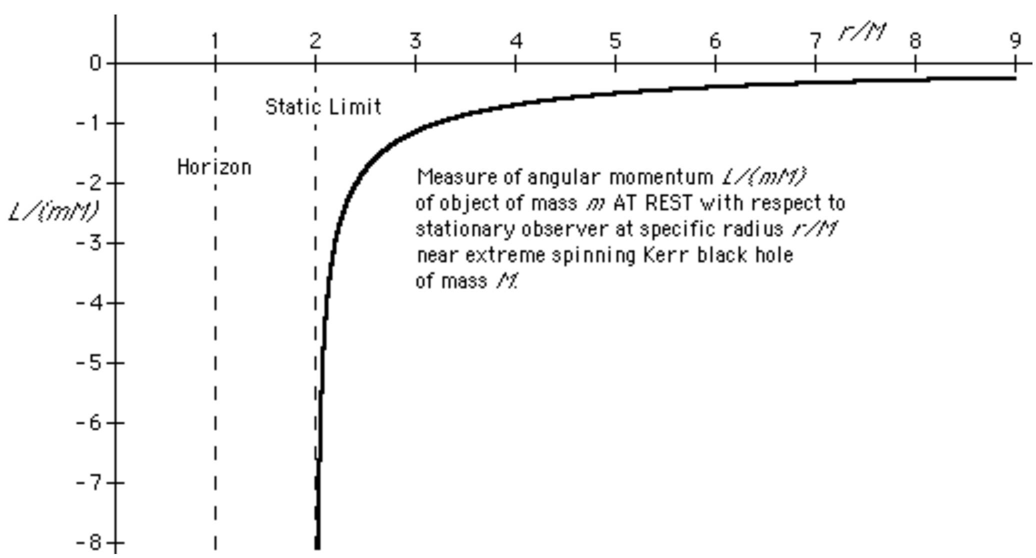
So no tangential force is needed for you to stand still on a stationary ring concentric to a spinning black hole.

The graph below also shows that the stationary observer has a different value of angular momentum at each different radius. (In our analysis we have neglected the difference in angular momentum between the head and feet of a stationary observer.) In order to change radius without moving sideways, you must change your angular momentum. Changing angular momentum *does* require a tangential force, but only temporarily, *while you are changing radius.* For example, suppose that you descend from a great distance along a radial line fixed with respect to the remote stars. As you move radially inward along this fixed line, the magnitude of your (negative) angular momentum must increase. So to stay on the fixed radial path of your descent without being swept sideways, you must fire a rocket tangentially in order to change your angular momentum, *but only while you continue to move inward.* Once you stop descending, for example by stepping onto a stationary ring, there is no sideways force and no need for a tangential rocket to maintain your position.

By how much will you have to increase your negative angular momentum as you descend along a fixed radial line? The answer comes from equation [21], page F-13 with $d\phi/d\tau = 0$. For an object at rest we have $d\tau = dt_{\text{shell}}$. Use equation [15] on the same page to eliminate $dt/d\tau = dt/dt_{\text{shell}}$ from equation [21]. Divide through by M. The result is the equation

$$\frac{L}{mM} = -\frac{2M}{r}\frac{dt}{d\tau} = -\frac{2M}{r}\frac{dt}{dt_{\text{shell}}} = -\frac{2M}{r}\left(1-\frac{2M}{r}\right)^{-1/2}$$

The figure below plots the quantity $L/(mM)$ for a stationary object as a function of r/M. Notice the result that the magnitude of the (negative) angular momentum increases without limit as you descend to the static limit at $r/M = 2$.



10 Negative Energy: The Penrose Process

Roger Penrose devised a scheme for milking energy from a spinning black hole. This scheme is called the **Penrose process** (see references). The Penrose process depends on the prediction that in some orbits inside the ergosphere a particle can have *negative* total energy. Before we detail the Penrose process, we need to describe negative total energy.

Negative Total Energy

What can negative total energy possibly mean? Negative energy is nothing new. In Newtonian mechanics the potential energy of a particle at rest far from Sun is usually taken to be zero by convention. Then a particle at rest near Sun has zero kinetic energy and negative potential energy, yielding a total energy less than zero. But in Newtonian mechanics the zero point of potential energy is arbitrary, and all reasonable choices of this zero point lead to the same description of motion. In contrast, special relativity determines the rest energy of a free material particle in flat spacetime, setting its rest energy equal to its mass. So the arbitrary choice of a zero point for energy is lost, and a particle far from a center of gravitational attraction always has an energy that is positive.

For Schwarzschild geometry the physical system differs from Newtonian. A particle at rest near the horizon of a nonspinning black hole has zero total energy (from equation [18] in Sample Problem 1, page 3-12). The meaning? That it takes an energy equal to its rest energy (= m) to remove this particle to rest at a large distance from the black hole (where it has the energy m). As a consequence, if the particle drops into the black hole from its stationary position next to the horizon, then the mass of the combined black-hole-particle system (measured by a far-away observer, Figure 4, page 3-11) does not change.

For Kerr geometry the physical system differs from that in Schwarzschild geometry. A particle can have a negative energy near a spinning black hole. The meaning? An energy *greater* than its rest energy (greater than m) is required to remove such a particle to rest at a great distance from the black hole. If the particle with negative energy is captured by the spinning black hole, the black hole's mass and angular momentum decrease. (See Section 11.) This process can be repeated until the black hole has zero angular momentum. Then it becomes a "dead" Schwarzschild black hole, from which only Hawking radiation can extract energy (box page 2-4).

Strategy of the Penrose Process

The strategy of the Penrose process is similar to the following unethical series of financial transactions:

1. You and I decide to share our money. Our combined net worth is positive.

2. I give you all my money, then borrow money from a bank and give that to you as well. My bank debt is a negative entry on my accounting balance sheet, so now my net worth is negative.

3. I declare bankruptcy and the bank is stuck with my debt.

The net result is the transfer of money from me and from the bank to you. The bank provides the mechanism by which I can enter a state of negative net worth.

The Penrose process is similar:

1. Starting at a distant radius, you and I together descend to a position inside the ergosphere.

2. We are moving together tangentially inside the ergosphere in the rotation direction. You push me away *violently* in a direction opposite to the direction of rotation. This push puts you into a new trajectory and puts me into a state of negative energy.

3. I drop into the black hole, which is stuck with my negative energy. You continue in your new trajectory, arriving at a distant radius with augmented energy.

The net result is the transfer of energy from me and from the black hole to you. The spinning black hole provides the mechanism by which I can enter a state of negative energy.

This entire strategy rests on the assumption that an object can achieve a state of negative energy in the space surrounding a spinning black hole. Is this assumption correct? Look again at expression [19] for the energy of a stone near an extreme Kerr black hole:

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} + \frac{2M^2}{r}\frac{d\phi}{d\tau} \qquad [19]$$

Can this energy be negative? Start to answer this question by finding the “critical” condition under which the energy is zero.

QUERY 26 **Conditions for zero energy.** Set *E/m* = 0 in equation [19] and show that the resulting expression for the bookkeeper rate of change of angle is

$$\left(\frac{d\phi}{dt}\right)_{\mathrm{E=0}} = \frac{2M-r}{2M^2} \qquad [36]$$

Under what conditions is this angular velocity negative? positive?

QUERY 27 **Bookkeeper tangential velocity for zero energy.** Now assume that the direction of motion is tangential and show that the bookkeeper velocity is given by the expression

$$v_{\mathrm{bkkpr\ E=0}} = R\left(\frac{d\phi}{dt}\right)_{\mathrm{E=0}} = \frac{R(2M-r)}{2M^2} \qquad [37.\ dr = 0]$$

QUERY 28 **Bookkeeper tangential velocities for negative energy.** Now redo the analysis for the circumstance that the particle energy is negative. Show that the condition is

$$v_{\mathrm{bkkpr\ E=neg}} < \frac{R(2M-r)}{2M^2} \qquad [38.\ dr = 0]$$

Figure 7 shows a plot of equation [37] along with plots of the positive and negative tangential velocities of light from Figure 2. The tangential motion of *any* particle must be bounded by the curves of tangential light motion. (Inside the ergosphere even light moving “in the negative tangential direction” moves forward, in the direction of rotation, according to the remote bookkeeper.) In addition, equation [38] tells us that a particle with negative energy must have a tangential velocity that lies below the heavy line in the Figure 7. The shaded area in that figure conforms to these conditions and shows the range of bookkeeper tangential velocities of a stone for which the stone has negative energy. Next we turn our attention away from the bookkeeper to what the ring rider measures (Query 29).

The essence of newer physics

Of all the entities I have encountered in my life in physics, none approaches the black hole in fascination. And none, I think, is a more important constituent of this universe we call home. The black hole epitomizes the revolution wrought by general relativity. It pushes to an extreme—and therefore tests to the limit—the features of general relativity (the dynamics of curved spacetime) that set it apart from special relativity (the physics of static, “flat” spacetime) and the earlier mechanics of Newton. Spacetime curvature. Geometry as part of physics. Gravitational radiation. All of these things become, with black holes, not tiny corrections to older physics, but the essence of newer physics.

—John Archibald Wheeler

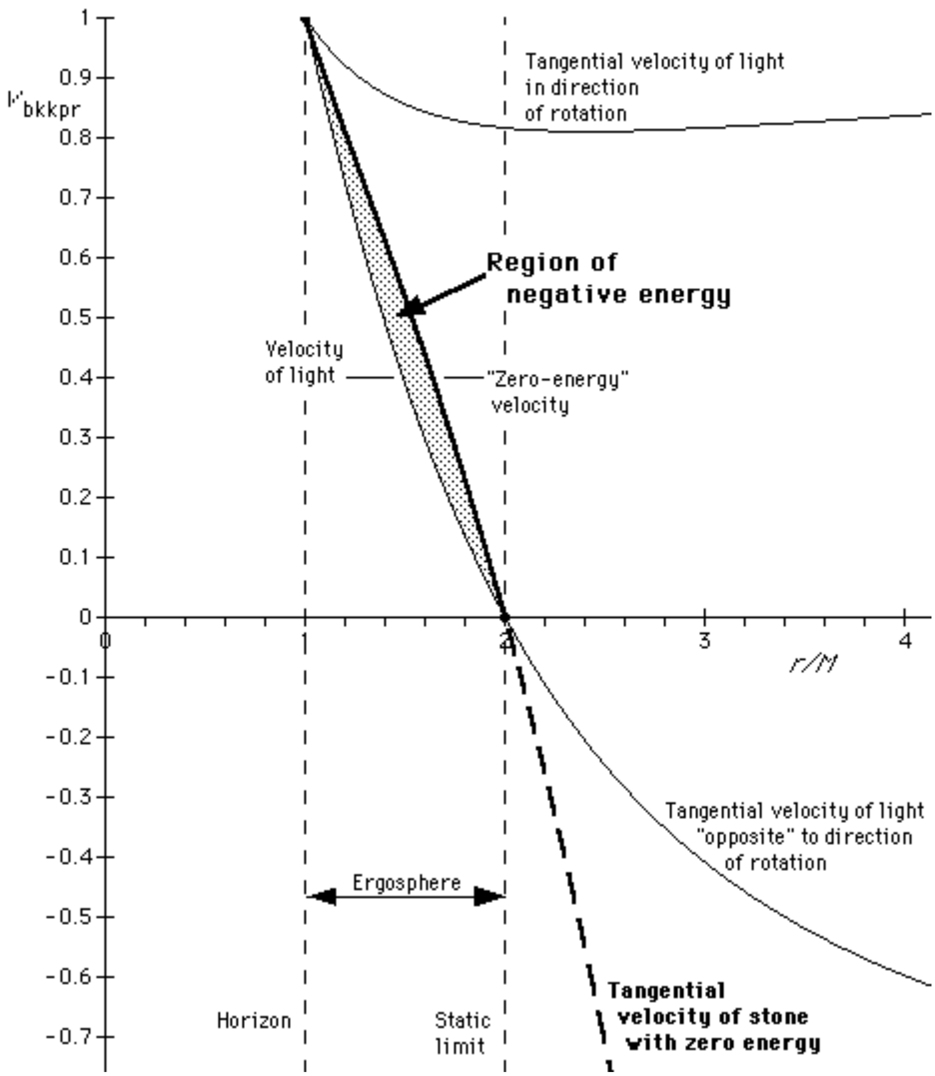


Figure 7 *Computer plot showing bookkeeper tangential velocities of light (thin curves) and tangential velocity of a stone with zero energy (thick curve), calculated using equation [37]. For r greater than 2M, the static limit, the particle cannot have zero energy (or negative energy), because it would have to be moving in a negative tangential direction with a speed greater than that of light in that direction. Only inside the ergosphere is this critical tangential velocity possible. The shaded area shows the range of bookkeeper velocities for which the stone has negative energy.*

QUERY 29 **Ring rider velocity for zero energy.** *Optional—messy algebra!* A stone moves tangentially along a rotating ring. For what values of the ring velocity v_{ring} will the energy measured at infinity be negative? Set $E/m = 0$ in equation [19]. Then make substitutions from equations [29] and [32] to convert variables to $d\phi_{\text{ring}}$ and dt_{ring}. Simplify using equation [4]. Show that the result is

$$v_{\text{ring, E}=0} = R\frac{d\phi_{\text{ring}}}{dt_{\text{ring}}} = -\frac{1}{2}\left(\frac{r}{M}-1\right)\frac{r}{M} \qquad [39]$$

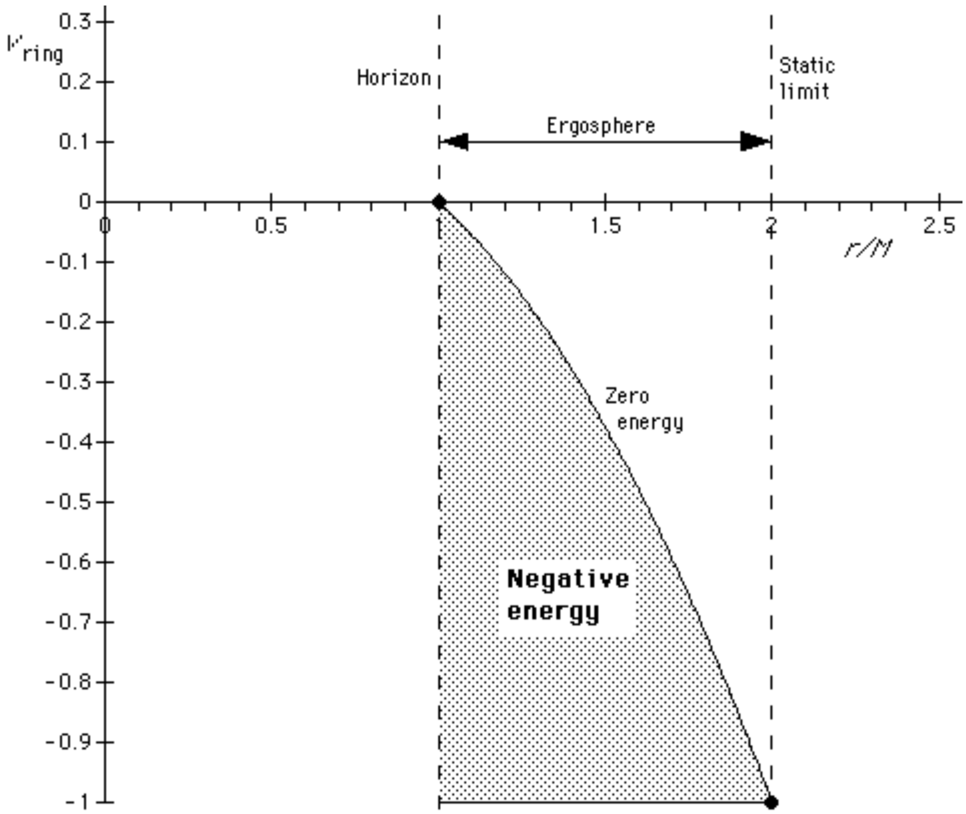


Figure 8 *Computer plot showing the range of ring velocities (shaded region) for which the energy measured at infinity is negative. Negative ring velocity means motion along the ring in a direction opposite to the direction of rotation of the black hole.*

Figure 8 plots equation [39] for ring velocity. Energy measured at infinity E/m will be negative for values of the ring velocity in the shaded region of the plot. The range of ring velocities for which energy is negative depends on the radius of the ring. Limiting cases are interesting: For a ring at the static limit, motion backward along the ring with the speed of light leads to zero energy. In contrast, for a ring near the horizon, any non-zero backward ring velocity, no matter how small, leads to negative energy.

11 Quasar Power

How much total energy can be extracted from a rotating black hole? In general relativity, energy is a seamless whole; we cannot separate the kinetic from the rest energy of a rotating object. Milking energy from a rotating black hole changes its mass M along with its angular momentum J. Analysis has identified a so-called **irreducible mass** M_{irr} that is the smallest residual mass that results when all the angular momentum is milked out of a rotating black hole. This irreducible mass M_{irr} of an uncharged rotating black hole with angular momentum parameter $a = J/M$ is given by the equation

$$M_{\mathrm{irr}}^2 = \frac{1}{2}[M^2 + M(M^2 - a^2)^{1/2}] \qquad [40]$$

or equivalently

$$M^2 = M_{\mathrm{irr}}^2 + \frac{J^2}{4M_{\mathrm{irr}}^2} \qquad [41]$$

(Wald, page 326. Misner, Thorne, and Wheeler, page 913) This result was discovered in Princeton by a 19-year-old Athenian, Demetrios Christodoulou, who never finished high school.

The final state is a nonrotating Schwarzschild black hole of mass M_{irr}. The net result is that a total energy $M - M_{\mathrm{irr}}$ has been extracted from an uncharged rotating black hole.

QUERY 30 **Irreducible mass of extreme Kerr black hole.** What is the irreducible mass of an uncharged extreme Kerr black hole of mass *M*? What fraction of the mass *M* of an extreme Kerr black hole can be extracted in the form of energy by an **advanced civilization** (defined as a civilization that can accomplish any engineering feat not forbidden by the laws of Nature)?

QUERY 31 **How much energy is available from the monster in our galaxy?** Imagine that the black hole of mass $M = 2.6 \times 10^6\ M_{\mathrm{Sun}}$ thought to exist at the center of our galaxy is an extreme Kerr black hole. How much total energy can be milked from it? Express your answer as a multiple of the mass M_{Sun} of our Sun.

From where do quasars get their power (box page F-4)? Probably not directly from the Penrose process (Section 10). One set of theories has the quasar radiation coming from the gravitational energy of matter descending toward the black hole as it orbits in an accretion disk. This matter interacts with other matter in the disk in a complicated manner not well understood. As debris in the disk moves toward the center, it is compressed along with its magnetic fields, is heated, and emits radiation copiously. The net result is to convert its gravitational energy into radiation with high efficiency (high compared with nuclear reactions on Earth). Note that the angular momentum of the black hole may actually be increased during this process, depending on the initial angular momentum of the gas and clouds that swirl into the black hole. Another theory derives the quasar output from the rotation energy of the black hole itself, employing magnetic field lines to couple black hole rotation energy to the matter swirling around exterior to the horizon of the black hole. Such a model leads to reduction in the rotation rate of the black hole.

QUERY 32 **Quasar output.** How much energy does a quasar put out each second? Suppose that the quasar emits energy at a rate 100 times the emission rate of our entire galaxy, which contains approximately 10^{11} stars similar to our Sun. How much light energy does Sun put out per second? Luminous energy from Sun pours down on the outer atmosphere of Earth at a rate of 1370 watts per square meter (called the **solar constant).** From the solar constant, estimate the energy production rate of our Sun in watts, then of our galaxy, and then of a quasar that emits energy at 100 times the rate of our galaxy. This rate corresponds to the total conversion to energy of how many Sun masses per Earth-year?

The details of the emission of radiation by quasars may be complicated, but the analysis in the present project provides the basis for an estimate of the energy *available* for such processes.

Suppose that each element of the accretion disk circles the black hole at the same rate of rotation as the local ring (an unrealistic assumption, since rotating with the ring does not place the particle in a stable circular orbit). As a given bit of debris moves inward, let it radiate energy sufficient to keep it at rest with respect to the local ring. For a bit of debris riding on the ring, the time $d\tau$ between ticks on its wristwatch is the same as time dt_{ring} between ticks of the ring clocks, since they are relatively at rest. Equation [19] for the energy of this bit of debris then becomes

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{dt_{\text{ring}}} + \frac{2M^2}{r}\frac{d\phi}{dt_{\text{ring}}} \qquad [42]$$

Now, the relation between ring time increments and bookkeeper time increments is given by equation [32]:

$$dt_{\text{ring}} = \frac{r - M}{R}dt \qquad [32]$$

QUERY 33 **Energy of stone riding on the ring.** Substitute equation [32] into equation [42], use equation [27] for the resulting $d\phi/dt$, and collect terms over a common denominator $R(r - M)$ to obtain

$$\frac{E}{m} = \frac{\left(1 - \frac{2M}{r}\right)R^2 + \frac{4M^4}{r^2}}{R(r - M)} \qquad [43. \text{ riding on ring}]$$

For the expression for R^2 in the numerator (only) substitute from equation [4] and simplify to show that, for a stone riding on the ring,

$$\frac{E}{m} = \frac{r - M}{R} \qquad [44. \text{ riding on ring}]$$

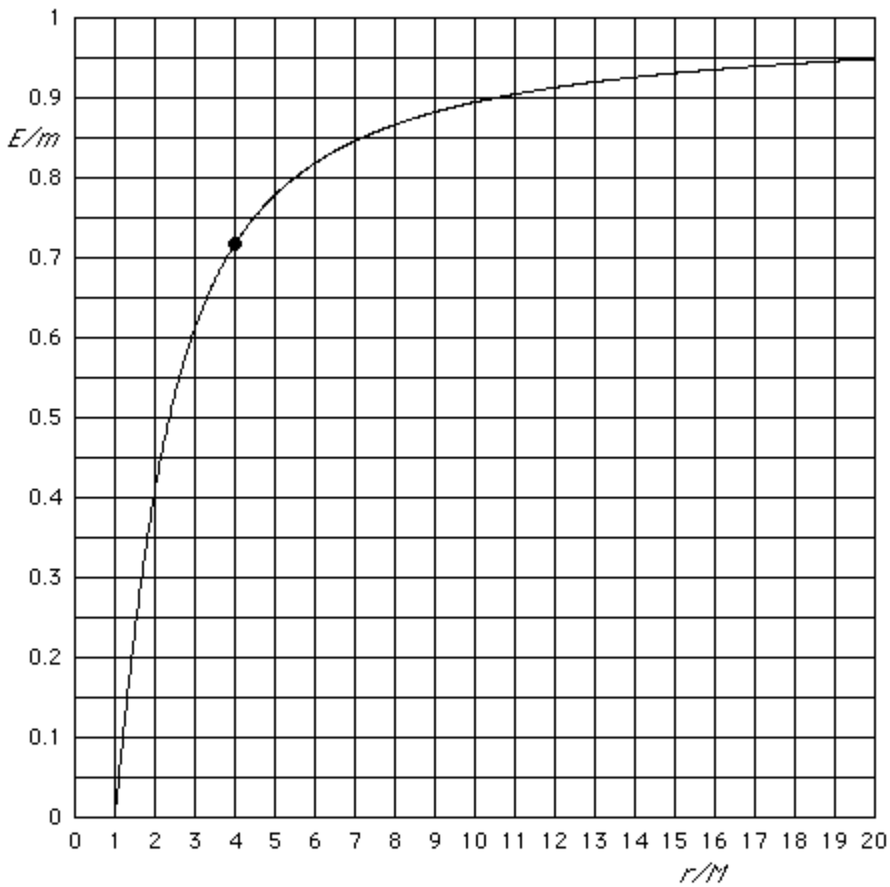


Figure 9 *Computer plot of energy measured at infinity for an object riding at rest with respect to an L = 0 ring rotating at various radii around an extreme Kerr black hole. Example shown by dot on the diagram: Stone riding at rest on a ring at r = 4M has total energy measured at infinity of E = 0.72m.*

Equation [44] is a simple expression but awkward to calculate because R is a function of r (equations [4] and [13]). However, the computer has no difficulty with these complications and plots the result in Figure 9.

QUERY 34 **Brilliant garbage.** A blob of matter starts at rest at a great distance from a black hole and gradually descends, riding at rest on each local ring and emitting any change of energy as radiation. Now this matter rides on the ring at $r = 2M$, the static limit. From Figure 9, determine what fraction of its original rest energy it has radiated thus far. In principle, what is the *maximum* fraction of its original rest energy that can be radiated before it disappears inward across the horizon of the black hole?

Figure 9 does not lead to a correct estimate of the emission rate of a quasar. In practice the rings do not rotate at the same rate as the accretion disk, and the accretion disk itself is not a perfectly efficient emitter of radiation. A few percent of the rest energy of swirling particles may be emitted in the form of radiation before they plunge across the horizon. Still, a few percent is far greater than the efficiency of nuclear reactors on Earth.

12 A “Practical” Penrose Process

Using results of Sections 10 and 11, we can devise a “practical” Penrose process by which energy can be milked from an extreme spinning black hole. Actually, this process is “practical” only for an advanced civilization, one that can accomplish any engineering feat not forbidden by the laws of Nature. *Outline of the strategy:* Equal quantities of matter and antimatter (say positrons and electrons united in positronium molecules, in bulk as liquid positronium) are carried down to a rotating ring just outside the horizon of an extreme Kerr black hole. There the matter and antimatter are combined (annihilated) to create two oppositely moving pulses of electromagnetic radiation. One pulse has negative energy and drops into the black hole, robbing the black hole of some of its mass-energy of rotation. The other pulse has positive energy and escapes to a distant observer who uses this energy for practical purposes. Now for the details.

The generalization of equation [44] for a particle moving along a rotating ring is given by the equation

$$\frac{E}{E_{\text{ring}}} = \frac{r-M}{R} + \frac{2M^2}{rR}v_{\text{ring}} \qquad [45]$$

where $v_{\text{ring}} = Rd\phi_{\text{ring}}/dt_{\text{ring}}$. Equation [45] comes from applying a boatload of algebra to equations [19], [29], and [32] and simplifying using equation [4]. In addition, the derivation of [45] employs the following results of special relativity:

$$E_{\text{ring}} = m\gamma_{\text{ring}} \qquad \text{[46. special relativity]}$$

where

$$\gamma_{\text{ring}} \equiv \frac{1}{(1-v_{\text{ring}}^2)^{1/2}} \qquad \text{[47. special relativity]}$$

A final transformation (time stretching) from special relativity tells us that

$$dt_{\text{ring}} = \gamma_{\text{ring}}d\tau \qquad \text{[48. special relativity]}$$

where $d\tau$ is the wristwatch time of the stone moving along the ring.

Note that in equation [45], v_{ring} can be positive or negative, corresponding to motion in either direction along the ring. Under some circumstances this results in negative energy for the particle.

Now apply some simplifying circumstances. First keep constant the value of $E_{\text{ring}} = m\gamma_{\text{ring}}$ in equation [46] while letting m go to zero and v_{ring} go to plus or minus one. The result signifies a pulse of electromagnetic radiation.

Second, apply equation [45] to a rotating ring very close to the horizon, as a limiting case. In other words $r \longrightarrow M$ and $R \longrightarrow 2M$. Equation [45] becomes

$$\frac{E}{E_{\text{ring}}} = \pm 1 \qquad \text{[49. light flash moving along ring as } r \rightarrow M\text{]}$$

With these equations we can analyze the following idealized method for milking energy from the black hole. Start with a mass m of matter and an equal mass m of antimatter. Total mass: $2m$.

Phase 1. Take the total load of mass $2m$ down to a position at rest on a ring of zero angular momentum near to the horizon of an extreme Kerr black hole, milking off the energy as it makes successive moves from rest on one ring to rest on the next lower ring.

QUERY 35 **Energy extracted in Phase 1**. When Phase 1 is completed, how much energy will have been milked off for use at a distant location?

Phase 2. Combine the matter and antimatter at rest with respect to the near-horizon ring and direct the resulting light pulses in opposite directions along the ring at the horizon.

QUERY 36 **Energies of tangential light flashes.** Just after Phase 2 is completed, what is the ring energy E_{ring} of each of the two light pulses moving along the ring as measured locally by a rider on that ring? What is the energy measured at infinity E of each of these flashes?

Now the light flash with negative energy drops across the horizon into the black hole, thereby reducing the angular momentum (and mass) of the spinning hole. In contrast, the light flash with positive energy flies out to a great distance and its energy is employed for useful purposes.

QUERY 37 **Total energy extracted.** In summary, what is the total useful energy made available to distant engineers as a result of this entire procedure? How much mass/energy was the input for this process?

QUERY 38 **Phase 1 reduction of angular momentum?** *Thought question, optional.* Does the energy extracted in Phase 1 by itself reduce the rotation rate of the black hole? In answering, recall the analogous extraction of energy from a nonrotating black hole (Exercise 6, Chapter 3).

13 Challenges

Nothing but algebra stands in the way of completing a full analysis of orbits of stones and light in the equatorial plane of the extreme Kerr black hole. The strategies required are analogous to those that led to similar results for the nonrotating Schwarzschild black hole (Chapters 4 and 5).

- Computing the orbits of a stone from equations that relate dr and $d\phi$ to the passage of wristwatch time $d\tau$ (similar to equations [21] and [22], page 4-9).
- Carrying out qualitative descriptions of different classes of orbits using an effective potential (similar to equation [32], page 4-18).
- Finding stable circular orbits, similar to those analyzed in the exercises of Chapter 4. (Stable orbits allow a more realistic analysis of the behavior and energy of a particle orbiting with the accretion disk.)
- Predicting orbits of light as done in Chapter 5.
- Predicting details of life inside the horizon, comparable to the analysis carried out for the Schwarzschild black hole in Project B, Inside the Black Hole. Such an analysis is probably fantasy, since inside the Cauchy horizon (choosing the minus sign in equation [2]) spacetime appears to be unstable, hence not described by the Kerr metric, and possibly lethal to incautious divers. (See *Research Note*, page F-5.)
- Verifying that an extreme spinning black hole cannot accept additional angular momentum. Can an object moving in the direction of rotation of an extreme black hole cross the horizon and thus increase the angular momentum of this structure which already has maximum angular momentum?

Much of the complicated algebra that lies on the way to these outcomes springs from the relation between the radius r and the reduced circumference R given by equation [4]. Once the algebra is mastered, results can be plotted using a simple computer graphing program.

- For readers with unfettered ambition or for those skilled in the use of computer algebra manipulation programs, the outcomes of this project can be rederived for a black hole that spins with angular momentum parameter $a = J/M$ *less* than its maximum value. Start with the metric [1] and use the more general reduced circumference R_a, defined by the equation (valid in the equatorial plane)

$$R_a^2 = r^2 + a^2 + \frac{2Ma^2}{r} \qquad [50]$$

  The resulting equations are easy to check at the extremes: They go to the Schwarzschild limit when $a \longrightarrow 0$ and to the expressions derived in this project when $a \longrightarrow M$.

- We have studied two important metrics: the Schwarzschild metric for a nonspinning black hole and the Kerr metric for a spinning black hole. You can apply the skills you have now mastered to analyze the consequences of a third metric, the so-called **Reissner-Nordstrøm metric** for an electrically charged *nonspinning* black hole. For a pair of events that occur near one another on a plane through the center of such a charged black hole, the Reissner-Nordstrøm metric has the form

$$d\tau^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)} - r^2 d\phi^2 \qquad [51]$$

Here Q is the electric charge of the black hole in units of length.

Good luck!

14 Basic References to the Spinning Black Hole

Introductory references to the spinning black hole

For the human and scientific story of the spinning black hole, read Kip S. Thorne, *Black Holes and Time Warps: Einstein's Outrageous Legacy,* W. W. Norton, New York, 1994, pages 46–54 and pages 286–299.

Bernard F. Schutz has an excellent analytic treatment in *A First Course in General Relativity,* Cambridge University Press, New York, 1985, pages 294–305.

Chapter 33 of Misner, Thorne, and Wheeler's *Gravitation*, W. H. Freeman and Company, San Francisco (now New York), 1973, is very thorough, with wonderful summary boxes, though beset with the mathematics of tensors and differential forms. It is also approximately 30 years old.

Chapter 12 of Robert M. Wald's *General Relativity* (University of Chicago Press, Chicago, 1984) is authoritative and straightforward. The mathematics is deep; you have to "read around the mathematics" to find the physical conclusions, which are clearly stated.

Section 12.7 of *Black Holes, White Dwarfs, and Neutron Stars* by Stuart L. Shapiro and Saul A. Teukolsky (John Wiley, New York, 1983) pages 357–364, covers the spinning black hole, mostly with algebra rather than tensors, and discusses orbits in some detail.

Steven Detweiler, editor, *Black Holes: Selected Reprints,* American Association of Physics Teachers, New York, 1982. This collection may be out of print but is available in some physics libraries.

Original references to the spinning black hole

The first paper: R. P. Kerr, "Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics," *Physical Review Letters,* Volume 11, pages 237–238 (1963).

Choice of coordinate system can make thinking about the physics convenient or awkward. Boyer and Lindquist devised the coordinates that illuminate our analysis in this project. Robert H. Boyer and Richard W. Lindquist, “Maximum Analytic Extension of the Kerr Metric,” *Journal of Mathematical Physics*, Volume 8, Number 2, pages 265–281 (February 1967). See also Brandon Carter, “Global Structure of the Kerr Family of Gravitational Fields,” *Physical Review*, Volume 174, Number 5, pages 1559–1571 (1968).

For completeness, the Newman electrically charged black hole: E. T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, “Metric of a Rotating, Charged Mass,” *Journal of Mathematical Physics*; Volume 6, Number 6, pages 918–919 (1965); also E. T. Newman and A. I. Janis, “Note on the Kerr Spinning-Particle Metric,” *Journal of Mathematical Physics*, Volume 6, Number 6, pages 915–917 (1965).

The Penrose process, to help you milk the energy of rotation from the spinning black hole: R. Penrose, “Gravitational Collapse: The Role of General Relativity,” *Revista del Nuovo Cimento*, Volume 1, pages 252–276 (1969).

15 Further References and Acknowledgments

Initial quote: S. Chandrasekhar, *Truth and Beauty: Aesthetics and Motivations in Science*, University of Chicago Press, 1987, pages 153–154.

Quote in reader objection, page F-3: Stephen Hawking, *Black Holes and Baby Universes*, Bantam Books, New York, 1993, pages 91–92.

Quote about Cauchy horizon, page F-5: E. Poisson and W. Israel, “Inner-Horizon Instability and Mass Inflation in Black Holes,” *Physical Review Letters*, Volume 63, Number 16, pages 1663–1666 (16 October 1989).

The value of the angular momentum of Sun (page F-6) was provided by Douglas Gough, private communication.

Quote on page F-23 from John Archibald Wheeler with Kenneth Ford, *Geons, Black Holes, and Quantum Foam, A Life in Physics*, W. W. Norton & Company, New York, 1998, page 312.

Stephan Jay Olson suggested using light flashes as part of the practical Penrose process in Section 12.

For more on the Reissner-Nordstrøm metric for a charged black hole (equation [51], page F-32), see entries in the Subject Index and the Bibliography and Index of Names in Misner, Thorne, and Wheeler’s *Gravitation*, W. H. Freeman and Company, San Francisco (now New York), 1973.