CHAPTER 2

Curving

It is not my purpose in this discussion to represent the general theory of relativity as a system that is as simple and as logical as possible, and with the minimum number of axioms; but my main object here is to develop this theory in such a way that the reader will feel that the path we have entered upon is psychologically the natural one, and that the underlying assumptions will seem to have the highest possible degree of security.

—Albert Einstein

1 "Distances" Determine Geometry

Describe an object with a table of distances between points.
Describe spacetime with a table of intervals between events.

Nothing is more distressing on first contact with the idea of curved spacetime than the fear that every simple means of measurement has lost its power in this unfamiliar context. One thinks of oneself as confronted with the task of measuring the shape of a gigantic and fantastically sculptured iceberg as one stands with a meterstick in a tossing rowboat on the surface of a heaving ocean.

Reproduce a shape using nails and string.

Were it the rowboat itself whose shape were to be measured, the procedure would be simple enough (Figure 1). Draw it up on shore, turn it upside down, and lightly drive in nails at strategic points here and there on the surface. The measurement of distances from nail to nail would record and reveal the shape of the surface. Using only the table of these distances between each nail and other nearby nails, someone else can reconstruct the shape of the rowboat. The precision of reproduction can be made arbitrarily great by making the number of nails arbitrarily large.

It takes more daring to think of driving into the towering iceberg a large number of pitons, the spikes used for rope climbing on ice. Yet here too the geometry of the iceberg is described—and its shape made reproducible—by measuring the distance between each piton and its neighbors.

The event is a nail driven into spacetime.

But with all the daring in the world, how is one to drive a nail into spacetime to mark a point? Happily, Nature provides its own way to localize a point in spacetime, as Einstein was the first to emphasize. Characterize the point by what happens there: firecracker, spark, or collision! Give a point in spacetime the name *event*.

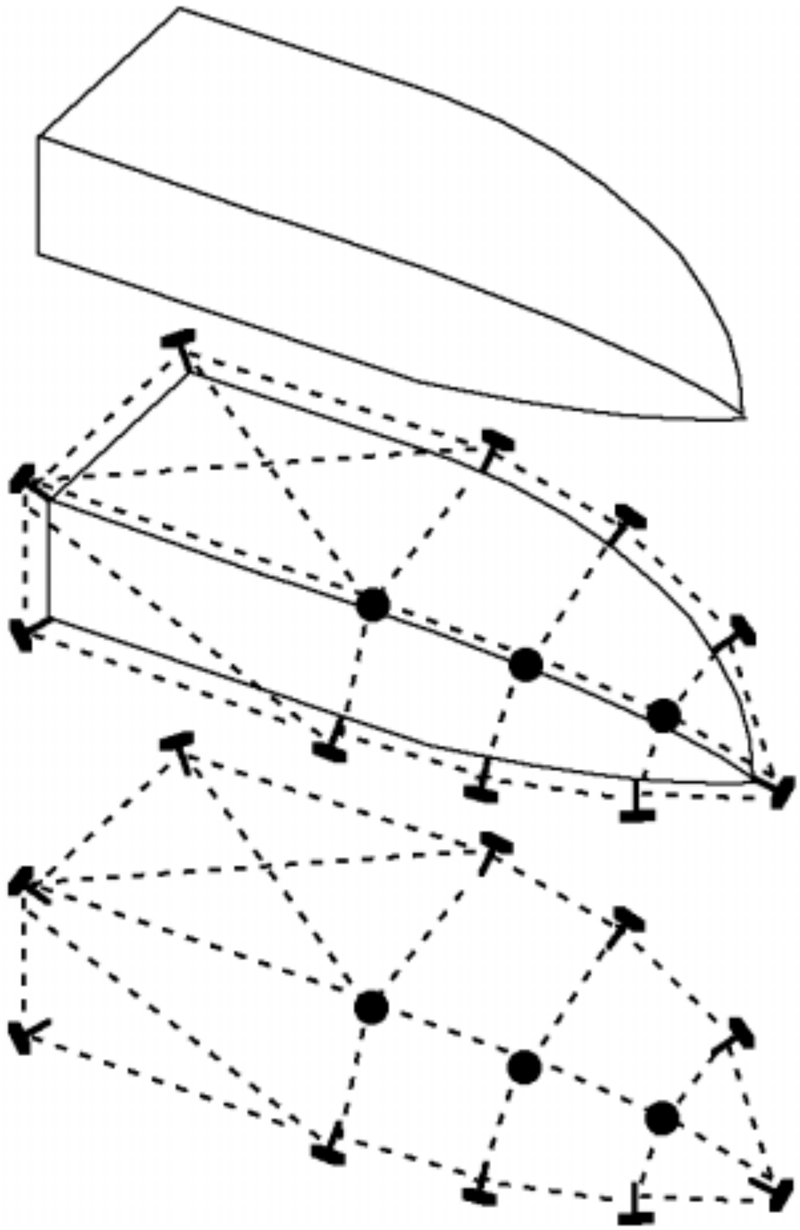

Figure 1 *Reproducing the shape of an overturned rowboat (top) by driving nails around its perimeter, then stretching strings between each nail and every nearby nail (middle). The shape of the rowboat can be reconstructed (bottom) using only the lengths of string segments—the distances between nails. To increase the precision of reproduction, increase the number of nails, the number of string segments, the table of distances.*

Interval: Separation between events in spacetime

Events are the nails, the pitons, the steel surveying stakes of spacetime. How can events describe the geometry of spacetime? Measure the "distance" between each event and every one of its neighboring events. We already know that for spacetime the "distance" between each pair of events means the *spacetime interval* between them (Chapter 1). The table of *distances* between points in space becomes a table of *intervals* between events in spacetime.

Events and intervals reproduce "shape" of spacetime.

The table of distances between points allows us to describe and reproduce the spatial geometry of a surface—whether plane or curved. The table of spacetime intervals between events allows us to describe and reproduce the "shape," the geometry of spacetime—whether the flat spacetime geometry described by special relativity or the curved spacetime geometry described by general relativity.

2 Reference Frames Are Secondary

Lab and rocket frames give different viewpoints on flat spacetime.
Different reference frames give different viewpoints on curved spacetime.

Curvature exists with or without reference frames.

Events themselves are the nails on which science hangs. Spacetime intervals between events evidence the geometry of spacetime, its curvature. This geometry, this curvature from point to point, exists whether one or another competing reference frame is used to describe it. Spacetime geometry exists—and can be described—even when no frame of reference is used at all!

We may—and will—choose to use several different reference frames to describe the same events near a gravitating star or planet. One frame spans the interior of an unpowered spaceship orbiting a spherically symmetric center of attraction. Another occupies the inside of a second unpowered spaceship plunging radially toward that center. A third reference frame consists of the inside of a powered spaceship, rockets blasting, that stands at rest outside the same heavenly body. (Or save rocket power by constructing and standing on a stationary spherical shell concentric to the star or planet.) There are many other possible frames. A central idea of general relativity is that reference frames are not fundamental—all are equally valid. People who use general relativity as a tool change reference frames more often than they change clothes. Each different frame illuminates some features of curved spacetime geometry, but rarely does any single reference frame reveal every important feature of that geometry.

Different frames offer different "vantage points" to study spacetime.

Special relativity uses laboratory and rocket frames as different vantage points to get an insight into flat spacetime that exists independent of any reference frame. In the same way we use alternative reference systems around a star to get insight into curved spacetime—a curved geometry that exists independent of any frame of reference. By using different frames for different purposes, we glimpse the spacetime geometry that lies behind all frames of reference.

You keep talking about "curvature" of spacetime. What is curvature?

The word *curvature* is an analogy, a visual way of extending ideas about three-dimensional space to the four dimensions of spacetime. Travelers detect curvature—in both three and four dimensions—by the gradual increase or decrease of the "distance" between "straight lines" that are initially parallel. In three space dimensions, the actual paths in space converge or diverge. Think of two travelers who start near one another at the equator of Earth and march "straight north." Neither traveler deviates to the right or to the left, yet as they continue northward they discover that the distance between them decreases, finally reaching zero as they arrive at the north pole. They can use this deviation to describe the curved spherical surface on which they travel. Similarly, in four-dimensional spacetime, travelers detect the deviation from parallelism of nearby worldlines of free particles, each of which follows an ideally straight spacetime path, often called a ***geodesic***. This curvature can be measured by the travelers and varies from place to place in spacetime.

Einstein: Coordinate systems are not fundamental.

We use frames of reference for our own convenience, for concreteness and economy of thought. But reference frames and their coordinates are not fundamental to Nature. Geometry is fundamental. It took Einstein seven years to achieve this basic insight. In a few sentences he summarizes the transition from special relativity to general relativity:

> *Now it came to me: . . . the independence of the gravitational acceleration from the nature of the falling substance, may be expressed as follows: In a gravitational field (of small spatial extension) things behave as they do in a space free of gravitation. . . . This happened in 1908. Why were another seven years required for the construction of the general theory of relativity? The main reason lies in the fact that it is not so easy to free oneself from the idea that coordinates must have an immediate metrical meaning.*

3 Free-Float Frame

Our old, comfy, free-float (inertial) frame carries us unharmed to the center of a black hole. Well, unharmed almost *to the center!*

No escape from inside the horizon of a black hole

We want to experience the spacetime geometry around a black hole, a star that has collapsed "all the way," without limit. General relativity predicts this fate for any too-massive collection of matter. General relativity predicts further that nothing, not even light, can escape from a black hole if the emitting satellite gets closer to the black hole than what is called the *horizon* (the radius of no return, defined more carefully in Section 9). If light cannot escape from an object, this object appears black from the outside. Hence the name "black hole."

"Capsule of flat spacetime"

No one can stop us from observing a black hole from an unpowered spaceship that drifts freely toward the black hole from a great distance, then plunges more and more rapidly toward the center. Over a short time the spaceship constitutes a "capsule of flat spacetime" hurtling through

Escape from a Black Hole? Hawking Radiation

Einstein's equations predict that nothing escapes from the so-called "horizon" of a black hole. In 1973, Stephen Hawking demonstrated a contrary conclusion using quantum mechanics. For years quantum mechanics had been known to predict that particle-antiparticle pairs, such as electrons and positrons, are continually being created and recombined in undisturbed space, despite the frigidity of the vacuum. These processes have, indirectly, important and well-tested observational consequences. Never in cold flat spacetime, however, do such events ever present themselves to direct observation. For this reason the pairs receive the name "virtual particles." When such a particle-antiparticle pair is produced near the horizon of a black hole, Hawking showed, one member of the pair will occasionally be swallowed by the black hole, leaving the other one to escape. Escaped particles form what is called **Hawking radiation**. The energy of the escaping particle comes from the black hole. Over time this loss of energy causes the black hole to "evaporate." The final stage may be a super-H-bomb explosion. For a black hole of several solar masses, however, the time required to achieve this explosive state exceeds the age of the Universe by a fantastic number of powers of ten. For this reason we ignore such emissions here. (See also the box on page 5-27.)

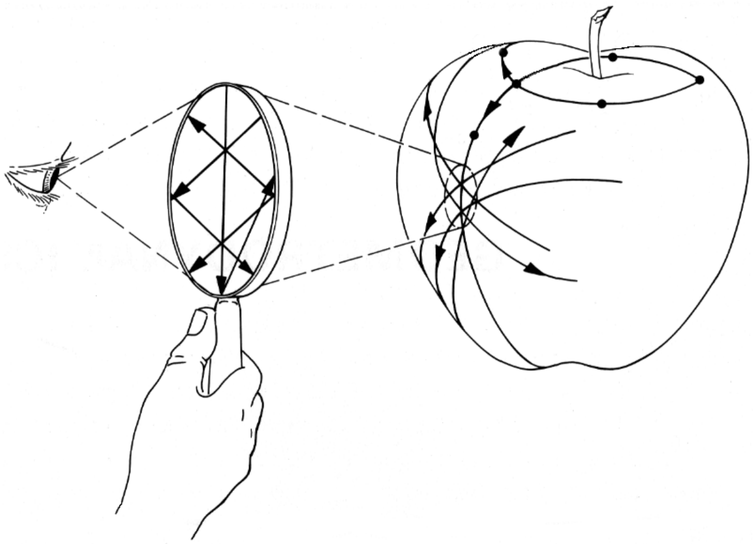

Figure 2 *The curved spacetime geometry of general relativity symbolized by the two-dimensional geometry of the surface of Newton's apple. The locally straight* (geodesic) *tracks followed by ants crawling on the apple's surface symbolize the tracks followed through spacetime by free particles. In any sufficiently localized region of spacetime, the geometry can be idealized as flat, as symbolized on the apple's two-dimensional surface by the straight-line course of the tracks viewed in the magnifying glass. In a region of greater extension, the curvature (curved two-dimensional space in the case of the apple; four-dimensional spacetime in the case of the real physical world) makes itself felt. On a larger scale, two tracks originally diverging from a common point later approach, cross, and go off in very different directions. In Newtonian theory this effect is ascribed to gravitational force acting at a distance from a massive body, symbolized here by the stem of the apple. According to Einstein, a particle gets its moving orders locally, from the geometry of spacetime right where it is. Its instructions are simple: "Go straight! Follow the straightest possible worldline (geodesic)." Physics is as simple as it could be locally. Only because spacetime is curved in the large do the tracks diverge, converge, and cross ("tidal accelerations").*

curved spacetime. It is a free-float frame like any other. Special relativity makes extensive use of such frames, and special relativity continues to describe Nature correctly for an astronaut in a local free-float frame, even as she falls through curved spacetime, through the horizon, and into a black hole. Keys, coins, and coffee cups continue to move in straight lines with constant speed in such a local free-float frame. (Figure 2 illustrates, by analogy, that paths curved in three space dimensions appear straight when we view small enough portions of these paths.) Collisions, creations, and annihilations of particles continue to follow the special relativity law of conservation of momentum-energy. What could be simpler?

Unavoidable relative accelerations near a star

However, as we approach the black hole the dimensions of our frame must be progressively constricted if we are to verify that it is free-float. In free fall near Earth, relative accelerations change the separation between two test particles, thus restricting the size of the spacetime region in which both are observed to be in free float (Chapter 1, Section 8). In imagination we can extend our near-Earth experience to regions exterior to more massive spherically symmetric objects: our Sun, a similar star, a white dwarf, a neutron star, a black hole. As we get closer and closer to each of these more and more compact spherically symmetric bodies, greater and greater become the relative accelerations between test particles. Near the center of a black hole these relative accelerations become lethal.

Relative accelerations are called **tidal accelerations**, because they are similar to the difference of our Moon's gravitational attraction on opposite sides of Earth that lead to tides. (See Section 2.3 of *Spacetime Physics*.)

Lethal effects of relative accelerations near black hole

Consider, for example, the plight of an experimental astrophysicist freely falling feet first toward a black hole. As the trip proceeds, various parts of the astrophysicist's body experience different gravitational accelerations. His feet are accelerated toward the center more than his head, which is farther away from the center. The difference between the two accelerations (the tidal acceleration) pulls his head and feet apart, growing ever more intense as he approaches the center of the black hole. The astrophysicist's body, which cannot withstand such extreme tidal accelerations, suffers drastic stretching between head and foot as the radial distance drops to zero.

But that is not all. Simultaneous with this head-to-foot stretching, the radial attraction toward the center funnels the astrophysicist's body into regions of space with ever-decreasing circumferential dimension. Tidal gravitational accelerations *compress* the astrophysicist on all sides as they *stretch* him from head to foot. The astrophysicist, as the distance from the center approaches zero, is crushed in width and radically extended in length. Both lethal effects are natural magnifications of the relative motions of test particles released from rest at opposite ends of free-float frames near Earth (Chapter 1, Section 8).

How small can a free-float frame be?

Confronted by tidal accelerations, how can we define a free-float frame falling into a black hole? At the center of the black hole we cannot; general relativity predicts infinite tidal accelerations there. However, short of the center, we employ the strategy used in free-float near Earth (Section 8 of Chapter 1): Limit the space and the time—the region of *spacetime*!—in which experiments are conducted. Very near the center we restrict ourselves to an ever smaller and more pinched local region of spacetime in which to define a free-float frame and in which to employ special relativity. How small can a free-float frame be? A single radioactive atomic nucleus can emit a detectable signal, for example a high-energy flash ("gamma ray"). In principle a reference frame can have space dimensions as tiny as that of the nucleus and time dimension equal to the emission time of the gamma ray. If gravitational tidal accelerations do not distort the nucleus "too much" within this spacetime region, the laws of special relativity accurately describe the nucleus in such a frame—the frame is effectively free-float for purposes of this experiment.

The constant, ever-present "force of gravity" that we experience on Earth is gone, eliminated as we step into a free-float frame. What remains of "gravity"? Only curvature of spacetime remains. What is this curvature?

Nothing but tidal acceleration. Curvature is tidal acceleration and tidal acceleration is curvature. Kip Thorne says it clearly: "Einstein and Newton, with their very different viewpoints on the nature of space and time, give very different names to the agent that causes test particles to accelerate toward or away from one another in a frame that is not quite free-float. Einstein calls it spacetime curvature; Newton calls it tidal acceleration. But there is just one agent acting. Therefore, *spacetime curvature and tidal accelerations must be precisely the same thing, expressed in different languages.*" (Quotation slightly edited; original in the references.)

Need global coordinates

One limitation of a free-float frame near a black hole is the tidal accelerations experienced by test particles as the frame falls toward zero radius. Another limitation is the large-scale consequence of tidal acceleration: No single free-float frame is large enough to describe relations between two events that occur on opposite sides of the central mass. Two such events might be the emission of two flashes at different times by an object whose orbit girdles a black hole. To relate such widely separated events, we need a global rather than a local coordinate system. Karl Schwarzschild provided the basis for such a global coordinate system around a spherical, nonspinning center of attraction. Schwarzschild coordinates apply approximately to slowly spinning bodies such as Earth and Sun and to nonspinning or slowly spinning neutron stars and black holes. But these coordinates also have limitations. Points of view provided by free-float and Schwarzschild-related coordinate systems—and by still other coordinate systems—probe deeply the geometry of empty spacetime around a star. We now begin the study of the coordinates used by Schwarzschild.

4 The *r*-coordinate: Reduced Circumference

How to measure the radius while avoiding the trap in the center

Spherically symmetric centers of attraction

Matter has, by virtue of gravitation, a marvelous ability to agglomerate into spherical centers of attraction. Nicolaus Copernicus is credited with the insight that replaced Earth as the only assumed center of attraction with multiple centers of gravity. Standing as witness to the simplicity of the spherical shape are Earth, Moon, planets, Sun, and stars. Each of these structures is compressed—more dense—near its center and less dense near its surface. But this density changes with radius only, not with angle around the center. Such structures earn the label **spherically symmetric**. (Strictly speaking, an astronomical object can be spherically symmetric only if it does not rotate on its axis. For our Sun and planets this rotation rate is small enough so that departures from spherical symmetry can be neglected in the interpretation of many observations.)

The closer the distribution of mass to exact spherical symmetry, the better the spacetime geometry around such a structure conforms to the wonderfully simple solution to the equations of general relativity discovered by Karl Schwarzschild in 1915. Schwarzschild's solution describes spacetime external to *any* isolated spherically symmetric body in the Universe.

Nicolaus Copernicus *Born Torun, Poland, February 19, 1473; died Frombork, Poland, May 24, 1543. The flower is an old symbol for medicine, which Copernicus learned at Padua. His medical skill was always at the service of the poor.*

The Dictionary of Scientific Biography says*: Whereas the pre-Copernican cosmos had known only a single center of gravity or heaviness, the physical universe acquired multiple centers of gravity from Copernicus, who thus opened the road that led to universal gravitation . . . [H]e put forward a revised conception of gravity, according to which heavy objects everywhere tended toward their own center—heavy terrestrial objects toward the center of the earth, heavy lunar objects toward the center of the Moon, and so on. [Copernicus wrote:]*

> *"For my part, I think that gravity is nothing but a certain natural striving with which parts have been endowed . . . so that by assembling in the form of a sphere they may join together in their unity and wholeness. This tendency may be believed to be present also in the sun, the moon, and the other bright planets, so that it makes them keep that roundness which they display.'"*

Schwarzschild's simple solution

What does this "Schwarzschild geometry" around Earth, star, or black hole look like? "What a nonsensical question!" we say at first. Whoever looks *at* space? We look *through* space. Or we thrust skeleton skyscrapers out into space, we push out into space the Buckminster Fuller framework of a great spherical building (Figure 3). Ha! Just such a Buckminster Fuller construction gives us at last a way to "see" what space looks like, as described in what follows.

To be specific, take the center of attraction to be a black hole. Let it have the same mass as Sun. Build around it, in imagination, an open spherical shell of rods fitted together in a mesh of triangles (Figure 3) similar to hemispherical jungle gyms found on playgrounds. This spherical shell, this scaffolding, is an alternative to our latticework of rods and clocks in local free-float frames. Mount clocks on this shell. The rods and clocks of this shell provide one system of spacetime coordinates to locate events.

Spherical shell of rods and clocks

We say to build this shell "in imagination," because neither steel nor tantalum nor any modern wonder material has a ratio of strength to weight adequate to support such a structure against the inward pull of gravity. However, the surface of a planet, moon, or star has itself the character of a shell. We walk around on such a shell every day: Earth's surface! In the absence of an actual spherical shell, we can use a spaceship that stands still above the surface by blasting its rockets inward.

We cannot measure radius directly.

How shall we define the size of the sphere formed by this latticework shell? Shall we measure directly its distance from its center? That won't do. Yes, in imagination we can stand on the shell. Yes, we can lower a plumb bob on a "string." But for a black hole, any string, any tape measure, any steel wire—whatever its strength—is relentlessly torn apart by the unlimited pull the black hole exerts on any object that dips close enough to its center. Even for Earth or Sun, the surface keeps us from lowering our plumb bob directly to the center.

Derive radius from measurement of circumference.

Then try another way to define the size of the spherical shell. Instead of lowering a tape measure from the shell, run a tape measure around it. Call the distance so obtained the *circumference* of the sphere. Divide this circumference by $2\pi = 6.283185\ldots$ to obtain a distance that would be the directly measured radius of the sphere if the space inside it were flat. But it isn't flat. Yet this procedure yields the most useful known measure of the size of the spherical shell.

The "radius" of a spherical object produced by this method of measuring has acquired a name, the **coordinate radius**, despite its being no true radius. We call it also the **reduced circumference**, to remind us that it is derived ("reduced") from the circumference:

$$\begin{aligned} \text{coordinate radius} &= r = \text{reduced circumference} \\ &= (\text{circumference})/2\pi \qquad [1] \\ &= r\text{-coordinate} \end{aligned}$$

The phrases *coordinate radius* and *reduced circumference* are such mouthfuls that we usually call it simply the ***r*-coordinate** and represent it by the symbol *r*. The *r*-coordinate is the radius computed from the sphere's circumference. This value of *r* is stamped on every shell for all to see.

Having constructed—in imagination—one spherical shell around our black hole and found its coordinate radius, its reduced circumference *r*, we construct inside it a second such framework of rods and likewise determine its radius. We find the reduced circumference *r* of the inner sphere to

Figure 3 *Geodesic globe named Spaceship Earth, the symbol of Disney Epcot Center in Orlando, Florida. Fifty meters in diameter, it contains a ride highlighting the history of communication from cave dwellers to the present. The spherical shells surrounding our black hole are openwork lattices, not a closed surface as shown here. © Disney Enterprises, Inc.*

be 1 kilometer less than that of the first one—based on tape-measure determinations of distance around the two spheres.

Directly measured separation between nested shells is greater than the difference in r-value.

Now, finally, we lower a plumb bob from the outer sphere and for the first time measure directly the true radial distance perpendicularly from the outer sphere to the inner one. Will we find a 1-kilometer radial distance between the two spheres? We would if space were flat. But it is *not* flat. Schwarzschild geometry tells us that the directly measured radial distance between the two nested spheres is *more* than 1 kilometer. That increase over the expectations of Euclidean geometry provides the most striking evidence in principle one can easily cite for the curvature of space we call gravitation. To examine such discrepancies is to see what space looks like around a black hole.

Small effect near Sun

Built around our Sun, the inner sphere cannot lie inside Sun's surface. Its r-coordinate can be no less than that of Sun's surface, which is approximately 695 980 kilometers. Around this inner shell we erect a second one—again in imagination—of *r*-coordinate 1 kilometer greater: 695 981 kilometers. The directly measured distance between the two would be not 1 kilometer, but 2 millimeters more than 1 kilometer.

How can we get closer to the center of a stellar object with mass equal to that of our Sun—but still be external to that object? A white dwarf and a neutron star each has roughly the same mass as our Sun, but each is much smaller. Therefore we can—in principle—conduct a more sensitive test of the nonflatness of space much closer to the centers of these objects while staying external to them. The effects of the curvature of space are much greater near the surface of a white dwarf or neutron star than near the surface of our Sun.

Huge effect near a black hole

Turn attention now to a black hole of one solar mass. Close to it the departure from flatness is much larger than it is anywhere in or around a white dwarf or a neutron star. Construct an inner sphere having an *r*-value, an *r*-coordinate, a reduced circumference of 4 kilometers. Let an outer sphere have an *r*-coordinate of 5 kilometers. In contrast to these two distances, defined by measurements around the two spheres, the directly measured radial distance between the two spheres is 1.723 kilometers, compared to the Euclidean-geometry figure of 1 kilometer (Sample Problem 2, page 2-28). At this location the curvature of space results in measurements quite different from anything that textbook Euclidean geometry would lead us to expect!

WHY is the directly measured distance between spherical shells greater than the difference in r-coordinates between these shells? Is this discrepancy caused by gravitational stretching of the measuring rods?

No, the quoted result assumes infinitely rigid measuring equipment. In practice, of course, a measuring rod held by the upper end will be subject to gravitational stretching. So think of flinging the rod up from below so that it comes to rest temporarily with its two ends next to the two shells and thereby measures the separation directly while in free float. Even in this case there will still be tidal forces on the rod. Strain gauges affixed along the rod can permit us to "calculate away" this stretching. For smaller and smaller separation between the shells the stretching can be reduced below any specified limit.

Don't avoid the issue! You have not answered the question: What CAUSES the discrepancy, the fact that the directly measured distance between spherical shells is greater than the difference in r-coordinates between these shells? WHY this discrepancy?

A deep question! Fundamentally, this discrepancy is evidence of space curvature resulting from the mass contained in the center of attraction. External to this center, the fabric of spacetime does not tear but transmits the ever-diluted curvature outward to influence locally every spherical shell, every test particle, every satellite in the surroundings.

5 Gravitational Red Shift

Rising light shows fatigue by increasing its period of oscillation.
Light rising from the horizon has infinite period—so it does not exist!

Time also enters into the (space*time*) curvature around a black hole. In no way is "time curvature" more apparent than in the behavior of a signal emitted from a clock bolted to a spherical shell near a gravitating body. Let this clock tick by emitting light in a radially outward direction. The emitted light increases its period of vibration as it climbs up out of the gravitational field.

The period of light increases as it climbs.

How does light increase its period of vibration? Every period (every back-and-forth undulation in the wave) of the light can be considered a measure of time, a "tick of the clock." Suppose that the light has a short period when emitted by the clock on the shell. The shell observer records that the emitting clock ticks rapidly; for him time is short from one tick to the next tick. When the light finally arrives at a remote observer, its period is longer. The received clock-tick signals are observed to be farther apart in time than the sent clock-tick signals. Light emitted from a shell clock still closer to the black hole suffers an even greater increase in its period—a greater "time between ticks"—when this light has climbed to infinity.

The period of the received light increases more and more as the emitter stands closer and closer to the black hole. Details of this increased period imply curvature not only of space but of time—curvature of *spacetime*! The increased period means also that the time dt_{shell} between two events—such as clock ticks—measured by an observer standing on a shell (or occupying a spaceship at rest, rockets blasting inward) will be different from the "far-away time dt" between these events as transmitted to and recorded by a clock remote from the gravitating body.

Gravitational red shift

Visible light with the longest period is red. The remote observer sees light emitted by the close-in clock to be "redder"—that is, of longer period—than it was at the point of emission. This effect thus earns the name **gravitational red shift.**

"Blackness" of a black hole

Why is a black hole black? Why cannot light escape from a black hole? After all, light cannot stop moving! Every local observer records the speed of light to be unity as it passes on its upward journey. The gravitational red shift result allows us to give a meaning to the phrase "cannot escape." Light of any period emitted from near the horizon (the threshold radius of no return) suffers a gravitational red shift to a very long period. The closer the clock is to the horizon, the farther toward infinity the period grows as the light climbs out of the black hole to a great distance. But a light signal with infinite period is no light signal at all! It cannot be detected. In this case almost no light has escaped from near the horizon of the black hole. For a clock *at* the horizon, as a limiting case, *no* light escapes to even 1 centimeter above the horizon (Chapter 5 exercises). Light is red shifted all the way to infinite period. This crisp result accounts for the blackness of a black hole (which is black except for Hawking radiation, a quantum phenomenon described in the box on page 2-4).

The gravitational red shift occurs between two clocks that are at different radii and both at rest with respect to the black hole or other center of gravitational attraction. Another and different red shift occurs due to the Doppler effect when two clocks move away from one another. An example is the red shift of light that we receive from nearby galaxies outside our own, thought to be due to the recession of these galaxies from us. (A generalization of the Doppler shift to curved expanding spacetime is the reddening of light from distant galaxies as the Universe expands—see Project G, The Friedmann Universe.) For observers on Earth this recessional red shift is in principle partly canceled by the **gravitational blue shift** of the light as it drops into the gravitational well surrounding Earth. However, for many everyday purposes the gravitational blue shift for Earth is negligible. (See exercise at the end of this chapter.)

The gravitational red shift is different from the Doppler shift due to relative motion.

We have described two consequences of spacetime curvature: the augmentation of distance between adjacent spherical shells and the increase in the period of light escaping outward from one of these shells. How these effects come about, and why they become so impressive at the horizon of a black hole, shows on an examination of the expressions describing the Schwarzschild solution to Einstein's great and still standard 1915 equation for the bending of spacetime geometry. Before we can write down these expressions in simple form, we need to describe the mass of the central body, not in the unperceptive conventional units of kilograms, but rather in the same geometric units we use to measure distance: meters or kilometers.

6 Mass in Units of Length

Want to make everything geometry? Then measure mass in meters!

Descriptions of spacetime near any gravitating body are simplest when the mass M of that body is expressed in units of distance—in meters or kilometers. This section is devoted to finding the conversion factor between, say, kilograms and meters.

Measure mass in meters.

Earlier when we wanted to measure space and time in the same units (Chapter 1, Section 2), we used the conversion factor c, the speed of light. The conversion from kilograms to meters is not so simple. Nevertheless, here too Nature provides a conversion factor, a combination of the speed of light and the **universal gravitation constant** $\boldsymbol{G}$ that characterizes the gravitational interaction between bodies.

Newton's theory of gravitation predicts that the gravitational force between two spherically symmetric masses M and m is proportional to the product of these masses and inversely proportional to the square of the distance r between their centers:

$$F = \frac{GM_{\mathrm{kg}}m_{\mathrm{kg}}}{r^2} \qquad \text{[2. Newton]}$$

Subscripts tell us that in this equation the masses M_{kg} and m_{kg} are in units of kilograms. In this equation G is the "constant of proportionality." The numerical value of this constant depends on the units with which mass

Numerical values of G and c: historical accident and distance are measured. Historically the units of mass and the units of distance were developed independently, without appreciation of their relationship. The numerical value of G was not built into Nature by law but arose by accident of human history, as the numerical value of the speed of light c likewise arose from historical accident alone. When we measure mass in kilograms and distance in meters, then G has the experimentally determined value

$$G = 6.6726 \times 10^{-11} \frac{\text{meter}^3}{\text{kilogram second}^2} \qquad [3]$$

Divide G by the square of the speed of light, c^2, to find the conversion factor that translates the conventional unit of mass, the kilogram, into what we have already found to be the natural geometric unit, the meter:

$$\frac{G}{c^2} = \frac{6.6726 \times 10^{-11} \dfrac{\text{meter}^3}{\text{kilogram second}^2}}{8.9876 \times 10^{16} \dfrac{\text{meter}^2}{\text{second}^2}} = 7.424 \times 10^{-28} \frac{\text{meter}}{\text{kilogram}} \qquad [4]$$

Now convert from mass M_{kg} measured in conventional units of kilograms to mass M in units of length—meters—by multiplication with this conversion factor:

$$M = \frac{G}{c^2} M_{\text{kg}} = \left(7.424 \times 10^{-28} \frac{\text{meter}}{\text{kilogram}}\right) M_{\text{kg}} \qquad [5]$$

Mass in units of meters unclutters equations.

Why make this conversion? First, it is an elegant way to proclaim that mass is fundamentally tied to geometry. Second, it allows us to get rid of the factors G and c^2 that would otherwise clutter up the equations to follow.

Wait a minute! Stars and planets are not the same as space. No twisting or turning on your part can make mass and distance the same. Therefore mass cannot be measured in units of distance. How can you possibly propose to measure mass in units of meters?

True, mass is not the same as distance. Neither is time the same as space: Clock ticks are different from meterstick lengths! Nevertheless, we have learned to measure both time and space in the same units: light-years of distance and years of time, for example, or meters of distance and meters of light-travel time. Using the same units for both space and time helps us to get rid of people-made complications and to recognize the unity we call spacetime. The conversion factor between time in seconds and space in meters is the speed of light c.

The same comments hold in the present case for measuring mass in units of length. Mass is not the same as length; no one claims it is. But we gain insight when we measure both in geometric units. When we express the mass of a star in meters, we can convert this figure to any other measure we want: grams, kilograms, or number of solar masses. For the translation from kilograms to meters, the conversion factor is not a mere power of the speed of light but includes the gravitational constant G. The factor that converts kilograms to meters is G/c^2. And the payoff of this conversion is similar to earlier payoffs: we see more simply how Nature works and we arrive more quickly at correct predictions. Mass, and therefore gravitation, is elevated (not reduced!) to geometry.

All right. Wonderful! Now go one step further and make the definition of the kilogram in terms of the meter an official international standard. Since 1983 the official international standard for the meter is the distance light travels in 1/299,792,458 second, thus tying the meter to a measurement of time. By defining (at some more enlightened future date) the kilogram in terms of the geometric unit meter, we link it also to a measurement of time. All other physical units—energy, momentum, electric charge—have long been defined in terms of time, length, and mass. By officially defining the kilogram in units of length, and therefore ultimately in units of time, we unify the world of measurement to a single quantity.

Your proposed unification is a good idea in principle but not yet satisfactory in practice. Measurement of mass is very precise. So is measurement of length and time. However, the conversion factor between mass and length, G/c^2, is not known with corresponding precision. The fault lies with the gravitational constant G, which is difficult to measure—presently accurate to 5 digits at most. Compare that with the nine-digit accuracy of the speed of light that allowed a redefinition of length in terms of time.

Why wait until G is known more accurately? Why not just define the kilogram in the unit of length using the conversion factor 7.424×10^{-28} meter/kilogram, this figure taken to be exact by definition?

There is no logical reason why G cannot be *defined* to have an exact value right now. However, convenience and accessibility are no less important criteria for standards than logical simplicity. The present standard of mass—a particular chunk of metal—can be accurately duplicated, providing secondary standards for calibration of the scales used in science and commerce. This standard is unlikely to be replaced until a way is discovered to measure the gravitation constant G much more accurately, with apparatus available in any well-equipped laboratory

Table 1 displays in both kilograms and meters the mass of Earth, the mass of Sun, and the mass of the huge spinning black hole believed to explain the activity observed at the center of our galaxy and a similar black hole in one other galaxy. (See the references.) Thus does the geometric language of relativity cut the stars down to size.

7 Satellite Motion in a Plane

Once moving in a plane, always moving in that plane

Orbits stay in a plane.

An isolated satellite zooms around a spherically symmetric massive body. Our very first look shows that this motion lies in a plane determined by the satellite's position, its direction of motion, and the center of the attracting body. We know that forever afterward the motion will remain confined to that same plane. Why? The reason is simple: symmetry! No distinction between “up out of" and “down below” that plane, so the satellite cannot choose either. Such a rise would provide immediate evidence that there is some further force at work beyond any exerted by the spherically symmetric body—evidence, in other words, that the satellite’s environment is not spherically symmetric with respect to that center of attraction.

Table 1 Masses of some astronomical objects

Object	Mass in kilograms	Geometric measure of mass	Equatorial radius
Earth	5.9742×10^{24} kilograms	4.44×10^{-3} meters or 0.444 centimeters	6.371×10^{6} meters or 6371 kilometers
Sun	1.989×10^{30} kilograms	1.477×10^{3} meters or 1.477 kilometers	6.960×10^{8} meters or 696 000 kilometers
Black hole at center of our galaxy	5.2×10^{36} kilograms (2.6×10^{6} Sun masses)	3.8×10^{9} meters (see references)	
Black hole in center of Virgo cluster of galaxies	6×10^{39} kilograms (3×10^{9} Sun masses)	4×10^{12} meters	

Locate satellite using r and ϕ.

The satellite moves in a plane, so we need two quantities, and only two, to specify its location at any instant. Adopt for one the *r*-coordinate, the reduced circumference of a circle cutting through the satellite. For the second coordinate take the **azimuthal angle** ϕ of the satellite's progression in the plane around the center of attraction (Figure 4).

Every astronaut, every satellite, every light pulse independently orbiting around a spherically symmetric body will remain in its own plane of motion, each position in the plane described by the reduced circumference *r* and the azimuthal angle ϕ in that particular plane. This limitation to a plane greatly simplifies the analysis of physical events described in the remainder of this book.

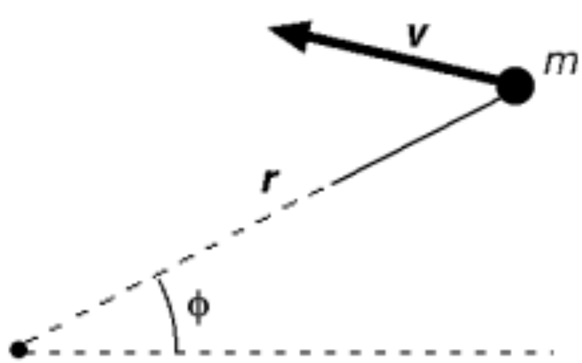

Figure 4 *A satellite moves in an orbit with instantaneous velocity v around a spherically symmetric body. This orbit lies in a plane and remains in that plane for all time. Satellite position on the plane is specified uniquely by two coordinates: we choose the r-coordinate and the azimuthal angle ϕ with respect to some arbitrary initial direction in the plane (horizontal dashed line in the figure). The inner part of the r-line is also dashed, because in the case of a black hole the radius cannot be surveyed directly.*

8 Metrics for Flat Spacetime

Rectangular space coordinates or polar space coordinates: Either can appear in a metric of flat spacetime.

Events and intervals form a universal language.

What gives us security as we move from flat spacetime geometry to Schwarzschild geometry? On what can we depend? What can we trust? Answer: *Events!* Events are the nails of reality on which all of science hangs (Section 1). And between event and event we seek the basic relation, the basic separation, the four-dimensional "distance" between firecracker explosion and firecracker explosion. We seek the *spacetime interval* between any pair of events.

When no large mass is in the vicinity we say that spacetime is *flat*. In flat geometry the expression for the wristwatch time τ between two events can be written in the usual rectangular coordinates described by Descartes ("Cartesian coordinates"). Let t, x, and y mark the separation between two events on a spatial plane when this separation is timelike (time separation greater than space separation). Then τ^2, the square of the wristwatch time between them, is given by the expression

Timelike spacetime interval

$$\tau^2 = t^2 - x^2 - y^2 \qquad \text{[6. flat spacetime]}$$

When, instead, the separation between the two events is spacelike, that is, when the space part of the separation predominates over the time part, we reverse the signs of the terms on the right of [6] to keep the combination positive. Give the resulting squared quantity the Greek letter σ (sigma):

Spacelike spacetime interval

$$\sigma^2 = -t^2 + x^2 + y^2 \qquad \text{[7. flat spacetime]}$$

The corresponding equations [1] and [8] of Chapter 1 for the spacetime interval earned the name *metric*; equations [6] and [7] are metrics too. A *metric* provides the method by which we *meter* or *measure* spacetime.

Polar coordinates are convenient when there is a center of attraction.

For describing the linear motion of one rocket with respect to another in flat spacetime, the Cartesian system of coordinates was perfect. Direction of relative motion: x. Direction transverse—perpendicular—to that relative motion: y. The Cartesian rectangular system is not so convenient as we prepare to describe spacetime around a spherically symmetric gravitating mass. Here the preeminent dimension is radial, toward and away from the center of attraction, with angle ϕ describing the location of an event on an imaginary circle of given radius r lying on a plane through that center. Rewrite the expression for the timelike interval (equation [6]) in polar coordinates. The resulting metric is

$$(d\tau)^2 = (dt)^2 - (dr)^2 - (rd\phi)^2 \qquad \text{[8. flat spacetime]}$$

The box on page 2-18 presents a derivation of the space separation part of this expression, namely $(dr)^2 + (rd\phi)^2$.

Spatial Separation in Flat Spacetime, Expressed in Polar Coordinates

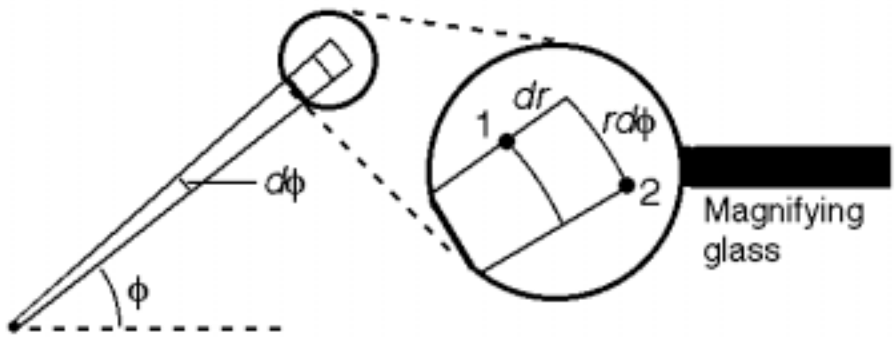

Figure 5 *Spatial separation between two points in polar coordinates.*

Let the coordinate separations between two events near one another be *dx* and *dy* in the *x* and *y* directions, respectively. Then the square of the spatial separation between two events is written

$$(\text{spatial separation})^2 = (dx)^2 + (dy)^2$$

Look for a similar expression for two events numbered 1 and 2 separated by the spherical polar coordinate increments *dr* and $d\phi$. See Figure 5.

Draw little arcs through events 1 and 2 to form a tiny rectangle, as shown in the magnified inset. The squared distance between events 1 and 2 is—approximately—the sum of the squares of two adjacent sides of the little rectangle. Each complete circle, if drawn here, would run through a total arc of 2π and possess a circumference of $2\pi r$. It was this circumference that was the starting point for our very definition of the reduced circumference *r*. The portion of each arc that is depicted in the figure extends only over the angle $d\phi$. It comprises only the fraction $d\phi/2\pi$ of the whole circle. By proportion, its length is $(d\phi/2\pi)$ times $2\pi r$, or $rd\phi$. This arc is so short that its length closely approximates the length of the corresponding straight line. We spell out this part of the reasoning because it goes over unchanged to the curved space geometry around a spherically symmetric body. Not so for the distance *dr!* Consider two points that lie at the same azimuth but have *r*-coordinates *r* and *r* + *dr*. Only in flat space is the distance between them equal to *dr*. Therefore only for flat space are we entitled to figure the distance in space between event 1 and event 2 by the formula

$$(\text{spatial separation})^2 = (dr)^2 + (rd\phi)^2$$

This squared spatial separation is the space part of the squared interval for flat spacetime, equation [8]. Notice that this derivation depends on $d\phi$ being small, so the small segment of arc $rd\phi$ is indistinguishable from a straight line.

Equation [8] is still true only for flat spacetime—the domain of special relativity. Why? Because the everyday world is still the everyday world, whether you view it while standing on your feet or standing on your head! Similarly, flat spacetime is flat whether the interval between events is expressed in Cartesian (rectangular) coordinates or in polar (spherical) coordinates. In brief, no massive body is yet positioned at the origin of this coordinate system

Now place the origin of the spherical coordinate system at the center of a nonspinning spherical object, approximated by Earth or Moon, Sun or white dwarf, neutron star or black hole. Examine the new spacetime geometry external to such a body. This new geometry is described by the Schwarzschild metric, introduced in Section 9.

9 The Schwarzschild Metric for Curved Spacetime

Spherically symmetric massive center of attraction?
Then the Schwarzschild metric describes curved spacetime around it.

The metric for the proper time between two timelike events in a plane in *flat* spacetime is given by equation [9]:

$$d\tau^2 = dt^2 - dr^2 - r^2 d\phi^2 \qquad \text{[9. flat spacetime]}$$

$r^2 d\phi^2$ term is still OK to describe spacetime near Earth.

How is this metric altered for two nearby events on a plane that passes through the center of a spherically symmetric massive body? The last term on the right stays the same because of the way we chose the r-coordinate. The r-coordinate is *defined* so that $2\pi r$ is the measured distance around a circle centered on the attracting mass; hence its name, *reduced circumference.* Measurement of the total circumference $2\pi r$ is the sum of measured distances ($r\, d\phi$) along many small segments of the circle. As a result, the last term on the right, $(r\, d\phi)^2$, remains correct for the Schwarzschild metric.

Timelike form of Schwarzschild metric

What about the time term and the radial term? How will they change near a black hole—or near Earth? The answer is embodied in the **Schwarzschild metric**. For two events close to one another the Schwarzschild metric introduces us to *curved empty spacetime* on a spatial plane through the center of a spherically symmetric (nonspinning) center of gravitational attraction:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2 d\phi^2 \qquad \text{[10. timelike form]}$$

The coordinates r, ϕ, and t appear in this equation. The angle ϕ has the same meaning in Schwarzschild geometry as it does in Euclidean geometry. We have defined r, the *reduced circumference*, so that $rd\phi$ is the incremental distance measured directly along the tangent to the shell. The time t is called **far-away time** and is measured on clocks far away from the center of attraction, as discussed in detail in Section 11.

The timelike Schwarzschild metric is so important that we write it for reference as equation [A] on the last page of this book.

Equation [10] is the timelike form of the Schwarzschild metric, for events in which the time separation predominates. In contrast, the spacelike form, describing a pair of events in which the space separation predominates, equation [10] is replaced by the equation

Spacelike form of Schwarzschild metric

$$d\sigma^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\phi^2 \qquad \text{[11. spacelike form]}$$

Equation [11] is placed for reference on the last page of this book as equation [B].

Sloppy Use of Differentials in Relativity

In going from equation [8] to equation [9], we have begun using differentials *dr, d*ϕ to describe the space separation between events and *dt* for the separation in time. Where did these differentials come from, and why do we suddenly start to use them? The analysis in the box on page 2-18 makes the approximation that the sides *dr* and *rd*ϕ of the little rectangle are *straight.* But the inner and outer sides are *not* straight: each is a portion of a circular arc. The approximation is sensible only if the little arc "looks like" a straight line, only if the angular separation *d*ϕ is a very small fraction of 2π, the angle for a complete circle. Our mathematician friends insist that the approximation is "correct" only in the limit of zero angle. Physicists tend to be a bit sloppy about applying mathematical differentials to nonzero (but still small) separations in real space and time.

But sloppy use of differentials by physicists goes farther than this. Equation [8] is usually written in the even more irresponsible form of equation [9]:

$$d\tau^2 = dt^2 - dr^2 - r^2 d\phi^2 \qquad \text{[9. flat spacetime]}$$

Compare equation [9] with [8]. Equation [9] is squeezed into the compact algebraic notation that by now has become standard. Legalistically it is wrong. On the left should appear $(d\tau)^2$, as it does in equation [8]. If we were credulous enough to take it seriously, $d\tau^2$ would give us not the square of the change in proper time, but rather the crazy idea of the small change in the *square* of proper time.

How did this sloppiness come about? Pure laziness. People got tired of writing down those extra parentheses, left them out, whispered a warning to their friends to write them back in—mentally at least—when putting the metric formula to use, and by now we're all in on the little secret. The same with the terms on the right-hand side, which should read $(dt)^2$, $(dr)^2$, and $(r\,d\phi)^2$, respectively, as they do in equation [8].

In late 1915, within a month of the publication of Einstein's general theory of relativity and just before his own death from battle-induced illness, Karl Schwarzschild (1873 –1916) derived this metric from Einstein's field equations. Einstein wrote to him, "I had not expected that the exact solution to the problem could be formulated. Your analytic treatment of the problem appears to me splendid."

The Schwarzschild metric appears here out of thin air. Where does it come from?

The Schwarzschild metric derives from Einstein's field equations for general relativity, equations that relate the "warping" of spacetime across a spacetime region to the mass and pressure in that region. Different distributions of mass lead to different metrics in the vicinity of the mass. Deriving a metric from the field equations is a major professional accomplishment. Einstein himself did not think it possible that anyone could carry out the task, even for a nonrotating, uncharged, spherically symmetric structure. The metric for a spinning black hole was not published until 1963, almost 50 years later. (See Project F, The Spinning Black Hole.)

Einstein's field equations themselves are not "derived," any more than Newton's laws of motion are derived. Indeed, a Newtonian prediction of the existence of a "horizon," the radius from which only light can escape, is given in the box on page 2-22. The field equations are, as Einstein was fond of saying, "the free invention of the human mind." This invention rests on Einstein's deep intuition for physical reality and symmetry—how Nature *must* behave. Of course the results must lead to correct predictions of experimental results, as they have repeatedly. In this book we start with the metric around each center of attraction. Each of these metrics is one step removed from the (underived) field equations. For a brief account of Einstein's development of the field equations and a description of their content, see Kip Thorne, *Black Holes and Time Warps*, pages 113–120.

The Schwarzschild description is complete.

Further investigation has shown that the Schwarzschild metric gives a *complete* description of spacetime external to a spherically symmetric, nonspinning, uncharged massive body (and everywhere around a black hole but at its central crunch point). *Every* (nonquantum) feature of spacetime around this kind of black hole is described or implied by the Schwarzschild metric. This one expression tells it all! Moreover, the vast majority of experimental tests of general relativity have been tests of the Schwarzschild metric. All test results have been consistent with Einstein's theory.

Different terms for "horizon"

At the radius $r = 2M$ something strange happens to the Schwarzschild metric. The time term goes to zero and the radial term increases without limit as r approaches the value $2M$ in both the timelike and spacelike versions, equations [10] and [11]. This value of r marks the location of the one-way surface through which anything may pass inward but nothing passes outward. This special value of the radial coordinate is given various names: the **Schwarzschild radius** or the **event-horizon radius**. The "membrane" at $r = 2M$ is called the **Schwarzschild surface** or the **Schwarzschild sphere**, the **Schwarzschild horizon**, the **event horizon**, or simply the **horizon**. (*Caution:* Some workers in the field refer to the geometric measure of mass M as the *gravitational radius*. Others reserve this name for $2M$. That is why we avoid the term in this book.)

Ways the Schwarzschild metric makes sense:

For us the Schwarzschild metric—one step from the field equations—is not derived but given. However, we need not accept it uncritically. Here we check off the ways in which it makes sense.

1. Depends only on r-coordinate.

First, the **curvature factor** $(1 - 2M/r)$ that appears in both the dt term and the dr term depends only on the r-coordinate, not on the angle ϕ. How come? Because we are dealing with a spherically symmetric body, an object for which there is no way to tell one side from the other side or the top from the bottom. This impossibility is reflected in the absence of any direction-dependent curvature factor multiplying dt^2 or dr^2.

2. Goes to flat spacetime metric for large *r*.

Second, as the r-coordinate increases without limit, the curvature factor $(1 - 2M/r)$ approaches the value unity, as it must. Why must it? Because an observer far from the center of attraction can carry out experiments in her vicinity without noticing the presence of the distant object at all. For her spacetime is locally flat. In other words, for large r the Schwarzschild metric [10] must go smoothly into the metric for flat spacetime [9].

3. Goes to flat spacetime metric for zero *M*.

Third, as the mass M goes to zero, the curvature factor $(1 - 2M/r)$ approaches the value unity, as it must. Why must it? Because a center of attraction with zero mass is the same as the absence of a massive body at that center, in which case equation [10] becomes equation [9], the expression for the interval in flat spacetime.

4. Confirms *dr* is less than the directly measured distance between shells.

Fourth, consider the factor for dr^2, namely $1/(1 - 2M/r)$. For r > 2M this factor has a value greater than one, which is consistent with our first "experiment in principle" around a black hole (Section 4). The directly measured separation $d\sigma$ is larger than that calculated from the difference dr in r-values between two adjacent Buckminster Fuller shells. Think of a rod held vertical to the shell, spanning the radial separation between two nested spherical shells. Set off two firecrackers, one at each end of this rod, at the same time, $dt = 0$. Take these explosions to be the two events whose separation is described by the metric [11]. The two explosion events have zero separation in azimuth, so $d\phi = 0$. Then the *proper distance* between the explosions is the distance that the shell observer measures directly; call it $dr_{shell} = d\sigma$. The spacelike equation [11] leads to

$$d\sigma = dr_{shell} = \frac{dr}{\left(1 - \frac{2M}{r}\right)^{1/2}} \qquad \text{[12. radial rod at rest on shell]}$$

Here dr is the difference in reduced circumference between two shells. Prior knowledge of the factor $(1 - 2M/r)^{1/2}$ in the denominator was used in describing our first "experiment in principle" that dr_{shell} is greater than

Newton Predicts the Horizon of a Black Hole?

A stone far from a black hole and initially at rest with respect to it begins to move toward the black hole. Gradually the stone picks up speed, finally plunging to the center. With what speed v does this stone pass a spherical shell at radius r? For low velocities and weak gravitational fields the speed is easily derived from Newtonian conservation of energy. In conventional units, the potential energy $V(r)$ of a particle of mass m_{kg} (measured in kilograms) in the gravitational field of a spherical body of mass M_{kg} is given by the expression

$$V(r) = -\frac{GM_{kg}m_{kg}}{r} \qquad \text{[13. Newton]}$$

Here G is the gravitational constant, and the zero of potential energy is taken to be at infinite radial distance r. A particle that starts at that great distance with zero velocity and therefore zero kinetic energy has a total energy zero for all later times and positions r given by the expression

$$E = 0 = \frac{1}{2}m_{kg}v_{conv}^2 - \frac{GM_{kg}m_{kg}}{r} \qquad \text{[14. Newton]}$$

where v_{conv} is velocity measured in the conventional units meters/second. From this equation,

$$v_{conv} = \left(\frac{2GM_{kg}}{r}\right)^{1/2} \qquad \text{[15. Newton]}$$

The particle moves radially inward at this speed. Divide through by c to give speed dr/dt with distance and time in the same units:

$$\frac{v_{conv}}{c} = v = \left(\frac{2GM_{kg}}{c^2 r}\right)^{1/2} \qquad \text{[16. Newton]}$$

But $(G/c^2)\, M_{kg} = M$, the central attracting mass expressed in units of length (equation [5]). The resulting speed is

$$v = \left(\frac{2M}{r}\right)^{1/2} \qquad \text{[17. Newton]}$$

Surprisingly, equation [17] is correct in general relativity too, but only when the speed is interpreted as the speed of the in-falling object *as measured by the shell observer* (Chapter 3, Section 5).

Escape Velocity

Equation [17] provides a prediction for the "radius of a black hole." Think of hurling a stone radially outward from radius r with the speed given by equation [17]. Then Newtonian mechanics, which runs equally well both forward and backward in time, predicts that this stone will coast to rest at a great distance from the center of attraction. Thus equation [17] tells us the **escape velocity**—the minimum velocity needed to escape from the gravitational attraction—for a stone launched outward from radius r. What is the maximum possible escape velocity? Here we elbow Newton aside and give the relativistic answer: The maximum escape velocity is the speed of light, $v = v_{conv}/c = 1$. Place this value in equation [17] to find the minimum radius from which an object can escape—the Newtonian prediction for the radius of the Schwarzschild horizon:

$$r_{horizon} = 2M \qquad \text{[18. Newton]}$$

According to general relativity this is the correct value—provided r is the reduced circumference! The physical interpretation, however, is quite different in the two theories. Newton predicts that a stone launched from the horizon with a speed less than that of light will rise some radial distance, slow, stop before escaping, and fall back. In contrast, Einstein predicts that nothing, not even light, can be successfully launched outward from the horizon (exercise in Chapter 5), and that light launched outward EXACTLY at the horizon will never increase its radial position by so much as a millimeter. (For historical details, see the box on page 3-3.)

dr, as this equation affirms. The change of this factor from place to place implies space curvature.

Equation [12] is so useful that we place it for reference as equation [D] in Selected Formulas at the end of this book.

Fifth, the curvature factor $(1 - 2M/r)$ in the numerator of the dt^2 term also has a value less than one, which is consistent with the gravitational red shift (Section 5). Think of a clock bolted to the shell at radius r. Choose two events to be two sequential ticks of this shell clock. Call dt_{shell} this wrist-watch time $d\tau$ between ticks of the shell clock. Between these two ticks the coordinate separations dr and $d\phi$ are both zero. The timelike equation [10] leads to

5. Confirms gravitational red shift.

$$d\tau = dt_{\text{shell}} = \left(1 - \frac{2M}{r}\right)^{1/2} dt \qquad \text{[19. clock at rest on shell]}$$

Here dt is the corresponding lapse of far-away time. From our second "experiment in principle" we know that the time dt_{shell} between pulses emitted by the clock is smaller at emission than their red-shifted value dt when received at a great distance. In brief, dt_{shell} is less than dt. This result is consistent with the less-than-one value of the curvature factor $(1 - 2M/r)$ in the time term of the Schwarzschild metric (equation [10]).

Equation [19] is placed for reference as equation [C] in Selected Formulas at the end of this book.

The Schwarzschild metric, equation [10], governs the motion of a free test particle external to any spherically symmetric, nonspinning, uncharged massive body. It applies with high precision to slowly spinning objects such as Earth or an ordinary star like our Sun. For the motion of a particle outside such an object, it makes no difference what the coordinates are inside the attracting sphere because the particle never gets there; before it can it collides with the surface of the star—collides with the fluid mass in hydrostatic equilibrium. The more compact the configuration, however, the greater the region of spacetime the test particle can explore. Our Sun's surface is 695 980 kilometers from its center. A white dwarf with the mass of our Sun has a radius of about 5000 kilometers, approximately that of Earth. The Schwarzschild metric describes spacetime geometry in the region external to that radius. A neutron star with the mass of our Sun has a radius of about 10 kilometers, so the test particle can come even closer and still be "outside," that is in a region described correctly by the Schwarzschild metric if the neutron star is not spinning.

The Schwarzschild metric applies only outside the surface of an object.

The ideal limit is not a star in hydrostatic equilibrium. It is a star that has undergone complete gravitational collapse to a black hole. Then the Schwarzschild metric, equation [10], can be applied almost all the way down to zero radius, $r = 0$. The wonderful thing about a black hole is that it has no surface, no structure, and no matter with which one will collide. A test particle can explore *all* of spacetime around a black hole without bumping into the surface—since there is no surface at all.

Black hole has no "surface."

How can a black hole have "no matter with which one will collide"? If it isn't made of matter, what is it made of? What happened to the star or group of stars that collapsed to form the black hole? Basically, how can something have mass without being made of matter?

The mass is all still there, inducing the curvature of adjacent spacetime. It is just crushed into a singularity at the center. How do we know? We don't! It is a prediction from the Schwarzschild metric. Can you verify this prediction? Only if you drop inside the horizon, perform experiments, and make measurements as you approach the singularity—and then neither you nor your reporting signal can make it back out through the horizon. Startling? Crazy? Absurd? Welcome to general relativity!

What is this "singularity" business, anyway? I've heard the term before, but I don't know what it means.

A singularity is a "nobody knows" phenomenon. Coulomb's law of electrical force between point charges has a $1/r^2$ factor in it, which goes to infinity at $r = 0$. But it doesn't really, because there is no such thing as a point charge in structures described by classical (non-quantum) physics. In the Schwarzschild metric the curvature factor $(1 - 2M/r)$ goes to zero at $r = 2M$, leading one term in the metric to blow up. However, it was discovered after long study that this singularity in the metric is due to the choice of coordinates and is not "real." Someone falling inward in free float feels no jolt as she passes $r = 2M$. (More on this smoothness at $r = 2M$ in Chapter 3 and Project B, Inside the Black Hole.) On the other hand, the singularity at $r = 0$ appears to be "real." That is, anything falling to the center of a black hole is crushed to zero volume—to a single point. That is the prediction of general relativity, which is a classical (non-quantum) theory. In contrast, quantum theory predicts that nothing—not even a single electron—can be confined to a point. So what's the truth? The truth is, nobody has figured it out yet! No one has developed a **theory of quantum gravity** that combines quantum mechanics and general relativity. Anyway, Nature has hidden away the singularity inside a one-way surface at $r = 2M$, so we cannot find out while remaining outside. This situation is often described by saying that all real singularities are "clothed," as if there is cosmic censorship. Are there any "naked" (uncensored) singularities not hidden by a one-way surface? None that we yet know about.

One cannot predict the future

If there are non-trivial singularities which are naked, i.e., which can be seen from infinity, we may as well all give up. One cannot predict the future in the presence of a spacetime singularity since the Einstein equations and all the known laws of physics break down there. This does not matter so much if the singularities are all safely hidden inside black holes but if they are not we could be in for a shock every time a star in the galaxy collapsed.

— Stephen Hawking

10 Picturing the Space Part of Schwarzschild Geometry

Freeze time; examine curved space.

How can one visualize the geometry around a black hole? In general relativity, every coordinate system is partial and limited, correctly representing one or another feature of curved spacetime and misrepresenting other features. Figures and diagrams that display these coordinate systems embody the same combination of clarity and distortion.

One partial visualization displays the *spatial* part of the Schwarzschild metric. Freeze time (set $dt = 0$) and limit ourselves to a spatial plane passing through the center of the black hole. Then the spacelike form of the Schwarzschild metric [11] becomes

Visualize the spatial part of the metric.

$$d\sigma^2 = \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\phi^2 \qquad [20.\ dt = 0]$$

Figures 6 and 7 represent this special case. The radius r of each circle is the r-coordinate, the reduced circumference, locating the intersection of a spherical shell with a spatial plane through the center of the black hole. The differential dr is the difference in reduced circumference between adjacent circles. We have *added* the vertical dimension in the diagram and scaled it so that the slanting distance upward and outward along the surface represents $d\sigma$, the proper distance between adjacent circles measured directly with a plumb bob and tape measure. The "funnel" surface resulting from this scaling condition is called a *paraboloid of revolution*, and the heavy curved line in Figure 7 is a *parabola*—actually half a parabola.

Figure 7 embodies the fact that $d\sigma$ is greater than dr, the demonstration in principle that evidenced the curvature of space around a black hole in the first place (Section 4). The figure further shows that the ratio $d\sigma / dr$ increases without limit as the radial coordinate decreases toward the critical value $r = 2M$ (vertical slope of the paraboloid at the throat of the funnel).

These figures embed curved-space geometry in the flat Euclidean three-space geometry perspective shown on the printed page. Therefore these figures are called **embedding diagrams**. But flat Euclidean geometry is *not* curved space geometry. Therefore we expect embedding diagrams to misrepresent curved space in some ways. They lie! For example, the vertical dimension in Figures 6 and 7 is an artificial construct. It is *not* an extra dimension of spacetime. *We* have added this Euclidean three-space dimension to help us visualize Schwarzschild geometry. In the diagram, only the paraboloidal surface represents curved-space geometry. Observers posted on this paraboloidal surface must stay on the surface, not because they are physically limited in any way, but because locations off the surface simply do not exist in spacetime.

"Embedding diagrams" help visualize space curvature.

Observers constrained to the paraboloidal surface cannot measure directly the radius of any circle shown in Figures 6 and 7. They must derive this radius—the reduced circumference—indirectly by measuring the distance around the circle and dividing this circumference by the quantity 2π. From the circumference of an adjacent circle they derive its different radius and calculate the difference dr in the reduced circumferences of the two circles. In contrast, they can directly measure $d\sigma$, the proper distance between these adjacent circles, and compare their result with the computed difference in reduced circumference dr. Result: $d\sigma$ is greater than dr. The ratio $d\sigma / dr$ becomes infinite at $r = 2M$ (where parabola is vertical in diagram).

Observers cannot measure dr directly.

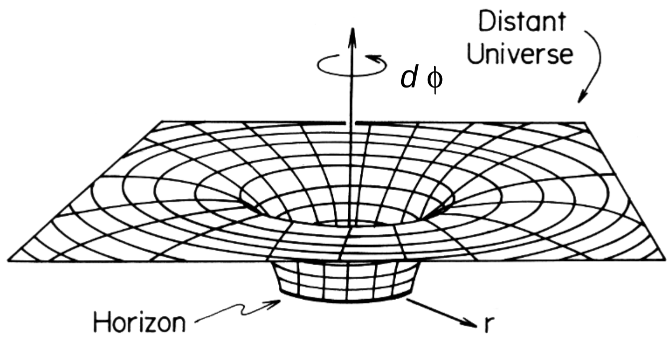

Figure 6 *Space geometry for a plane sliced through the center of a black hole, the result "embedded" in a three-dimensional Euclidean perspective. All of the curvature of empty space (space free of any mass–energy whatsoever) derives from the mass of the black hole. Circles are the intersections of the spherical shells with the slicing plane. WE add the vertical dimension to show that $d\sigma$ is greater than dr in the spatial part of the Schwarzschild metric, as shown more clearly in Figure 7.*

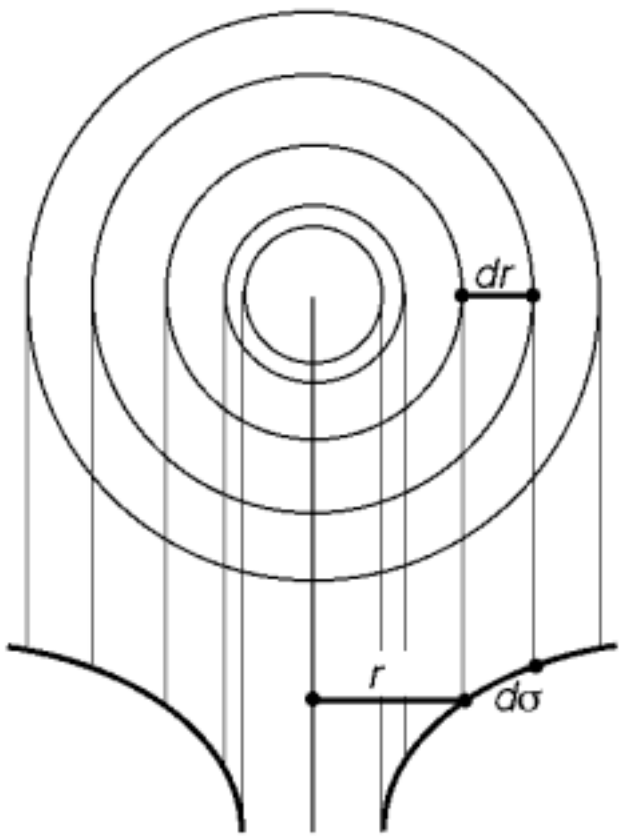

Figure 7 *Projections of the embedding diagram of Figure 6, showing how the directly measured radial distance $d\sigma$ between two adjacent spherical shells is greater than the difference dr in r-coordinates. Real observers exist only on the paraboloidal surface (shown edge-on as the heavy curved line). They can measure $d\sigma$ directly but not r or dr. They derive the r-coordinate (the* reduced circumference) *of a given circle by measuring its circumference and dividing by 2π. Then dr is the computed difference between the reduced circumferences of adjacent circles.*

SAMPLE PROBLEM 1 Limits of Small Curvature

The curvature factor $(1 - 2M/r)$ in the Schwarzschild metric marks the difference between flat and curved spacetime. How far from a center of attraction must we be before this curvature becomes extremely small?

A. As a first example, find the value of the radius r from the center of our Sun ($M \approx 1.5 \times 10^3$ meters) such that the curvature factor differs from the value unity by one part in a million. Compare the value of this radius with the radius of Sun ($r_S \approx 7 \times 10^8$ meters).

B. As a second example, find the radial distance from Sun such that the curvature factor differs from the value unity by one part in 100 million. Compare the value of this radius with the average radius of the orbit of Earth ($r \approx 1.5 \times 10^{11}$ meters).

SOLUTION

A. We want $1 - \frac{2M}{r} \approx 1 - 10^{-6}$, which yields

$$r \approx \frac{2M}{10^{-6}} = 2 \times 1.5 \times 10^3 \times 10^6 \text{ meters} = 3 \times 10^9 \text{ meters} \qquad [21]$$

This radius is approximately four times the radius of Sun.

B. This time we want $1 - \frac{2M}{r} \approx 1 - 10^{-8}$, so

$$r \approx \frac{2M}{10^{-8}} = 2 \times 1.5 \times 10^3 \times 10^8 \text{ meters} = 3 \times 10^{11} \text{ meters} \qquad [22]$$

which is approximately twice the radius of Earth's orbit.

Curvature of spaceTIME is needed to describe orbits.

The embedding diagrams, Figures 6 and 7, represent one cut through the spatial part of the Schwarzschild geometry. Time does not enter, since $dt = 0$. There being no place on this surface for changing time, it depicts nothing moving. Therefore this representation has nothing to tell us directly about the motion of particles and light flashes through the spacetime of Schwarzschild geometry (in spite of all the steel balls you have seen rolling on such surfaces in science museums!). In Chapters 3 and 4 we describe trajectories near a black hole, including trajectories that plunge through the Schwarzschild surface at $r = 2M$ "into" the black hole. But first, Section 11 describes the meaning of "far-away time" t in the Schwarzschild metric.

11 Far-Away Time

Freeze space; examine curved spacetime.

It is not enough to know the geometry of space alone. To know the grip of spacetime that tells planets how to move requires knowing the geometry of spacetime. We have to know not merely the distance between two nearby points, P, Q, in space but the interval between two nearby *events*, A, B, in spacetime.

Far-away time t measured at large r.

The Schwarzschild metric uses what we call **far-away time *t***. There can be many remote clocks recording far-away time t. These remote clocks form a latticework that extends in all directions from the isolated black hole. Far from the influence of the black hole, these clocks are in a region of flat spacetime, so they can be synchronized with one another using light flashes similar to the synchronization pulse for free-float frames described in Chapter 1 (Section 9). However, in the present case the synchronizing

SAMPLE PROBLEM 2 Sample of "Radial Stretching"

Verify the statement at the end of Section 4 that for a black hole of one solar mass, the directly measured radial distance calculates as 1.723 kilometers between a shell at $r = 4$ kilometers and a shell at $r = 5$ kilometers. In Euclidean geometry, this measured distance would be 1 kilometer.

SOLUTION

The mass of Sun to four significant figures is $M = 1.477$ kilometers. Express all masses and distances in kilometers. Use the increments of the Schwarzschild metric to obtain

$$dr_{\text{shell}} = \frac{dr}{\left(1 - \frac{2M}{r}\right)^{1/2}} = \frac{1 \text{ kilometer}}{\left(1 - \frac{2.954 \text{ kilometer}}{r}\right)^{1/2}} \qquad [23]$$

Which radius r do we use in the denominator of the right-hand expression? If we use $r = 4$ kilometers, the result is

$$dr_{\text{shell}} = 1.956 \text{ kilometer} \qquad [24.\ r = 4 \text{ km}]$$

On the other hand, if we use $r = 5$ kilometers, the result is

$$dr_{\text{shell}} = 1.563 \text{ kilometer} \qquad [25.\ r = 5 \text{ km}]$$

The trouble here is that the term $2M/r$ changes significantly over the range $r = 4$ kilometers to $r = 5$ kilometers. The radial stretch factor differs from radius to radius. The results in equations [24] and [25] bracket the answer. An exact calculation requires that we sum all the increments of dr_{shell} from $r_1 = 4$ kilometers to $r_2 = 5$ kilometers. This "summation" is an integration. The result of the integration will be Δr_{shell} between the values 1.563 kilometer and 1.956 kilometer:

$$\Delta r_{\text{shell}} = \int_{r_1}^{r_2} \frac{dr}{\left(1 - \frac{2M}{r}\right)^{1/2}} = \int_{r_1}^{r_2} \frac{r^{1/2} dr}{(r - 2M)^{1/2}} \qquad [26]$$

This integral is not in a common table of integrals. So make the substitution $r = z^2$, from which $dr = 2z dz$. Then the integral and its solution become

$$\int_{z_1}^{z_2} \frac{2z^2 dz}{(z^2 - 2M)^{1/2}} = z(z^2 - 2M)^{1/2} + 2M \ln\left|z + (z^2 - 2M)^{1/2}\right|\Big|_{z_1}^{z_2} \qquad [27]$$

Here ln is the natural logarithm (to the base e) and | | stands for absolute value. Substitute the values (units omitted)

$$\begin{aligned} 2M &= 2.954 \\ z_1 &= \sqrt{4} = 2 \\ z_2 &= \sqrt{5} = 2.236 \end{aligned} \qquad [28]$$

and recall that in general ln(B) – ln(A) = ln(B/A). The result is

$$\Delta r_{\text{shell}} = 1.723 \text{ kilometer} \qquad [29]$$

This value, given at the end of Section 4, lies between the bracketing values in equations [24] and [25] for the fixed choices r_1 and r_2.

pulse or pulses must stay in the remote region, not travel through regions where the value of the curvature factor $(1 - 2M/r)$ differs significantly from the value unity. We call the time *far-away time* as read on these clocks at rest with respect to the attracting body. The technical term is **ephemeris time**. Often we say ***t*-coordinate** and give it the symbol t. The increment dt of far-away time appears on the right side of the Schwarzschild metric.

By definition, the time lapse dt between two events is that recorded on a remote clock by an observer far from the attracting mass.

The relation between dt_{shell} and dt can be read directly from the Schwarzschild metric. Think of an Earth clock mounted in a fixed position on the surface of Earth, which we consider to be nonrotating for purposes of this example. The spatial position of the Earth clock does not change between

Your Own Personal Far-Away Clock

If he wishes, an observer at rest on a spherical shell deep in the gravitational pit of a black hole (but outside the horizon!) can have, in addition to his regular shell clock, a second clock that reads far-away time t directly. To this end he needs to carry out two tasks: (1) adjust the rate at which his personal far-away clock runs and (2) synchronize his personal far-away clock with a remote clock that really is far away.

1. **Rate adjustment.** By turning the fast-slow screw on his personal far-away-time clock, he adjusts it to run fast by the factor $1/(1 - 2M/r)^{1/2}$ compared with his regular proper clock, this factor reckoned using the known mass M of the black hole and his measured reduced circumference r. No such rate adjustment is required by the British resident of New York City who always carries a second wristwatch (far-away-time clock) set to Britain's Greenwich time.

2. **Synchronization.** He synchronizes his personal far-away-time clock by some such procedure as the following: (a) Send radially outward to a remote clock an "inquiring" light pulse requesting the time.
(b) Upon receiving the inquiry, the remote clock immediately sends a reply flash that encodes its time.
(c) When he receives the reply flash, the inner observer assumes that the encoded time is halfway in time between the events of emission of the inquiry flash and reception of the reply flash—and sets his personal far-away-time clock accordingly.

By placing personal far-away clocks on all shells, one can in effect extend the far-away latticework of rods and clocks down to the horizon of a black hole (or down to the surface of a nonrotating star, planet, white dwarf, or neutron star). The far-away time of any event is recorded by these clocks, and the value of the r-coordinate is stamped on every shell.

ticks. Hence dr and $d\phi$ are both equal to zero. Both ticks occur at the clock. Therefore the interval between the ticks on the same clock is the proper time $d\tau$ read on the clock: $dt_{\text{shell}} = d\tau$. Two events that occur at the same place evidently have a timelike relation, so choose the timelike version of the Schwarzschild metric. The result was displayed in equation [19]:

Relation between dt_{shell} and dt

$$dt_{\text{shell}} = \left(1 - \frac{2M}{r}\right)^{1/2} dt \qquad [19]$$

Equation [19] tells us that an observer remote from Earth records a time separation dt between the arrival of the two pulses that is different from the time recorded on the Earth surface clock that emits the two pulses. In the Schwarzschild metric the curvature factor for time is identical with the curvature factor for space except for one circumstance. In the case of space, *divide* by the curvature factor less than one and get outward distances greater than expected from radial coordinates. In the case of time, *multiply* by the same less-than-one curvature factor and get time lapses near a black hole *less* than expected from the readings on far-away clocks.

Hold it! How can the time dt between two events always be the time lapse as recorded "on a remote clock by an observer far from the center of attraction"? What about two events that occur close to the center of attraction? For example, suppose a clock at rest on Earth's surface ticks twice and we on Earth read off the change in clock time. How is the time lapse between these ticks to be recorded by your remote observer?

There are two equivalent ways to determine far-away time lapse between two events occurring on Earth's surface: (1) Compare the reading of a clock on Earth's surface to the reading of a far-away clock by sending a light signal between them. The Earth-surface clock sends a light signal outward with each tick. The two signals are separated by time dt_{shell} as recorded on the shell clock. An observer remote from Earth receives the two signals and times their separation dt using her clock. (Since time *sep-*

aration is being measured, the flight time of the signals cancels out.) (2) Alternatively, have a far-away clock on the shell, as described in the box on page 2-29.

Gravitational red shift of "climbing light"

Instead of using two separate pulses to make the comparison of shell time with far-away time (equation [19]), use a light wave. Every period (every back-and-forth undulation of the wave) of the light emitted by the close-in clock can be considered as a measure of the time dt_{shell} between its ticks. When this signal is received by a remote observer, the period dt is longer, as given in equation [19]. Visible light with a longer period is more red. As described in Section 5, the *gravitational red shift* is named after this 1915 prediction—and 1960 finding —that the remote observer sees light emitted by the close-in clock to be redder than it was at the point of emission.

Gravitational blue shift of "falling light"

If the signal originates farther from the center of gravity and is sent inward toward the center, the received period decreases. The receiver detects a shorter period than the "proper period" of the sender. The light is shifted toward the blue. We call this the **gravitational blue shift**.

Near a black hole such effects are very much greater than they are near Earth. When the light originates at the black hole horizon ($r = 2M$) and is sent outward radially, the square root of the curvature factor, $(1 - 2M/r)^{1/2}$, becomes zero. Far from the black hole the period dt of the received light measured by the far-away observer becomes infinite, no matter how short is the period dt_{shell} of the emitted light measured by the emitting shell observer. But a light signal with an infinite period is no light signal at all! As described earlier, this is the sense in which no light can escape from the horizon of a black hole—and makes the name "black hole" so descriptive.

Come on! A clock is a clock. You say a lot about exchanging signals between clocks, but nothing about the real time recorded on a real clock. Which observer's clock records the REAL time between a pair of events?

We learned in special relativity that there are measured and verified differences in the time between two events as recorded in different frames in uniform relative motion. Similarly, in general relativity there are measured and verified differences in the time between two events as recorded by different observers near a black hole, even when these observers are relatively at rest. In both special and general relativity *you cannot tell by observation* whether these differences are due to the method of exchanging signals or due to clock rates themselves. The phrase "real time" does not have a unique meaning independent of the means by which that time is measured.

Every schoolboy in the streets of Göttingen

Many not close to his work think of Einstein as a man who could only make headway by dint of pages of complicated mathematics. The truth is the direct opposite. As the great mathematician of the time, David Hilbert, put it, "Every schoolboy in the streets of Göttingen understands more about four-dimensional geometry than Einstein. Yet . . . Einstein did the work and not the mathematicians." The amateur grasped the simple central point that had eluded the expert.

—John Archibald Wheeler

SAMPLE PROBLEM 3 Shining Upward

What happens when light emitted from one shell is absorbed at another shell? In particular, let light be emitted from the shell at $r_1 = 4M$ and absorbed at the shell $r_2 = 8M$. By what fraction is the period of this light increased by the gravitational red shift?

SOLUTION

Equation [19] relates the period dt_{shell} of light measured by a shell observer at r to the period dt measured by a remote observer. But we want the period measured by a second shell observer at a different radius. One way to find the period at the second shell is to use equation [19] twice, once for each observer, and make the remote time lapse dt equal in both cases. Ask the remote observer to hold up a mirror that reflects the light from the inner shell back down to the second shell. This procedure must give the same result as direct transmission between the two shells. Use equation [19] twice.

$$\frac{dt_{\text{shell 1}}}{\left(1-\frac{2M}{r_1}\right)^{1/2}} = dt = \frac{dt_{\text{shell 2}}}{\left(1-\frac{2M}{r_2}\right)^{1/2}} \qquad [30]$$

From which

$$\frac{dt_{\text{shell 2}}}{dt_{\text{shell 1}}} = \frac{\left(1-\frac{2M}{r_2}\right)^{1/2}}{\left(1-\frac{2M}{r_1}\right)^{1/2}} \qquad [31]$$

Substitute $r_1 = 4M$ and $r_2 = 8M$ to yield

$$\frac{dt_{\text{shell 2}}}{dt_{\text{shell 1}}} = \frac{\left(1-\frac{1}{4}\right)^{1/2}}{\left(1-\frac{1}{2}\right)^{1/2}} = \frac{0.866}{0.707} = 1.22 \qquad [32]$$

The period of the light is increased (redshifted) by the factor 1.22 as it climbs from $r = 4M$ to $r = 8M$. This factor would shift, for example, spectral yellow light to deep red.

12 Three Coordinate Systems

(1) Free-float, (2) Spherical shell, (3) Schwarzschild bookkeeping.
Live locally in the first two; span spacetime with the third.

Many possible reference frames

Ride in an unpowered satellite as you fall toward a black hole. Or stand on the scaffolding of a spherical shell and observe this satellite up close as it streaks past. Or analyze the satellite motion using the reduced circumference r, angle ϕ, and far-away time t. Each of these observations requires a different set of spacetime coordinates, a different point of view from which to examine and analyze the motion of the satellite and the structure of spacetime around the black hole. Can a person exist in each of these frames, and if so what is this existence like? How do we describe satellite motion in each one of these frames? And how is the description in one frame related to the description in another frame? We conclude this chapter with brief answers to these questions.

Free-float frame

Nowhere could life be simpler or more relaxed than in a free-float frame, such as an unpowered spaceship falling toward a black hole. The speed of this spaceship increases with time as observed by a sequence of shell observers past which it plunges. For those of us who ride inside, however, the spaceship serves as a special-relativity capsule in which we can be oblivious to the presence of the black hole. Up-down, right-left, back-forth: every direction is the same. We observe that keys, coins, and coffee cups remain at rest or if pushed move with constant speed in a straight line. In this free-float frame we use special relativity to compute the spacetime interval between nearby events and analyze collisions as if we were in interstellar space devoid of gravity.

Free-float frame is only local.

However, the simplicity of our free-float frame is only local. We detect curvature of spacetime by the tide-producing relative accelerations among two or more free test particles situated far enough apart or observed for a long enough time to reveal the nonflat nature of nonlocal spacetime. Tidal accelerations drive toward one another objects that lie separated along some directions in the free-float frame; tidal accelerations drive apart particles that lie separated along another direction (Figure 4 of Chapter 1). Detecting the presence of tidal accelerations identifies our reference frame as not free float. To make these tidal accelerations undetectable—by instruments of given sensitivity—we either narrow the spatial extent of our free-float frames or limit the time duration of any particular experiment or both. These constraints are minor in free-float frames situated near Earth or far from any gravitating body. They become progressively and inexorably confining as we approach the center of a black hole. Near this center, general relativity predicts that tidal accelerations tear apart every physical object.

The free-float frame is familiar, simple, and universal. It is the only one of our three frames in which humans can exist near a black hole. Even a body made of steel would be crushed by the "gravitational force" while standing on a spherical shell near the horizon. In contrast, for a large enough black hole the tidal forces can be tolerated by the human body even inside the horizon, at least at a sufficiently great distance from the central singularity. (See Project B, Inside the Black Hole.)

The spherical shell

Earth's surface is a "spherical shell."

We live on a (nearly) spherical shell: the surface of Earth. This shell is our home. We habitually construct latticeworks—called buildings—with mutually perpendicular axes on which we mount synchronized clocks. To sit or stand on our spherical shell—Earth's surface—forces us away from the natural motion of a free particle. This departure from natural motion we experience as a "force of gravity" pointing toward the center of Earth. In everyday life we simply include this "force" with other forces in order to get on with the practical analysis of events around us. This approximation works admirably well for the small space and time regions of everyday experience. It works also for high-speed particle interactions in a laboratory, which are over so quickly that the "force of gravity" has little effect. For such experiments the Earth frame is effectively free-float, and analysis using special relativity gives good results even for observations from our shell frame. (See Chapter 1, Section 8.)

Shell observer uses special relativity.

Is special relativity sufficient for the shell observer? Yes, at least locally in space and time. This conclusion is supported by the form of the metric for a shell observer. Substitute into the Schwarzschild metric (equation [10]) the expression for $dr_{\text{shell}}{}^2$ from equation [12] and the expression for $dt_{\text{shell}}{}^2$ from equation [19]. The result is

The Metric as Micrometer

Figure 8 *A micrometer caliper, used to measure small distances, such as the thickness of metal sheet. A calibrated screw on the right meters the gap between cylinders at the left.*

What *is* the metric? What is it *good* for? Think of a **micrometer caliper** (Figure 8), a device used by metalworkers and other practical workers to measure small distances. The worker *owns* the caliper and *chooses* which distance to measure.

The metric is a "four-dimensional micrometer" for measuring the small spacetime separation between a chosen pair of events. You *own* the metric. You *choose* the events whose separation you wish to measure with the metric. The "metric micrometer" translates bookkeeper coordinate increments dr, $d\phi$, and dt into proper time $d\tau$ or proper distance $d\sigma$ between the pair of events *you* choose.

I. *One possible choice for two events*: Two sequential ticks of a clock bolted to a spherical shell. Then $dr = d\phi = 0$ and the proper time lapse $d\tau$ is that read directly on the shell clock, dt_{shell}. The result is equation [19], page 2-23, for the relation between shell time and far-away time.

II. *A second possible choice of two events:* Events at the two ends of a stick held at rest radially between two adjacent shells. Choose $dt = d\phi = 0$, then the proper distance is the directly measured length of the stick dr_{shell}. The result is equation [12], page 2-22, for the relation between directly measured distance between shells and their radial separation dr in the Schwarzschild coordinate.

III. *A third possible choice:* Two ticks on the wristwatch of a particle in free fall inward along a radius. Then $d\phi = 0$ and the proper time is read directly on this wristwatch, which leads to several important results in Chapter 3.

And so on. There is an infinite set of event pairs near one another that you can choose for measurement using your four-dimensional micrometer—the metric.

What advice will the "old spacetime machinist" give to her younger colleague about the practical use of the metric micrometer? She might say the following:

1. Think *events* and separations between pairs of events, not fuzzy concepts like "length" or "location."

2. Do not confuse results from one pair of events with results from another pair of events.

3. Whenever possible, choose the pair of events so that the differential of one or more coordinates is zero.

4. Whenever possible, identify the proper time or proper distance with someone's direct measurement.

5. When a light flash can move directly from one event to another event in the time between them, then the proper time between the events is zero: $d\tau = 0$.

$$d\tau^2 = dt_{\text{shell}}^2 - dr_{\text{shell}}^2 - r^2 d\phi^2 \qquad [33]$$

Shell frame is also local.

The right side of equation [33] contains only coordinate increments measured directly by the shell observer. (Recall that radius r is *defined* so that $rd\phi$ is the directly measured distance along the surface of the shell—Section 4 and the box on page 2-18.) The metric [33] *looks* like that of flat spacetime. But spacetime is *not* flat on a shell near a black hole, and this limits the usefulness of equation [33] to *local* measurements. After all, dt_{shell} and dr_{shell}, along with $rd\phi$, are all functions of the radius r. Still, equation [33] has its uses. For example, it implies that the shell observer measures light ($d\tau = 0$) to have the speed unity—most easily seen by substituting the incremental distance squared $ds_{\text{shell}}{}^2$ (with a minus sign) for the last two terms in on the right of [33]. Of course every experiment that takes place on the shell is influenced by the apparent "gravitational force" due to the fact that the shell is not a free-float frame.

Free-float and shell observers use special relativity to exchange data.

Except for this gravitational force, the **local shell frame** implied by equation [33] is similar to the local free-float frame of a passing plunging observer. In both frames the local speed of light is unity and special relativity correctly describes brief, local experiments. Moreover, the shell observer and a passing free-float observer can use the Lorentz transformation of special relativity to exchange data on local events that are close together in spacetime. We shall use this ability to exchange data many times in the chapters and projects that follow.

In two important ways the local free-float frame is more general than the local shell frame: (1) Special relativity can be made to work well for a longer time in a free-float frame by making the spatial extent smaller, whereas on the shell direct effects of the gravitational force cannot be reduced by any such ruse. (2) The free-float observer can cross the horizon and continue her experiments without interruption, at least until tidal forces overwhelm her. In contrast, inside the horizon neither shells nor shell observers can exist, and equation [33] is useless there.

Different rates for clocks separated vertically

Less familiar than "gravitational" effects to most Earth inhabitants is the difference in rates between clocks separated vertically in the gravitational field, an effect that is a daily experience for anyone designing or predicting the performance of the Global Positioning System, which uses atomic clocks in satellites (see Project A, Global Positioning System).

Vertical separations are affected by curvature.

Least familiar of all effects of general relativity for the shell observer is the difference between the radius of Earth, directly measurable in principle, and the reduced circumference obtained by dividing the circumference by 2π. For two concentric spherical shells near a black hole, the directly measured radial distance between them is greater than the difference in r-values of their reduced circumferences.

These effects in clock rates and vertical separations limit the region of spacetime—space *and* time—in which to analyze experiments using special relativity expressed in shell coordinates. Now we move to a set of coordinates that are global in extent but farther removed from the reality of most experiments carried out near Earth, Sun, or black hole.

Bookkeeper coordinates r, ϕ, and t

Schwarzschild coordinates span full region of spacetime.

A free-float observer makes observations that span only a little patch of spacetime. A local shell observer has similar limitations. In contrast, the coordinates r, ϕ, and t, called **Schwarzschild coordinates**, satisfy the need for a global description of events, a description that encompasses, for example, two events located so far apart in space that they lie on opposite sides of the black hole. The reduced circumference r, the azimuthal angle ϕ, and the far-away time t span all of spacetime surrounding a black hole. They report measurements made in a distant frame at rest with respect to the center of gravitational attraction. Latin was the international language of medieval Europe; the coordinates r, ϕ, and t form the international language for describing events that take place near a black hole.

Schwarzschild observer is a bookkeeper.

The **Schwarzschild observer** is a bookkeeper, an archivist, a top-level accountant who rarely measures anything herself. Instead she spends her time examining reports from local shell and free-float observers and combining them to describe events that span spacetime around a black hole. Before accepting a report, this perfectionist demands that coordinate separations between events described in the report be translated into her language: increments in reduced circumference r, azimuthal angle ϕ, and far-away time t. Therefore we call her the **Schwarzschild bookkeeper**.

An orbiting satellite rapidly emits two sequential flashes as it streaks past two shells concentric to a black hole. The local shell observer measures directly the small separations between the emissions of these flashes: time separation dt_{shell} measured by nearby clocks bolted to the shell and vertical separation dr_{shell} measured with a tape measure. The shell observer also measures the change in azimuthal angle $d\phi$ in the plane of the orbit and verifies by direct meterstick measurement that this increment of angle corresponds to the tangential separation $rd\phi$, where r is the reduced circumference stamped on every shell by the original builders. The shell observer converts dt_{shell} to dt using equation [19] and dr_{shell} to dr using equation [12]—both special cases of the Schwarzschild metric. The shell observer then reports the resulting separations dt, dr, and $d\phi$ to the Schwarzschild bookkeeper.

Schwarzschild bookkeeper traces out orbit.

Now the Schwarzschild bookkeeper swings into action. She knows the space and time coordinates r, ϕ, and t at the beginning of this increment of time. To these coordinates she adds increments dr and $d\phi$ for each lapse of far-away time dt reported by a local shell observer. The result is a table, a diagram, or what we call a **Schwarzschild map** that traces the satellite through spacetime as expressed in her coordinates r, ϕ, and t. Such a map is shown in Figure 9.

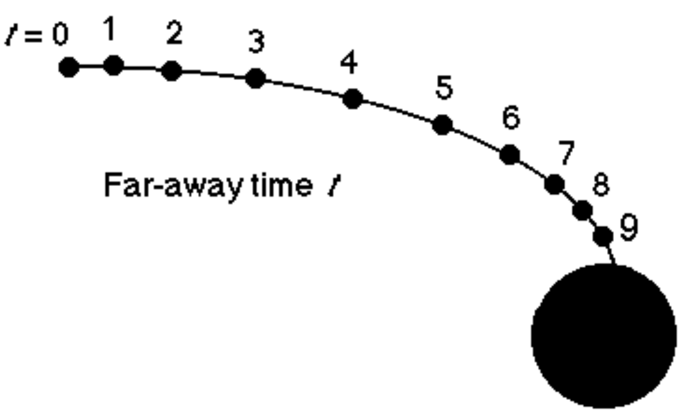

Figure 9 *Schematic Schwarzschild map of the trajectory of an object that plunges into a black hole. Only every hundredth flash is numbered and shown; adjacent flashes are very close together in space and time so they can be observed directly by one or another free-float observer or shell observer. NO ONE observes directly the trajectory shown on this map. Question: Why are numbered event dots closer together near both ends of the trajectory than in the middle of the trajectory? (Answer in Chapter 3.)*

Schwarzschild map is a bookkeeping device.

Notice that the Schwarzschild map of Figure 9 is a summary, an artifact, a bookkeeping device. It indicates events each of which is recorded locally.

Nobody observes this entire trajectory directly. The price paid for the universal language of r-coordinate and t-coordinate is the loss of direct experience. No one lives in or on a road map, but we use road maps to describe the territory and plan our trips. Similarly, coordinates r, ϕ, and t are calibrations on a Schwarzschild map of spacetime. These coordinates simply and precisely locate events in the entire spacetime region outside the surface of any spherically symmetric gravitating body. The Schwarzschild map guides our navigation near a black hole.

What if an astronaut riding in the satellite wants to transmit to the Schwarzschild bookkeeper data about separation between events she observes inside her unpowered spaceship? She begins by using special relativity to transform her coordinate separations to values on the passing shell. Then the shell observer can transmit the results to the far-away bookkeeper, as before.

r, ϕ, t are bookkeeping coordinates.

In summary, the full range of coordinates r, ϕ, and t are primarily for bookkeeping. A computer can replace the bookkeeper. Then nobody lives in the coordinates r, ϕ, and t, nobody works there, nobody takes data directly using the wide span of these three coordinates. They form an accounting system, a bookkeeping device, a data bank, a spreadsheet, a tabulating mechanism, an international language, the basis for a spacetime map that describes events and motions in the entire spacetime region surrounding Earth, Moon, Sun, or black hole. For this reason, we often call r, ϕ, and t **bookkeeper coordinates**. The strength of bookkeeper coordinates is universality; their weakness is isolation of most data entries from direct experience.

People can live on a shell and in a free-float frame.

In contrast, people can, in principle, live and work in free-float frames and on spherical shells, taking and analyzing data as if they were in flat spacetime, but unfortunately they can do so only for limited patches of spacetime.

Don't tell me I cannot experience directly the entire trajectory shown in Figure 9! Station me a great distance from a black hole. Then I can view the satellite directly with my eyes as it orbits the black hole or plunges toward it.

True, you see the entire orbit—at least until the satellite reaches the horizon of the black hole. But what you see by eye are not the coordinates r, ϕ, and t of this trajectory. First of all, there is a time delay between emission of a flash by the satellite and the instant at which you see this flash with your eye. The relative delay increases as the satellite moves farther from you or deeper into the gravitational pit.

Second, there is an effect we have not yet mentioned (see Figure 10): Light is deflected in a gravitational field. In 1919 Arthur Eddington verified Einstein's prediction for starlight passing Sun, making Einstein an instant worldwide celebrity. The tiny deflection near Sun becomes dramatic near a black hole (Chapter 5 and Project D, Einstein Rings). As a result, you typically do not see the satellite where it was but in some other direction.

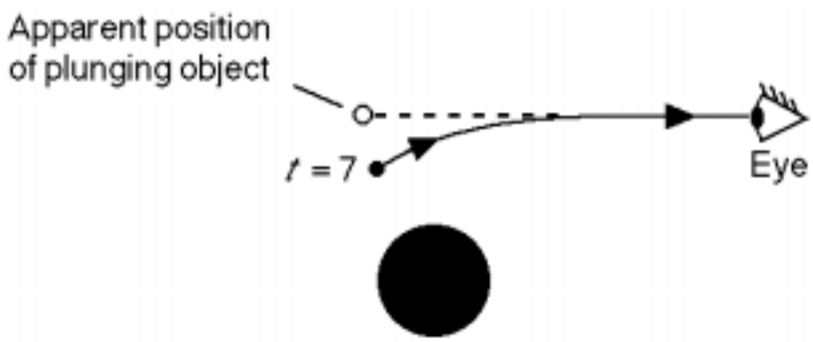

Figure 10 *Schwarzschild map of trajectory of light flash (solid curve) emitted at far-away time t = 7 from the plunging object (solid dot) whose trajectory is shown in Figure 9. The light is deflected as it moves outward, leading a remote observer to see the flash emission at a different location (dashed line and open dot).*

Visual appearance is distorted.

Visual appearance can be misleading near a black hole. This misinformation of visual appearance is not new in principle. When we set up the original latticework of rods and clocks that formed the reference frame of special relativity, we limited the observer to collecting data from the recording clocks (Chapter 1, Section 9). We expressly warned the observer about reporting events that he views by eye. Why warn him? Because the speed of light is finite. Light from a distant event can arrive at the observer's eye long after light from a nearer event that actually occurred later as recorded on the latticework of clocks. It is not easy to analyze events when their order is scrambled in the process of observation and recording.

Similar light-delay problems occur in viewing by eye objects in orbit around a black hole. Added to light delay is the visual misinformation about direction due to the deflection of light that results from the curvature of spacetime (Figure 10).

With knowledge of how light moves in the neighborhood of a black hole (Chapter 5), you may be able to reconstruct the Schwarzschild map from your visual observations. But such a reconstruction is quite different from seeing the Schwarzschild map directly.

Schwarzschild metric provides a universal language to fix the location of events.

In the following chapters we use bookkeeper coordinates r, ϕ, and t to describe and predict orbits of satellites and the trajectories of light flashes near a black hole. We use these coordinates to draw Schwarzschild maps of the trajectories. Behind the Schwarzschild map of any orbit stand observations made by free-float observers or shell observers, or predictions of their observations. The Schwarzschild metric is central in the translation of coordinates back and forth between direct observers (shell observers and free-float observers) and between each of these observers and the Schwarzschild bookkeeper.

Schwarzschild lattice

Using what we have learned about spacetime near a nonrotating, spherically symmetric massive body, we can in principle set up a **Schwarzschild lattice**, at rest with respect to the center of gravitational attraction, from which one can read directly the Schwarzschild coordinates r, ϕ, t of an event right down to the horizon. The value of the reduced circumference r is stamped on every spherical shell. Scatter over each shell a set of clocks that read far-away time (box, page 2-29). For a given plane in which a particle orbits, mark the angle ϕ on the shell, with zero angle in some chosen direction. The shell observer and the free-float observer near an event can then read the Schwarzschild coordinates r and ϕ of this event directly from the place on the shell at which the event occurs and the time t on the far-away clock mounted on the shell next to that event.

Construct Schwarzschild lattice from horizon outward to infinity.

This Schwarzschild lattice can stretch from near the horizon outward indefinitely in every direction (from an isolated body). Then the remote Schwarzschild bookkeeper receives, records, and manipulates Schwarzschild coordinates without the need for translation from shell or free-float coordinates. Everyone involved understands the Schwarzschild metric and its predictions, leading to a tidy, agreed-upon system that describes the location of all events. This collection of shells and clocks can then be called the “Schwarzschild observer.”

I understand the idea of the three different coordinate systems described in this section, but I don’t understand why we can’t just use the Schwarzschild lattice alone. It provides time and space measures that each experimenter can agree to by looking at a nearby Schwarzschild clock, measuring tangential distances directly, and reckoning radial separations by subtracting the r-values stamped on each shell. Then we need no translating measurements from one coordinate system to another. The Schwarzschild lattice works fine all by itself. Get rid of all other coordinate systems!

Good point. No one can stop us from using Schwarzschild coordinates alone to design our experiments and predict results. And these predictions will describe what we observe. Then everything is totally consistent and convenient for experimenters scattered throughout the entire region outside the horizon of a black hole. There is a price for this convenience, however; are you willing to pay the following price? Our standard of time is based on the properties of particular atoms. Near a black hole (and near Earth!) an atomic clock “runs slow” when measured using the far-away Schwarzschild time coordinate. And the directly measured radial distance between shells is greater than the difference in r-values between these shells (while directly measured tangential distances are indeed correctly predicted by change in Schwarzschild coordinates). Does curved spacetime cause measuring rods to seem “rubbery,” having different apparent lengths when oriented tangentially than when oriented radially? Does curved spacetime force “atomic time” to run at a different rate near a center of attraction than far from this center? Typically, such questions about “reality” are of no interest to people in the field. Whatever point of view leads to correct predictions is fine with them! And using the Schwarzschild lattice, with its seemingly rubbery measuring rods and time-changed atoms, leads to correct predictions. On the other hand, we may be more comfortable assuming that an atomic clock runs at its regular rate when observed at rest near a black hole. In this case we naturally adopt the shell frame for local observations, with its local measuring rods and atom-defined clock times. Each alternative reference frame has its own advantages and brings different perspectives to the structure of spacetime. In our opinion, life as a general relativist, however long, is more fun when you learn to jump mentally from frame to frame! For more, see Kip Thorne’s *Black Holes and Time Warps,* Chapter 11, What Is Reality?

13 Summary

The Schwarzschild metric

In general, the metric provides a complete description of spacetime: the curvature of spacetime and the results of measurements carried out with rods and clocks. The metric for flat spacetime is the one that dominated our study of special relativity. However, special relativity cannot describe spacetime globally in the vicinity of a massive object. General relativity can do so, earning the name Theory of Gravitation.

The Schwarzschild metric describes spacetime exterior to the surface of any nonrotating, uncharged, spherically symmetric massive object. It describes spacetime everywhere around a nonrotating, uncharged black hole.

Several conventions make the Schwarzschild metric easy to understand and use:

1. Satellite motion in a plane. A light flash or test particle that moves through Schwarzschild geometry stays in a single spatial plane that passes through the center of the black hole. Describing motion on this plane requires only two space dimensions plus the time.

2. Polar coordinates. Motion with respect to a center is simply described using polar coordinates r and ϕ. For example, the metric for flat spacetime with two spatial dimensions goes from the Cartesian form

$$d\tau^2 = dt^2 - dx^2 - dy^2 \qquad \text{[34. flat spacetime]}$$

to the polar form

$$d\tau^2 = dt^2 - dr^2 - r^2 d\phi^2 \qquad \text{[9. flat spacetime]}$$

3. Mass in units of meters. We measure the mass M of a planet, star, or black hole in units of meters. Equation [5] makes the conversion from mass M_{kg} in kilograms to mass M in meters, using G, the gravitational constant of Newtonian mechanics and c, the speed of light:

$$M = \frac{G}{c^2} M_{\text{kg}} = \left(7.424 \times 10^{-28} \frac{\text{meter}}{\text{kilogram}}\right) M_{\text{kg}} \qquad [5]$$

In length units, the mass of Sun is 1.477 kilometers and the mass of Earth is 0.444 centimeters.

4. Radius as reduced circumference. The presence of the black hole renders impossible the direct measurement of the radial coordinate r of an object or satellite. Instead, define the radius as $r = (\text{circumference})/2\pi$, where the circumference is measured around the great circle of a stationary spherical shell concentric to the black hole or center of attraction. As a reminder of this process, we often call r the *reduced circumference*.

5. Time *t* measured on far-away clocks. To avoid the effects of curvature on clocks, calculate the time, called *bookkeeper time* or *far-away time*, that would be measured on clocks located in flat spacetime far from the attracting body. Give far-away time the symbol t. Light flashes are used for comparison of clock rates and also for communication between a far-away clock and a clock in curved regions of spacetime.

Predictions from the Schwarzschild metric

With these simplifying conventions the Schwarzschild metric in its timelike form can be written

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2 d\phi^2 \qquad [10]$$

This metric "measures" the separation of a pair of events that have a timelike relation and that occur near one another in spacetime. Various choices of these two events lead to predictions verified by experiment:

Prediction 1. Gravitational red shift. Let the two events be sequential ticks of a clock at rest on a spherical shell near a black hole. *At rest* means that the space separation between events is zero: $dr = d\phi = 0$. The proper time $d\tau$ (defined as the time between the events in a frame in which they occur at the same place) is just the time dt_{shell} read on the shell clock. Then the Schwarzschild metric tells us the relation between shell-time lapse and the lapse of far-away time:

$$dt_{\text{shell}} = \left(1 - \frac{2M}{r}\right)^{1/2} dt \qquad [19]$$

Instead of describing ticks on a clock, this equation can measure the period of a steady light wave emitted outward from a spherical shell at radius r. The equation predicts that the period dt measured by a remote observer is *greater* than the period dt_{shell} measured by the observer at the emitting clock. For visible light, longer period means redder light, so the general name for this effect is the *gravitational red shift*.

Prediction 2. Curvature of space. Let the two events occur at the ends of a measuring rod radially oriented with ends at two concentric spherical shells. And let these two events occur at the same far-away time. To analyze these two spacelike events, use the spacelike form of the Schwarzschild metric:

$$d\sigma^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\phi^2 \qquad [11]$$

For this example, $dt = d\phi = 0$, and the proper distance $d\sigma$ between them (defined as the separation between two events in a frame in which they occur at the same time) is just the radial separation measured by a shell observer:

$$dr_{\text{shell}} = \frac{dr}{\left(1 - \frac{2M}{r}\right)^{1/2}} \qquad [12]$$

The shell observer measures the distance dr_{shell} between shells to be greater than the difference dr between the reduced circumferences of the two shells.

Reference frames

General relativity allows use of any coordinate system whatsoever. We choose three coordinate systems convenient for our purposes: local free-float frames, local frames on spherical shells, and the global frame that employs Schwarzschild coordinates r, ϕ, t. Observers can take measurements directly in free-float frames and on spherical shells, but these measurements are local. In contrast, Schwarzschild coordinates describe events that can span all of spacetime near a massive body, but no one observer can make these measurements directly. Instead we speak of the *Schwarzschild bookkeeper* who records and analyzes events measured by others.

Shell observers and passing free-float observers compare their local measurements using special relativity, including the Lorentz transformation. Shell observers and the Schwarzschild bookkeeper compare their measurements using equations [12] and [19]. The tangential distance $rd\phi$ is the same in both systems.

One can construct in imagination a *Schwarzschild lattice* of spherical shells, each stamped with the reduced circumference r, angle ϕ, and covered with clocks reading far-away time t. The Schwarzschild lattice can in principle start near the horizon and extend outward indefinitely (from an isolated body). The Schwarzschild coordinates of any event outside the horizon can then be read directly using this lattice. This collection of shells and clocks can collectively be called the "Schwarzschild observer."

Black holes just didn't "smell right"

During the 1920s and into the 1930s, the world's most renowned experts on general relativity were Albert Einstein and the British astrophysicist Arthur Eddington. Others understood relativity, but Einstein and Eddington set the intellectual tone of the subject. And, while a few others were willing to take black holes seriously, Einstein and Eddington were not. Black holes just didn't "smell right"; they were outrageously bizarre; they violated Einstein's and Eddington's intuitions about how our Universe ought to behave. . . . We are so accustomed to the idea of black holes today that it is hard not to ask, "How could Einstein be so dumb? How could he leave out the very thing, implosion, that makes black holes?" Such a reaction displays our ignorance of the mindset of nearly everybody in the 1920s and 1930s. . . . Nobody realized that a sufficiently compact object must implode, and that the implosion will produce a black hole.

—Kip Thorne

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Woodcut of Copernicus, page 2-8, by Tobias Stimmer taken from "The Universe as Home for Man" by John Archibald Wheeler, in T*he Nature of Scientific Discovery*, edited by Owen Gingerich, Smithsonian Institution Press, Washington, D.C., 1975. Caption from *Dictionary of Scientific Biography*, edited by Charles Gillispie, Scribner's, New York, 1971. The order of sentences has been rearranged.

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