CHAPTER 1

Speeding

The important thing is not to stop questioning. Curiosity has its own reason for existing. One cannot help but be in awe when he [or she] contemplates the mysteries of eternity, of life, of the marvelous structure of reality. It is enough if one tries merely to comprehend a little of this mystery every day. Never lose a holy curiosity.

—Albert Einstein

1 Special Relativity

Key idea: Concepts useful in exploring the very fast help us to examine spacetime near very massive objects.

Special relativity: fast objects
General relativity: spacetime near massive objects

We use relativity to explore the boundaries of Nature. **Special relativity** describes the very fast. **General relativity**—the **Theory of Gravitation**—describes matter and motion near massive objects: stars, galaxies, black holes. General relativity also describes the Universe as a whole. This chapter discusses a few key concepts of special relativity useful in exploring general relativity. The treatment here is not designed to be an introduction to special relativity; for introductory treatments see Section 11, Readings in Special Relativity, and detailed references to our own introductory treatment at the end of each section.

2 Wristwatch Time

Everyone agrees on the wristwatch time between two events.

Begin relativity with wristwatch time between two ticks.

What is the root of relativity? Is there a single, simple idea that launches us along the road to understanding? Alice's adventures in wonderland begin when a rabbit rushes past her carrying a pocket watch. Our adventure in relativity begins when a small stone flies past us wearing a wristwatch.

The wristwatch ticks once at #1 and once at #2 (Figure 1). Wristwatch ticks may be one second apart—or one microsecond. Measure the distance s and time t between these ticks in a particular **free-float** or **inertial** reference frame. (The free-float frame is described in Section 8. Briefly, it is one in which Newton's first law holds: a free particle at rest remains at rest and one in motion continues that motion at constant speed in a straight line.) Special relativity warns us that a different observer passing us in uniform relative motion typically records a different value of spatial separation s and a different value of time lapse t between these two ticks. That is the bad news. The good news is a central finding of special relativity:

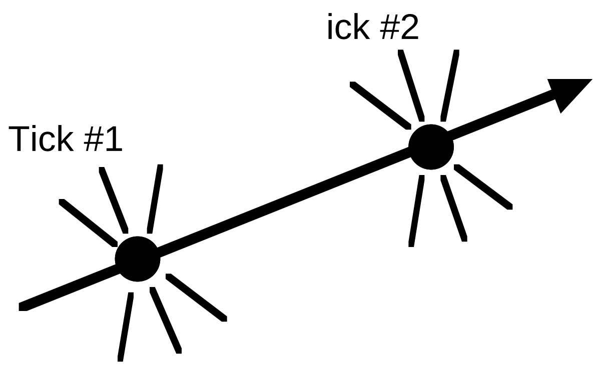

***Figure 1** Straight-line uniform-speed trajectory of a stone through space. The stone wears a wristwatch that ticks and emits a flash at #1 and then ticks again and emits a second flash at #2. These two ticks are a distance s apart and have a time separation t as measured in the frame of reference for which this diagram is drawn.*

All inertial observers, whatever their state of relative motion and whatever values they measure for s and t, agree on the value of the time τ between ticks as recorded on the wristwatch carried by the stone. The formula is simple:

$$\tau^2 = t^2 - s^2 \qquad [1]$$

Define wristwatch time.

We use the Greek letter τ (tau) for the **wristwatch time** between these two watch ticks. The wristwatch time is often called the **proper time** or, more formally, the **timelike spacetime interval** ("timelike" because the time separation t is greater than the space separation s). All observers agree on the value of the wristwatch time between two events. In contrast, the value of t and the value of s between these events will typically differ from frame to frame. Call t the **frame time** and s the **frame distance** between this pair of events. Wristwatch time τ can be used to describe the separation between *any* pair of events for which t is greater than s. It tells the observer in any frame what the time lapse will be on a wristwatch that moves uniformly from one event to the other.

Measure space and time in the same units.

For simplicity, the units of space and time are the same, such as light-years and years, or meters of distance and meters of light-travel time. In both cases the speed of light c is the conversion factor between measures of space and time. For example, the relation between seconds and meters of light-travel time is

$$t(\text{in meters}) = ct_{\text{second}} \qquad [2]$$

The metric: Key to all relativity

Equation [1], which connects the wristwatch time between two adjacent ticks to their space and time separations in a given frame, is called the **metric**. The metric (with a minus sign between squared quantities) tells us the *separation* between events in spacetime, just as the Pythagorean Theorem (with a plus sign between squared quantities) tells us the *distance* between points in a space described by Euclidean geometry. The metric is central in both special and general relativity. In describing physical systems for which it can be derived, the metric provides the answer to every possible question about (nonquantum) features of spacetime. And with a simple extension it also predicts the trajectories of particles and light.

The fact that all free-float observers agree on the wristwatch time τ earns it the label **invariant**. *Invariant* means that all observers calculate the same value, independent of reference frame. In relativity every invariant quantity is a diamond, to be treasured.

Wristwatch time is an INVARIANT.

How fast does the stone travel between ticks? The stone's speed depends on the reference frame. For the frame of Figure 1, the speed (assumed to be constant) is $v = s/t$. Measure distance s and time lapse t in the same unit. For example, a spaceship travels half a light-year of distance during one year of time; its speed is then 0.5 year / year and the units cancel. As another example, if an elementary particle moves 0.7 meter in one meter of light-travel time its speed is 0.7. Hence the speed v has no units. In this book the symbol v represents the speed of an object as a fraction of the speed of light.

Velocity v is a fraction of the speed of light.

Fuller Explanations: *Spacetime Physics*, Chapter 1, Spacetime: Overview; Chapter 3, Same Laws for All; Chapter 6, Regions of Spacetime.

SAMPLE PROBLEM 1 Wristwatch Times

PROBLEM 1A. An unpowered spaceship moving at constant speed travels 3 light-years in 5 years, this time and distance measured in the rest frame of our Sun. What is the time lapse for this trip as recorded on a clock carried with the spaceship?

SOLUTION 1A. The two events that start and end the spaceship's journey are separated in the Sun frame by s = 3 light-years and t = 5 years. Equation [1] gives the resulting wristwatch time:

$$\begin{aligned} \tau^2 &= t^2 - s^2 = 5^2 - 3^2 = 25 - 9 = 16 \text{ years}^2 \\ \tau &= 4 \text{ years} \end{aligned} \qquad [3]$$

which is *less* than the time lapse as measured in the Sun frame.

PROBLEM 1B. An elementary particle is created in the target of a particle accelerator and arrives at a detector 4 meters away and 5 meters of light-travel time later, as measured in the laboratory. The wristwatch of the elementary particle records what time between creation and detection?

SOLUTION 1B. The events of creation and detection are separated in the laboratory frame by s = 4 meters and t = 5 meters of light-travel time. Equation [1] tells us that

$$\begin{aligned} \tau^2 &= t^2 - s^2 = 5^2 - 4^2 = 25 - 16 = 9 \text{ meters}^2 \\ \tau &= 3 \text{ meters} \end{aligned} \qquad [4]$$

Again, the wristwatch time for the particle is less than the time recorded in the laboratory frame.

3 Proper Distance

Everyone agrees on the proper distance between two events.

Two firecrackers explode 1 *meter apart* and *at the same time*, as measured in a particular free-float frame. In this frame these explosions are **simultaneous**. No stone can travel fast enough to be present at both of these explosions without moving at an infinite velocity, which is impossible. Therefore equation [1] is useless to define a wristwatch time τ between these two events.

SAMPLE PROBLEM 2 Speeding to Andromeda

At approximately what constant speed v must a spaceship travel so that the occupants age only 1 year during a trip from Earth to the Andromeda galaxy? Andromeda lies 2 million light-years distant from Earth.

SOLUTION

The word *approximately* in the statement of the problem tells us that we can make some assumptions. We assume that a single free-float frame can stretch all the way from Sun to Andromeda, so special relativity applies. We also predict that the speed v of the spaceship measured in the Sun frame is very close to unity, the speed of light. That allows us to set $(1 + v) \approx 2$ in the last of the following steps:

$$\tau^2 = t^2 - s^2 = t^2\left(1 - \frac{s^2}{t^2}\right) = t^2(1 - v^2)$$
$$= t^2(1+v)(1-v) \approx 2t^2(1-v) \quad [5]$$

Equate the first and last expressions to obtain

$$1 - v \approx \frac{\tau^2}{2t^2} \quad [6]$$

Now, we assumed that v is very close to the speed of light. It follows that the time t for the trip in the Sun frame is very close to the time that light takes to make the trip: 2 million years. Substitute this value and also demand that the wristwatch time on the spaceship (the aging of the occupants during their trip) be $\tau = 1$ year. The result is

$$1 - v \approx \frac{\tau^2}{2t^2} = \frac{1\ \text{year}^2}{2 \times 4 \times 10^{12}\ \text{year}^2}$$
$$= \frac{10^{-12}}{8} = 1.25 \times 10^{-13} \quad [7]$$

Equation [7] expresses the result in sensible scientific notation. However, your friends may be more impressed if you report the speed as a fraction of the speed of light: $v = 0.999999999999875$. This result justifies the assumptions we made about the value of v and the time for the trip as measured in the Sun frame. *Additional question*: What *distance* does the spaceship rider measure between Earth and Andromeda?

Use simultaneous explosions to measure length of a rod.

Simultaneous explosions are thus useless for measuring time. But they are perfect for measuring length. *Question:* How do you measure the length of a rod, whether it is moving or at rest in your frame? *Answer:* Set off two firecrackers at the two ends and *at the same time* ($t = 0$) in your frame. Then *define* the rod's length in your frame as the *distance* s between this pair of explosions.

Special relativity warns us that a different observer passing us in uniform relative motion typically will *not* agree that the two firecrackers exploded at the same time. That is the bad news (and the idea most difficult to understand in all of special relativity). But there is good news: All inertial observers, whatever their state of relative motion, can calculate the distance σ between explosions as recorded in the frame in which they do occur simultaneously. The new metric is a variation of the old metric [1]:

$$\sigma^2 = s^2 - t^2 \quad [8]$$

Proper distance is an INVARIANT.

The Greek letter σ (sigma) labels what we call the **proper distance** between such events or, more formally, the **spacelike spacetime interval** ("spacelike" because the space separation s is greater than the time separation t). All free-float observers agree on the value of the proper distance—the proper distance is an *invariant*. In contrast, the value of t and the value of s between these events typically differ, respectively, as measured in different frames. Proper distance σ can be used to describe the separation between *any* pair of events for which s is greater than t. It tells the observer in any frame what the distance σ is between the events as measured in a frame in which they occur at the same time.

We attach special significance to the length of a rod measured in the frame in which it is at rest. Let a firecracker explode at each end of a rod at the same time in its rest frame. We call the distance between these explosions the **proper length** of the rod. Any other inertial observer, whatever her state of relative motion, can calculate the proper length of the rod from equation [8] using the time t and distance s that she measures between these particular explosions in her own reference frame.

As in equation [1], the units of space and time in equation [8] are the same, such as light-years and years—or meters of distance and meters of light-travel time.

The name **spacetime interval** is the collective name for the timelike spacetime interval (equation[1]) and the spacelike spacetime interval (equation [8]).

Fuller Explanations: What happens to equations [1] and [8] when s and t have the *same* magnitude? Find the answer in *Spacetime Physics*, Chapter 6, Regions of Spacetime.

4 The Principle of Extremal Aging

The Twin Paradox leads to a definition of natural *motion.*

To get ready for curved spacetime (whatever that may mean), look further at the motion of a free particle in **flat spacetime**, the arena of the free-float frame (Section 8) in which special relativity correctly describes motion.

How does a free particle move in flat spacetime? We say: "What a ridiculous question! Everyone knows that a free particle moves with constant speed in a straight line—at least as observed in a free-float frame." Ah yes, but *why* does a free particle move straight with constant speed? What lies behind this motion? Our answer for flat spacetime will be a trial run for the description of motion in curved spacetime, the arena of general relativity.

Twin Paradox predicts the motion of a free particle.

A deep description of motion arises from the famous **Twin Paradox**. Recall that one identical twin relaxes on Earth while her twin sister frantically travels to a distant star and returns. When the two meet again, the stay-at-home twin has aged more than her traveling sister. (This outcome can be predicted by extending Sample Problems 1 and 2 to include return of the traveler to the point of origin.) Upon being reunited, the "identical twins" are no longer identical. Very strange! But (almost) no one who has studied relativity doubts the difference in age, and experiments with fast-moving particles verify it.

Being at rest is one *natural* motion.

Which twin has the motion we can call *natural*? Isaac Newton has a definition of natural motion. He would say, "A twin at rest tends to remain at

SAMPLE PROBLEM 3 How Slow Is "Speeding"?

A. Answer "yes" or "no" to questions (a) through (e):

Is the stay-at home twin older when they get together again if the traveling twin

(a) streaks to the Andromeda galaxy (2 million light-years distant) and back?

(b) soars to Alpha Centauri (4 light-years distant) and back?

(c) flies to the planet Pluto and back?

(d) hurries to Earth's Moon and back?

(e) strolls next door to the neighbor's house and back?

B. In case (e) of part A, what is the approximate difference in aging between the twins if the traveling twin strolls at 1 meter per second and the next door neighbor's house is 100 meters away?

SOLUTION

A. In principle, one should reply "yes"—the stay-at-home twin will be older—for *all* cases in part A. Part B examines the actual value of the aging difference for small relative velocity.

B. Solve equation [1] for s^2 and apply it to the outward trip from the twins' house to the neighbor's house. The word *approximately* in the statement of the problem gives us permission to make assumptions.

Usually we do not notice results of the Twin Paradox in our everyday lives, so it seems reasonable to assume that the frame time t is very nearly the same as the wristwatch time τ for the stroll next door. This allows us to set $(t + \tau) \approx 2t$ in the following steps. We also set $t = s/v$ in one of the steps.

$$s^2 = t^2 - \tau^2 = (t+\tau)(t-\tau)$$
$$s^2 \approx 2t(t-\tau) = 2\frac{s}{v}(t-\tau) \qquad [9]$$

Equate the first and the last of the expressions in the last line of [9] and multiply through by $v/(2s)$ to obtain

$$t - \tau \approx \frac{sv}{2} \qquad [10]$$

We need to express the velocity v as a fraction of the speed of light. A speed of 1 meter per second is equal to

$$v = \frac{1 \text{ meter/second}}{c} = \frac{1 \text{ meter/second}}{3 \times 10^8 \text{ meter/second}} \qquad [11]$$
$$= 3.3 \times 10^{-9}$$

Substitute this value of v into equation [10] to yield the time difference for one leg of the round trip:

$$t - \tau \approx \frac{100 \text{ meters} \times 3.3 \times 10^{-9}}{2} \qquad [12]$$
$$= \frac{3.3 \times 10^{-7}}{2} \text{ meters of light-travel time}$$

The round trip difference will be twice this value, or 3.3×10^{-7} meters of light-travel time. Divide the result by the speed of light to obtain the time difference in seconds:

$$\begin{pmatrix}\text{time difference} \\ \text{for round trip}\end{pmatrix} \approx \frac{3.3 \times 10^{-7} \text{ meter}}{3 \times 10^8 \text{ meter/second}} \qquad [13]$$
$$= 1.1 \times 10^{-15} \text{ second}$$

(This result justifies our assumption that the two times t and τ are very nearly equal.) So after her stroll next door and back, the traveling twin will be approximately 10^{-15} seconds younger than her stay-at-home sister. To measure this tiny time difference exceeds the sensitivity of even the most accurate atomic clock. That is why we do not notice relativistic effects in our everyday lives! Nevertheless, Nature witnesses the difference by selecting the stay at home twin as the one whose motion (or whose *lack* of motion in this frame) is *natural*.

rest." So it is the stay-at-home twin who moves in the natural way. In contrast, the out-and-back twin suffers the forces required to change her state of motion—from outgoing motion to incoming motion—so that the two sisters can meet again in person. The motion of the traveling twin is forced, *not natural.*

Moving uniformly is another *natural* motion.

Viewed from a second relatively moving free-float frame, the stay-at-home twin moves with constant speed in a straight line. Hers is also *natural* motion. Newton would say, "A twin in motion tends to continue this motion at constant speed in a straight line." So the motion of the stay-on-

Earth twin is also natural from the viewpoint of a second frame in uniform relative motion—or from any frame moving uniformly with respect to the original frame. In *any* such frame, the time lapse on the wristwatch of the stay-at-home twin can be calculated from the metric (equation [1]).

Natural motion in general: Extremal wristwatch time

The lesson of the Twin Paradox is that the natural motion of a free object between two events in flat spacetime is the one for which the wristwatch worn by the object has a maximum time reading between those two events. Purists insist that we say not *maximum* reading but rather *extremal* reading: either maximum or minimum. This book contains only examples of maximum wristwatch time for natural motion. Still, let's try to keep the purists happy! Replace the two words *maximum* and *minimum* with the single word *extremal*. The result is the **Principle of Extremal Aging**.

> ***Principle of Extremal Aging:*** *The path a free object takes between two events in spacetime is the path for which the time lapse between these events, recorded on the object's wristwatch, is an extremum.*

Principle of Extremal Aging: works for general relativity too.

It turns out that the Principle of Extremal Aging describes motion even when spacetime is not flat. The Principle of Extremal Aging accompanies us into curved spacetime, into the realm of general relativity. But for now we stay in flat spacetime and use the Principle of Extremal Aging to derive relativistic expressions for energy and momentum.

5 Energy in Special Relativity

The Principle of Extremal Aging tells us the energy of a free particle.

Derive energy from the metric plus the Principle of Extremal Aging.

Combining the metric (Section 2) with the Principle of Extremal Aging (Section 4) leads to the relativistic expression for energy in flat spacetime—the formula for energy used in special relativity. Here is the plan in outline: A free stone following its natural path carries a wristwatch that emits three flashes. We consider all three flashes to be fixed in space and the emission times for the first and last flashes also to be fixed. We then adjust the time of the middle flash so that the *wristwatch time* from the first flash to the last flash is an extremum. The outcome is the expression for a quantity that is the same along every segment of the path—this quantity is *conserved*. We identify the conserved quantity as the energy. Now fill in some details.

Three flashes: When will the middle flash occur?

Think of a free stone flying along a straight line in space as observed in an inertial frame (Figures 2 and 3). The stone emits three flashes #1, #2, and #3 bracketing two adjacent segments of its trajectory, segments labeled A and B in the figures. These two segments need not be the same length. Fix the *positions* of all three flash emissions in space, fix also the *times* for flash emissions #1 and #3, then ask: At what time t will the free stone pass location #2 and emit the second flash? Find this intermediate time t by demanding that the total wristwatch time from #1 to #3 be an extremum. In other words, use the Principle of Extremal Aging to find the time for the middle flash. The result leads to a conserved quantity, the energy of the stone.

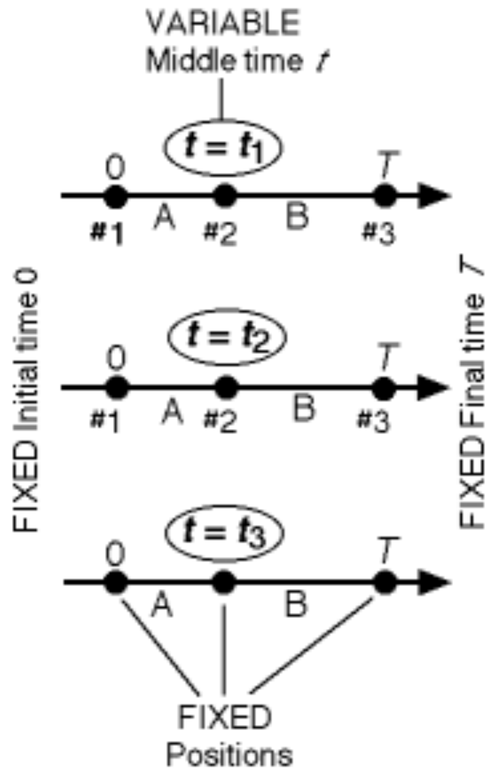

Figure 2 *Three alternative cases of a stone moving along a straight line in space as it emits three flashes, #1, #2, and #3. The space locations of emissions are the same in all three cases, as are the times of first and last emissions #1 and #3. But emission time for the middle flash #2 is different for the three cases. We ask: At what time will a free stone following a natural path pass the intermediate point and emit flash #2? We answer this question by demanding that the total wristwatch time τ from first to last flash emissions be an extremum. From this requirement comes an expression for the energy of the stone as a constant of the motion.*

Now for the full step-by-step derivation.

1. Let t be the frame time between flash #1 and flash #2 and let s be the frame distance between these two flashes. Then the metric [1] tells us that the wristwatch time τ_A along segment A is

$$\tau_A = (t^2 - s^2)^{1/2} \qquad [14]$$

   To prepare for the derivative that leads to extremal aging, differentiate this expression with respect to the intermediate time *t:*

$$\frac{d\tau_A}{dt} = \frac{t}{(t^2 - s^2)^{1/2}} = \frac{t}{\tau_A} \qquad [15]$$

2. Next, let T be the fixed time between flashes #1 and #3 and S be the fixed distance between them. Then the frame time between flash #2 and flash #3 is $(T - t)$ and the frame distance between them is $(S - s)$. Therefore the wristwatch time τ_B along segment B is

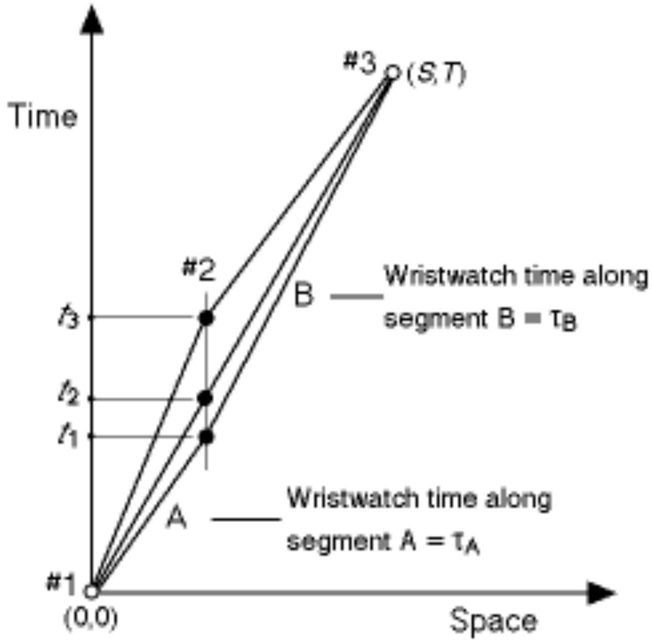

Figure 3 *Three alternative cases of a stone moving along a straight line in space as it emits three flashes, #1, #2, and #3. These are the same three cases shown in Figure 2, but here we plot the stone's path in space* and time. *Such a spacetime plot is called a* ***worldline****. On each of three alternative worldlines, flash emissions #1 and #3 are fixed in space and time. Flash emission #2 is fixed in space (horizontal direction in figure) but its time is varied (up and down in the figure) to find an extremum of the total wristwatch time $\tau = \tau_A + \tau_B$ from #1 to #3. The result is an expression for a quantity that is a constant of the motion: the energy of the stone.*

$$\tau_B = [(T-t)^2 - (S-s)^2]^{1/2} \qquad [16]$$

Again, to prepare for the derivative that leads to extremal aging, differentiate this expression with respect to the intermediate time *t:*

$$\frac{d\tau_B}{dt} = \frac{-(T-t)}{[(T-t)^2-(S-s)^2]^{1/2}} = -\frac{T-t}{\tau_B} \qquad [17]$$

3. The total wristwatch time τ from event #1 to event #3 is the sum of the wristwatch time τ_A between events #1 and #2 plus the wristwatch time τ_B between events #2 and #3:

$$\tau = \tau_A + \tau_B \qquad [18]$$

4. Now ask: *When*—at what frame time *t*—will the a stone, following its natural path, pass the intermediate point and emit the second flash #2? Answer with the Principle of Extremal Aging: Time *t* will be such that the aging (τ in equation [18]) is an extremum. To find this extremum set the derivative of τ with respect to *t* equal to zero. Take the derivative of both sides of [18] and substitute from equations [15] and [17]:

Use Principle of Extremal Aging to find the time for the middle flash.

$$\frac{d\tau}{dt} = \frac{d\tau_A}{dt} + \frac{d\tau_B}{dt} = \frac{t}{\tau_A} - \frac{T-t}{\tau_B} = 0 \qquad [19]$$

5. The last equality in equation [19] leads to the equation

$$\frac{t}{\tau_A} = \frac{T - t}{\tau_B} \qquad [20]$$

Quantity whose value is the same for adjacent segments

6. In expression [20] the frame time t is the time for the particle to traverse segment A. Call this time t_A. The time $(T - t)$ is the frame time for the particle to traverse segment B. Call this time t_B. Then equation [20] can be rewritten in the simple form

$$\frac{t_A}{\tau_A} = \frac{t_B}{\tau_B} \qquad [21]$$

7. The locations of segments A and B were chosen arbitrarily along the straight path in space of the particle moving in a region of flat spacetime. Equation [21] holds for *all* pairs of adjacent segments placed *anywhere* along the path. We did not specify where segment A was to begin. Nothing stops us from beginning the analysis with the second segment B and adding to it a third segment C with which to compare it (which may have a different length than either of the first two segments). Then equation [21] applies to the second and third segments. But if the value of the expression is the same for the first and second segments and also the same for the second and third segments, then it must be the same for the first and third segments. Continuing in this way, envision a whole series of adjacent segments, labeled A, B, C, D, . . . , for each of which equation [21] applies, leading to the set of equations

$$\frac{t_A}{\tau_A} = \frac{t_B}{\tau_B} = \frac{t_C}{\tau_C} = \frac{t_D}{\tau_D} = \ldots \qquad [22]$$

In brief, here is a quantity that is a constant of the motion for the free particle—a quantity that has the same value along any segment of the natural path of a free particle moving in flat spacetime. Then equation [22] tells us that

$$\frac{t}{\tau} = \text{a constant of the motion} \qquad [23]$$

E/m = *t*/τ is a constant of the motion.

What is this quantity? It is related to the relativistic expression for the total energy of the particle. If we have already studied special relativity, we know that

$$\frac{t}{\tau} = \frac{t}{[t^2 - s^2]^{1/2}} = \frac{t}{t\left[1 - \left(\frac{s}{t}\right)^2\right]^{1/2}} = \frac{1}{(1 - v^2)^{1/2}} = \frac{E}{m} \qquad [24]$$

where m is the mass of the particle. Equation [24] gives the energy per unit mass of a particle that moves with constant speed.

OBJECTION: Baloney! Everyone knows that a free particle moves with constant speed along a straight path in space as observed in a free-float frame. So as this motion proceeds, every possible expression that depends only on $v = s/t$ is also a constant of the *motion, for example the expression v^{12}, which is certainly not the correct expression for energy! Your derivation proves nothing!*

RESPONSE: You are *almost* right. Any function of velocity $v = s/t$ is indeed constant for the special case of a free particle in flat spacetime. And if v is constant, so is t/τ, as witnessed by Equation [24]. But notice the priorities used in the derivation: The Principle of Extremal Aging has highest priority; the expression for energy comes out of this principle. Of all the quantities that remain constant because v is constant, the Principle of Extremal Aging picks out $t/\tau = E/m$ as primary. (The following section shows that a similar analysis picks out the relativistic expression for momentum as a constant of the motion.) Chapter 3 contains a new and more general expression for energy in curved spacetime. In that case the velocity is *not* constant—yet that more general expression for energy is correct and a constant of the motion nevertheless. Our derivation of the expression for E/m in flat spacetime is thus a trial run for the derivation of the energy of a particle in the curved spacetime around a center of gravitational attraction.

If the particle changes speed, then it changes energy. In that case it makes sense to talk about *instantaneous speed* and to use calculus notation. Let the pair of flash emissions in Figure 1 be separated by the incremental frame coordinates dt, ds, and incremental wristwatch time $d\tau$. The equation for E/m then becomes

Particle energy in special relativity

$$\frac{E}{m} = \frac{dt}{d\tau} \qquad [25]$$

Ordinarily we use the ratio E/m in equations, instead of E alone. Why? Because it emphasizes two important principles: (1) Only spacetime relations between events appear on one side of equations such as [24] and [25], reminding us that it is *spacetime geometry* that leads to these expressions, not some weird property of matter. (2) The ratio E/m has no units. Therefore, whoever uses these equations has total freedom in choosing the unit of E and m, as long as it is the *same* unit. The same unit in the numerator and denominator of [25] may be kilograms or the mass of the proton or million electron-volts or the mass of Sun. If you insist on using conventional units, such as joules for energy E and kilograms for mass m, then a conversion factor c^2 intrudes into our simple equation:

$$\frac{E_{\text{joules}}}{m_{\text{kg}}c^2} = \frac{dt}{d\tau} \qquad [26]$$

Rest energy: famous formula

Now view the particle from a reference frame in which the particle is at rest. In this rest frame there is zero distance s between sequential flash emissions. Equation [1] says that for $s = 0$ the frame time t and wristwatch time τ have exactly the same value. For a particle at rest, then, equation [26] reduces to the most famous equation in all of physics:

$$E_{\text{joules rest}} = m_{\text{kg}}c^2 \qquad [27]$$

Note that equation [27] describes the *rest energy* of a particle. For a particle in motion, the energy is given by equation [26].

In equation [27], c has the *defined* value 2.99792458×10^8 meters / second. An equation of the same form is correct if E is measured in ergs, m in grams, and c in centimeters / second.

Fuller Explanations: Energy in flat spacetime: *Spacetime Physics*, Chapter 7, Momenergy.

6 Momentum in Special Relativity

The metric plus the Principle of Extremal Aging give us an expression for momentum.

The relativistic expression for momentum is derived by a procedure analogous to the one used to derive the relativistic expression for energy. The figures look similar to Figures 2 and 3, but in this case the *time t* for the intermediate flash emission is *fixed*, while the *position s* for this event is *varied* right and left to yield an extremum for the total wristwatch time from the first flash to the third flash. (You carry out the derivation of momentum in the exercises at the end of this chapter.) The result is a second constant of the motion for a free particle:

$$\frac{s}{\tau} = \frac{s}{[t^2 - s^2]^{1/2}} = \frac{s/t}{[1-(s/t)^2]^{1/2}} = \frac{v}{(1-v^2)^{1/2}} = \frac{p}{m} \qquad [28]$$

Equation [28] gives the momentum per unit mass for a particle moving with constant speed. If the particle changes speed, then once again we use calculus notation:

Particle momentum in special relativity

$$\frac{p}{m} = \frac{ds}{d\tau} \qquad [29]$$

Equation [29] has the same form as in Newton's nonrelativistic mechanics, except here the incremental wristwatch time $d\tau$ replaces the Newtonian lapse dt of "universal time."

Fuller Explanations: Momentum in flat spacetime: *Spacetime Physics*, Chapter 7, Momenergy.

7 Mass in Relativity

Everyone agrees on the value of the mass m of the stone.

Find mass from energy and momentum.

An important relation among mass, energy, and momentum follows from the metric and our new expressions for energy and momentum. Suppose a moving stone emits two flashes very close together in space ds and in time dt. Then equation [1] gives the increase of wristwatch time $d\tau$:

$$(d\tau)^2 = (dt)^2 - (ds)^2 \qquad [30]$$

Divide through by $d\tau^2$ and multiply through by m^2 to obtain

$$m^2 = m^2\left(\frac{dt}{d\tau}\right)^2 - m^2\left(\frac{ds}{d\tau}\right)^2 = \left(m\frac{dt}{d\tau}\right)^2 - \left(m\frac{ds}{d\tau}\right)^2 \qquad [31]$$

or, substituting expressions [25] and [29] for energy and momentum,

$$m^2 = E^2 - p^2 \qquad [32]$$

Energy (also momentum) may be different for different observers . . .

In equation [32], mass, energy, and momentum are all expressed in the same units, such as kilograms or electron-volts. In conventional units, the equation has a more complicated form:

$$(mc^2)^2 = E^2_{\text{conv}} - p^2_{\text{conv}}c^2 \qquad [33]$$

where the subscript "conv" means "conventional units."

. . . but mass is an invariant, the same for every observer.

Equations [32] and [33] are central expressions in special relativity. The particle energy E will typically have a different value when measured in different frames that are in uniform relative motion. Also the particle momentum p will typically have a different value when measured in different frames that are in uniform relative motion. However, the values of these two quantities in *any* given free-float frame can be used to determine the value of the particle mass m, which is independent of the reference frame. Particle mass m is an *invariant*, independent of reference frame, just as the time $d\tau$ recorded on the wristwatch between ticks in equation [1] is an invariant, independent of the reference frame.

The mass m of key, car, or coffee cup defined in equation [32] is the one we use throughout our study of both special and general relativity. Such a **test particle** responds to the structure of spacetime in its vicinity but has small enough mass not to affect this spacetime structure. (In contrast, the large mass M of a planet, star, or black hole does affect spacetime in its vicinity.) Wherever we are, we can always climb onto a local free-float frame (Section 8) and apply special-relativity expression [32] or some other standard method to measure the mass m of our test particle.

Fuller Explanations: Mass and momentum-energy in flat spacetime: *Spacetime Physics*, Chapter 7, Momenergy.

No Mass Change with Velocity!

The fact that no object moves faster than the speed of light is sometimes "explained" by saying that "the mass of a particle increases with speed." This interpretation can be applied consistently, but what could it mean in practice? Someone riding along with a faster-moving stone detects no change in the number of atoms in the stone, nor any change whatever in the individual atoms, nor in the binding energy between atoms. Our viewpoint in this book is that mass is an *invariant*, the same for all free-float observers when they use equations [32] or [33] to reckon the mass. In relativity, invariants are diamonds. Do not throw away diamonds! For more on this subject, see *Spacetime Physics,* Dialog: Use and Abuse of the Concept of Mass, pages 246–251.

8 The Free-Float Frame Is Local

In practice there are limits on the space and time extent of the free-float (inertial) frame.

The free-float (inertial) frame is the arena in which special relativity describes Nature. The power of special relativity applies strictly only in a frame—or in each one of a collection of overlapping frames in uniform relative motion—in which a free particle released from rest stays at rest and a particle launched with a given velocity maintains the magnitude and direction of that velocity.

Limits of local free-float frames imply the need for general relativity.

If it were possible to embrace the Universe with a single free-float (inertial) frame, then special relativity would describe that universe and general relativity would not be needed. But general relativity *is* needed precisely because typically inertial frames are inertial in only a limited region of space and time. Inertial frames are **local**. The free-float frame can be realized, for example, inside various "containers," such as (1) an unpowered spaceship in orbit around Earth or Sun or (2) an elevator whose cables have been cut or (3) an unpowered spaceship in interstellar space. Riding in these free-float frames for a short time, we find no evidence of gravity.

Free-float frame cannot be too large.

Well, *almost* no evidence. The enclosure in which we ride cannot be too large or fall for too long a time without some unavoidable changes in relative motion being detected between particles in the enclosure. Why? Because widely separated test particles within a large enclosed space are differently affected by the nonuniform gravitational field of Earth—to use the Newtonian way of speaking. For example, two particles released side by side are both attracted toward the center of Earth, so they move closer together as measured inside a falling long narrow horizontal railway coach (Figure 4, left). Moving toward one another has nothing to do with gravitational attraction between these test particles, which is entirely negligible.

As another example, think of two test particles released far apart vertically but one directly above the another in a long narrow vertical falling railway coach (Figure 4, right). For vertical separation, their gravitational accelerations toward Earth are in the same direction, according to the Newtonian analysis. However, the particle nearer Earth is more strongly attracted to Earth and gradually leaves the other behind: the two particles move far-

Elevator Safety

Could the cables snap and send an elevator plummeting down the shaft?

This is every rider's worst fear, but experts say there's no need to worry. You're being supported by four to eight cables, each of which could support the weight of the car by itself. In fact, the only time an elevator has been known to go into freefall—with all of its cables cut—was during World War II, when an American bomber accidentally hit the Empire State Building [in New York City]. The plane's crew died, but the lone elevator passenger survived.

—*Good Housekeeping Magazine*, February 1998, page 142.

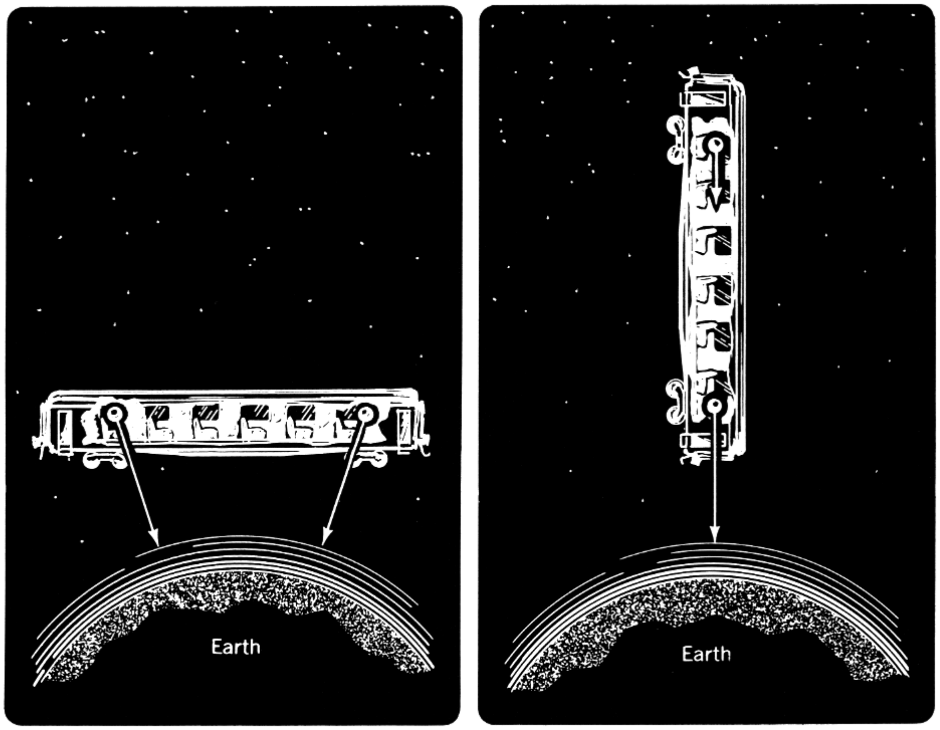

Figure 4 *Einstein's old-fashioned railway coach in free fall. Left: horizontal orientation. Right: vertical orientation.*

ther apart as observed inside the falling coach. Conclusion: The large enclosure is not a free-float frame.

A rider in either railway car shown in Figure 4 sees the pair of test particles *accelerate* toward one another or away from one another. These relative motions earn the name **tidal accelerations,** because they arise from the same kind of nonuniform gravitational field—this time the field of Moon—that account for ocean tides on Earth.

Now, we want the laws of motion to look simple in our free-float frame. Therefore we want to eliminate all relative accelerations produced by external causes. "Eliminate" means to reduce them below the limit of detection so that they do not affect measurements of, say, the velocity of a particle in an experiment. We eliminate the problem by choosing a room that is sufficiently small. Smaller room? Smaller relative motions of objects at different points in the room!

Reduce space or time extension to preserve free-float frame.

Let someone have instruments for detection of relative motion with any given degree of sensitivity. No matter how fine that sensitivity, the room can always be made so small that these perturbing relative motions are too small to be detectable in the time required for the experiment. Or, instead of making the room smaller, shorten the time duration of the experiment to make the perturbing motions undetectable. For example, very fast particles emitted by a high-energy accelerator on Earth traverse the few-meter span of a typical experiment in so short a time that their deflection in

Earth's gravitational field is negligible. The result: The frame of the laboratory at rest on Earth's surface is effectively free-float for purposes of analyzing these experiments.

Both space and time enter into the specification of the limiting dimensions of a free-float frame. Therefore—for a given sensitivity of the measuring devices—a reference frame is free-float only within a limited region of *spacetime.*

Test for free-float property within the frame itself.

An observer tests for a free-float frame by releasing particles from rest throughout the space and noting whether they remain effectively at rest during the time set aside for our particular experiment. Wonder of wonders! Testing for free float can be carried out entirely within the frame itself. The observer need not look out of the room or refer to any measurements made external to the room. A free-float frame is "local" in the sense that it is limited in space and time—and also "local" in the sense that its free-float character can be determined from within, locally.

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One way to get rid of "gravitational force" is to jump from a high place toward a trampoline below. That is to say, a locally free-float frame is always available to us. But no contortion or gyration whatsoever will eliminate the *relative* accelerations of test particles that indicate the limits of the free-float frame. These relative accelerations are the central indicators of the *curvature of spacetime*. They stand as warning signs that we are reaching the limits of special relativity.

General relativity requires more than one free-float frame.

How can we analyze a pair of events widely separated near Earth, near Sun, or near a neutron star, events too far apart to be enclosed in a single free-float frame? For example, how do we describe the motion of an asteroid whose orbit completely encircles Sun, with an orbital period of many years? The asteroid passes through many free-float frames but cannot be tracked using a single free-float frame. Special relativity has reached its limit! To describe accurately motion that oversteps a single free-float frame, we must turn to general relativity—the Theory of Gravitation—as we do in Chapter 2.

Fuller explanations: *Spacetime Physics,* Chapter 2, Floating Free, and Chapter 9, Gravity: Curved Spacetime in Action.

9 The Observer

Ten thousand local witnesses

Detect each event locally, using a latticework of clocks.

How, in principle, do we record events in space and time? Nature puts an unbreakable speed limit on signals—the speed of light. This speed limit causes problems with the recording of widely separated events, because we do not *see* a remote event until long after it has occurred. To avoid the light-velocity delay, adopt the strategy of detecting each event using equipment located right next to that event. Spread event-detecting equipment over space as follows. Think of assembling metersticks and clocks into a cubical latticework similar to a playground jungle gym (Figure 5). At every intersection of the latticework fix a clock. These clocks are identical and measure time in meters of light-travel time.

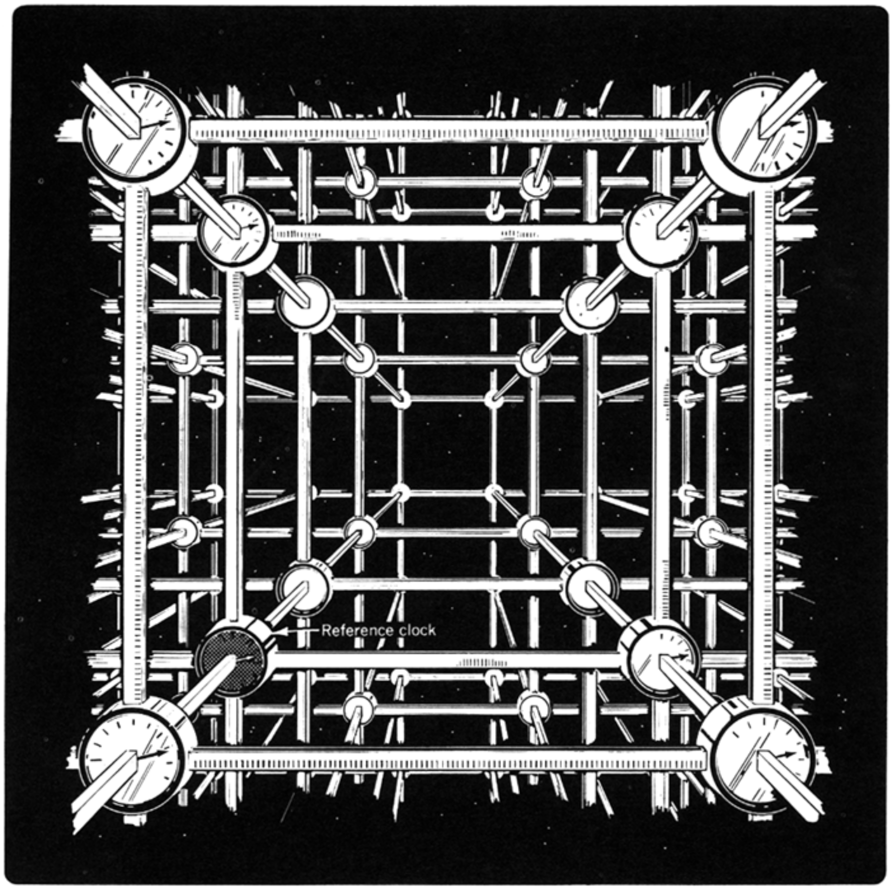

Figure 5 *Latticework of metersticks and clocks*

These clocks should read the *same time.* That is, the clocks need to be **synchronized** in this frame. There are many valid ways to synchronize clocks. Here is one: Pick one clock as the standard, the **reference clock**. At midnight the reference clock sends out a **synchronizing flash** of light in all directions. Prior to emission of the synchronizing flash, every other clock in the lattice has been stopped and set to a time (in meters) later than midnight equal to the straight-line distance (in meters) of that clock from the reference clock. Each clock is then started when it receives the reference flash. The clocks in the latticework are then said to be *synchronized.*

Synchronize clocks in the lattice.

Use the latticework of synchronized clocks to determine the location and time at which any given event occurs. The spatial position of the event is taken to be the location of the clock nearest the event and the time of the event is the time recorded on that clock. The location of this nearest clock is measured along three directions, northward, eastward, and upward from the reference clock. The spacetime location of an event then consists of four numbers, three numbers that specify the space position of the clock nearest the event and one number that specifies the time the event occurs as recorded by that clock.

Measuring the space and time location of an event

Specify the location of an event as the location of the clock nearest to it. With a latticework made of metersticks, the location of the event will be uncertain to some substantial fraction of a meter. For events that must be located with greater accuracy, a lattice spacing of 1 centimeter or 1 millimeter would be more appropriate. To track an Earth satellite, lattice spacing of 100 meters might be adequate.

The lattice clocks, when installed by a foresighted experimenter, will be **recording clock**s. Each clock is able to detect the occurrence of an event (collision, passage of light flash or particle). Each reads into its memory the nature of the event, the time of the event, and the location of the clock. The memory of all clocks can then be read out and analyzed later at some command center.

The "observer" is all the recording clocks in one frame.

In relativity we often speak about the **observer**. Where is this observer? At one place or all over the place? Answer: The word *observer* is a shorthand way of speaking about the whole collection of recording clocks associated with one free-float frame. This is the sophisticated sense in which we hereafter use the phrase "the observer measures such-and such."

What happens to our latticework of clocks in the vicinity of Earth or Sun or neutron star or black hole? Suppose one of these centers of attraction is isolated in space and we stay far away from it. Then there is no problem in setting up an extensive latticework that starts far from the center and stretches even farther away in all directions. Such an extensive *far-away lattice* can represent a single valid free-float frame. And in studying general relativity we often speak of a **far-away observer**.

The far-away lattice is not free float when extended to near Earth or black hole.

But there are problems in extending the far-away latticework of clocks down toward the surface of any of these structures. A free particle released from rest near that center does not remain at rest with respect to the far-away lattice. A single free-float frame no longer provides a simple description of motion.

Many local frames are required near Earth or black hole.

To describe motion near a center of gravitational attraction we must give up the idea of a single global free-float frame, one that covers all space and time around Earth or black hole. Replace it with many local frames, each of which provides only a small part of the global description. A world atlas binds together many overlapping maps of Earth. Individual maps in the atlas can depict portions of Earth's surface small enough to be essentially flat. Taken together, the collection of maps bound together in the world atlas correctly describes the entire spherical surface of Earth, a task impossible using a single large flat map for the entire Earth. For spacetime near nonrotating Earth or black hole, the task of binding together individual localized free-float frames is carried out by the *Schwarzschild metric*, introduced in Chapter 2. The Schwarzschild metric frees us from limitation to a single free-float frame and introduces us to curved spacetime.

Fuller Explanations: *Spacetime Physics*, Chapter 2, Section 2.7, Observer.

10 Summary

The wristwatch time τ between two events, the time recorded on a watch that moves uniformly from one event to the other, is related to the separation s between the events and the time difference t between them as measured in a given frame. For space and time measured in the same units, this relation is given by the equation

$$\tau^2 = t^2 - s^2 \qquad [1]$$

The wristwatch time τ is an *invariant*, the same calculated by all observers, even though t and s may have different values, respectively, as measured in different reference frames. Equation [1] is an example of the *metric*.

Of all possible paths between an initial event and a final event, a free particle takes the path that makes the wristwatch time along the path an extremum. This is called the *Principle of Extremal Aging.*

From the metric and the Principle of Extremal Aging one can derive two quantities that are constants of the motion for a free particle. One constant of the motion is the energy per unit mass *E/m:*

$$\frac{E}{m} = \frac{dt}{d\tau} \qquad [25]$$

The second constant of the motion is the momentum per unit mass *p/m:*

$$\frac{p}{m} = \frac{ds}{d\tau} \qquad [29]$$

The spacetime arena for special relativity is the *free-float (inertial) frame*, one in which a free test particle at rest remains at rest and a free test particle in motion continues that motion unchanged. We call a region of spacetime *flat* if a free-float frame can be set up in it.

In principle one can set up a latticework of synchronized clocks in a free-float frame. The position and time of any event is then taken to be the location of the nearest lattice clock and the time of the event recorded on that clock. The *observer* is the collection of all such recording clocks in a given reference frame.

Most regions of spacetime are flat over only a limited range of space and time. Evidence that a frame is not inertial (so that its region of spacetime is not flat) is the relative acceleration ("tidal acceleration") of a pair of free test particles with respect to one another. If tidal accelerations affect an experiment in a region of space and time, then we say that spacetime region is *curved,* and special relativity cannot validly be used to describe this experiment. In that case we must use *general relativity, the theory of gravitation,* which correctly describes the relations among events spread over regions of space and time too large for special relativity.

Note on terminology: In this book we use the convention recommended by the International Astrophysical Union that names for objects in the solar system be capitalized and used without the article. For example, we say "orbits around Sun" or "the mass of Moon." This provides a consistent convention; one would not say "orbits around *the* Mars." We also capitalize the words *Nature* and *Universe* out of respect for our cosmic home.

11 Readings in Special Relativity

Spacetime Physics, Introduction to Special Relativity, Second Edition, Edwin F. Taylor and John Archibald Wheeler, W. H. Freeman and Co., New York, 1992, ISBN 0-7167-2327-1. Our own book, to which reference is made at the end of several sections in Chapter 1 and elsewhere in the present book.

Special Relativity, A. P. French, W. W. Norton & Co., New York, 1968, Library of Congress 68-12180. An introduction carefully based on experiment and observation.

A Traveler's Guide to Spacetime, An Introduction to the Special Theory of Relativity, Thomas A. Moore, McGraw-Hill, Inc., News York, 1995, ISBN 0-07-043027-6. A concise treatment by a master teacher.

Flat and Curved Space-Times by George F. R. Ellis and Ruth M. Williams, Clarendon Press, Oxford, 1988, ISBN 0-19-851169-8. A leisurely, informative, and highly visual trip through special relativity is followed by treatment of curved spacetime. See more on this book in the section Readings in General Relativity at the end of the present book.

Space and Time in Special Relativity, N. David Mermin, Waveland Press, Inc., Prospect Heights, IL, 1989, ISBN 0-8813-420-0. Rigorous and mildly eccentric.

Understanding Relativity: A Simplified Approach to Einstein's Theories, Leo Sartori, University of California Press, Berkeley, 1996, ISBN 0-520-20029-2. Thoughtful and complete.

Relativity, The Special and General Theory, Albert Einstein, Crown Publishers, New York, 1961, ISBN 0-517-025302. A popular treatment by the Old Master himself. Published originally in 1916. Enjoyable for the depth of physics, the humane viewpoint, and the charm of old-fashioned trains racing past embankments.

Relativity Visualized, Lewis Carroll Epstein, Insight Press, 1985, ISBN 0-935218-05-X. Eccentric visual treatment of special and general relativity.

Of historical interest

Relativity and Its Roots, Banesh Hoffmann, Scientific American Books, New York, 1983, ISBN 0-7167-1510-4. History of the subject by one of Einstein's collaborators.

The Principle of Relativity, A. Einstein, H. A. Lorentz, H. Weyl, H. Minkowski, Dover Publications, Inc., New York, 1952, Standard Book Number 486-60081-5. Translations of many of the original papers. See the following reference for a more recent translation of Einstein's special relativity paper.

Albert Einstein's Special Theory of Relativity: Emergence (1905) and Early Interpretation (1905–1911), Arthur I. Miller, Addison-Wesley Publishing Co., Inc., 1981, ISBN 0-201-04680-6. Careful historical analysis of Einstein's original special relativity paper "On the Electrodynamics of Moving Bodies," the setting in which it was produced, and early consequences for the scientific community. Includes a modern, corrected translation of the paper itself.

12 Reference

Initial quote: Personal memoir of William Miller, an editor of *Life* magazine, quoted in the issue of May 2, 1955. See *The Quotable Einstein*, edited by Alice Calaprice, Princeton University Press, 1996, page 199.