The Principle of Least Action

Based on *part* of Chapter 19, Volume II of *The Feynman Lectures on Physics*
Addison-Wesley, 1964: pages 19-1 thru 19-3 & 19-8 thru 19-9.

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July 2000

0. The Action Software

The set of exercises on Action will combine hand calculations with use of a
computer program called **ACTION** that allows more rapid and graphical analysis.
An Appendix to this unit has a set of introductory exercises for the Action program
(for the case of zero potential -- no gravitational field) to help you get used to its
various features. You will probably find it helpful to go through these appendix
exercises (with the prefix Z) before beginning the exercises in the body of this unit
(with the prefix Q).

I. Introduction

Almost all of the nonrelativistic *classical* mechanics of single particle motion can be
derived from the Principle of Least Action. In particular, the Principle of Least
Action predicts particle motion in the presence of a potential, such as that due to a
gravitational field. And the Principle of Least Action lends itself to the transition
from nonrelativistic classical mechanics to nonrelativistic quantum mechanics.

In this exercise we apply the *classical* Principle of Least Action to a much simplified
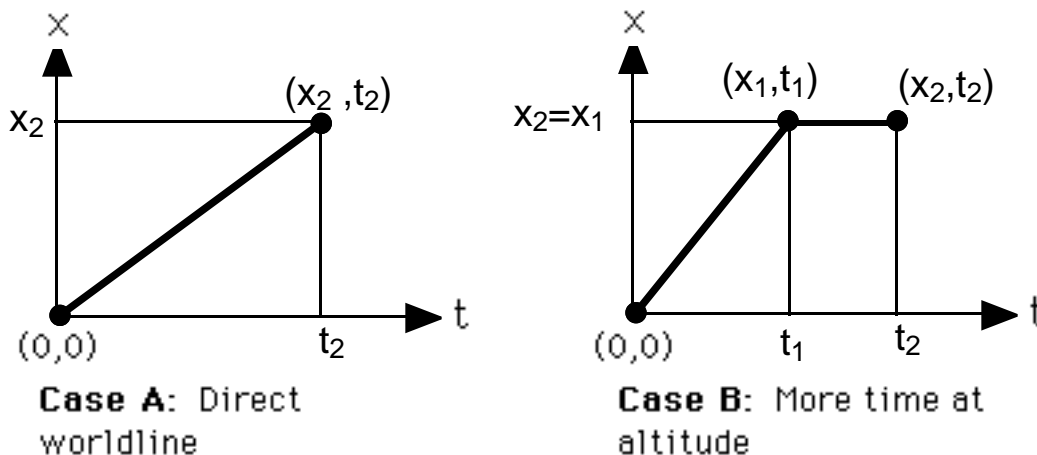analysis of the motion of a one-kilogram stone moving vertically upward in a
gravitational field.

Figure 1. Two possible alternative worldlines between the events (0,0) and (x_2,t_2) that
occur at different heights x in a gravitational field.

Figure 1 shows a diagram of two alternative paths of a stone in a gravitational field that acts vertically downward. There are several important things to say about these motions:

1. The motion in *space* is one-dimensional; the stone moves vertically upward. Feynman uses *x* as the vertical space coordinate. Most of us would ordinarily use the variable *y* for this position.

2. The diagrams in Figure 1 are *spacetime* diagrams, with time plotted along the horizontal axis. The path of a stone plotted on a spacetime diagram is called a ***worldline***. A worldline is NOT a path in space, but rather a plot of location as a function of time. The worldline tells much more about motion than a picture showing the trajectory in space alone. In particular, the slope of the worldline tells us the *velocity* of the stone at every point in its motion. On each straight segment of the worldlines of Figure 1, the stone is moving with constant velocity and therefore with constant kinetic energy.

3. A point on a spacetime diagram gives us not only the location of the stone in space but also the time at which it reaches that location. Such a spacetime point is called an ***event.***

4. Descriptions of the two motions represented in Figure 1 are as follows:

  Case A: The stone starts at the origin and rises with constant velocity to height x_2.

  Case B: The stone starts at the origin and rises with constant velocity greater than that of Case A, arriving at height $x_1 = x_2$ sooner, where it stops at that height for an additional time.

5. Neither of these motions is the natural motion of a stone as it flies in a gravitational field. But it is not hard to calculate the Action for these unnatural paths, so we use them to inform ourselves.

6. Typically, energy is *not* conserved along candidate worldlines shown in Figure 1. In particular, a stone rising vertically with constant velocity has a constant kinetic energy but an increasing potential energy. Therefore its total energy is increasing during that portion of the worldline. Only for the correct, "natural" path followed by a rising stone is energy conserved. The Action program will help us find the correct "natural" path that exhibits conservation of energy.

II. The Action for Case A
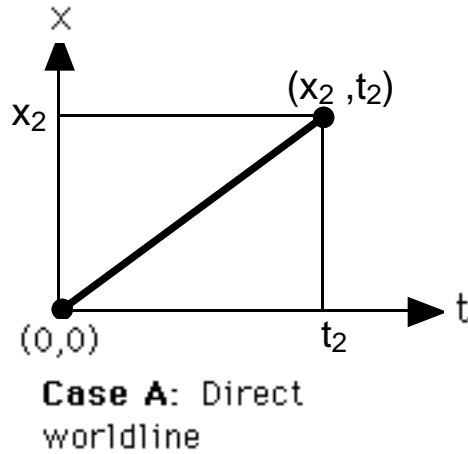We find an expression for the Action in Case A.

Figure 2: Direct worldline starting at origin.

The formal definition of the *action* is given on page 19-3 of the Feynman reading:

$$\text{Action} = S = \int_{t_1}^{t_2}(KE - PE)\,dt = \int_{t_1}^{t_2}KE\,dt - \int_{t_1}^{t_2}PE\,dt \tag{1}$$

where t_1 and t_2 are the times of the initial and final events. For Case A, time $t_1 = 0$:

$$\text{Action} = S = \int_{0}^{t_2}(KE - PE)\,dt = \int_{0}^{t_2}KE\,dt - \int_{0}^{t_2}PE\,dt \tag{2}$$

These integrals, like all integrals, are an indicator that something is being *summed.* For the first integral on the right side of equation (2), the sum adds up a lot of small contributions *KE dt.* Multiply the kinetic energy *KE* during each short interval of time by that interval of time *dt* and add up all of these little products. The result is an approximation of the integral:

$$\int_{0}^{t_2}KE\,dt \approx (KE_a dt_a + KE_b dt_b + KE_c dt_c + ...) \tag{3}$$

Case A shows a straight segment of worldline with a constant *slope,* which means a constant *velocity* along that segment. This means also a constant *kinetic energy* along this straight segment. So along this segment of worldline the constant kinetic energy, call it *KE,* can be factored out of the bracket on the right side of equation (2).

What is left is just the sum of all the little time elements between $t = 0$ and $t = t_2$.

This sum is just equal to t_2:

$$\int_0^{t_2} KE\,dt \approx KE\left(dt_a + dt_b + dt_c + ...\right) = KE\,t_2 \qquad \text{(straight worldline)} \qquad (4)$$

This gives an exact expression for the first integral on the right side of equation (2) for the case of a straight worldline.

The potential energy contribution to the Action, the second integral on the right side of equation (2), is almost as simple, but not quite. The potential energy has the form:

$$PE = mgx \qquad (5)$$

where m is the mass of the stone, g is the acceleration of gravity, and x is the height of the stone above the ground. The PE integral in equation (2) corresponds to the summation:

$$\int_0^{t_2} PE\,dt \approx mg\left(x_a dt_a + x_b dt_b + x_c dt_c + ...\right) \qquad (6)$$

No one can stop us from making all increments of time dt equal:

$$\int_0^{t_2} PE\,dt \approx mg\left(x_a dt + x_b dt + x_c dt + ...\right) = mg\left(x_a + x_b + x_c + ...\right)dt \qquad (7)$$

Now suppose the time interval t_2 is divided into N equal segments of size dt. Then we can write

$$dt = \frac{t_2}{N} \qquad (8)$$

Substitute this into equation (7) and perform a sleight of hand by placing N under the sum of x-terms:

$$\int_0^{t_2} PE\,dt \approx mg\left[\frac{x_a + x_b + x_c + ...}{N}\right]t_2 \qquad (9)$$

The expression in the square brackets has a simple interpretation. The numerator is the sum of the values of x at N equally-spaced locations along the worldline segment. Dividing by N gives a result that approximates the *average value* of x over this segment. But this segment is a straight line, so the average value of x is just its value at the midpoint of the segment:

$$\left[\frac{x_a + x_b + x_c + ...}{N}\right] = \frac{x_2}{2} \qquad (10)$$

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With this substitution, equation (9) becomes:

$$\int_{0}^{t_2} PE \, dt \approx mg\frac{x_2}{2}t_2 = PE_{avg}\,t_2 \tag{11}$$

where PE_{avg} is the average of the potential energy over the worldline of Case A. Again, equation (11) is exact for a straight worldline. So for this straight segment of worldline from (0,0) to (x_2, t_2), the Action has the value:

$$\text{Action} = S = \int_{0}^{t_2}(KE - PE)\,dt = \left(KE - PE_{avg}\right)t_2 \qquad \text{(Case A)} \tag{12}$$

Or, expressed in terms of the endpoints of the worldline:

$$\text{Action} = S = \frac{1}{2}m\frac{x_2^2}{t_2^2}t_2 - mg\frac{x_2}{2}t_2 \qquad \text{(Case A)} \tag{13}$$

Here are the first exercises:

Q1. Using a hand calculator, find the numerical value of S (for a mass $m = 1$ kg) for Case A, given the specific values:

$x_2 = 25$ meters

$t_2 = 7.5$ seconds $\tag{14}$

Q2. Check your answer to Q1 by setting up the same straight worldline using the ACTION software. (You may want to do the introductory exercises in the Appendix before trying this question.)

NOTE #1: You must set up the GRAVITY potential using the POTENTIAL menu at the top of the screen that appears when you press the NEW CASE button.

NOTE #2: Pay attention to the scales on the horizontal and vertical axis, which are labeled in a funny way.

NOTE #3: To help you correctly place event-dots, try out various settings in the GRID menu.

NOTE #4: Under the ENDPOINTS menu choose Gravity Case to enter the above endpoints directly. Or choose Digital Entry to do the same digitally.

NOTE #5: The values of Action derived as answers to the Q-questions in this section are NEGATIVE numbers. The Action can be a negative number,

and the LEAST action can mean the MOST NEGATIVE number. (Go figure!) Example: S = – 1234 has a lower value than S = – 900.

> **NOTE #6: Do not worry if the ACTION software does not yield a number exactly equal to your calculated value. The ACTION software is accurate to only a few percent.**

III. Generalization: Action for a Straight Segment of Worldline

Now we are going to make a great simplification by approximating *every* worldline as a series of straight segments. This makes it easy for a computer to calculate the Action for each straight segment, then add up the Action for all segments to yield the total Action. One such segment is shown in Figure 3:
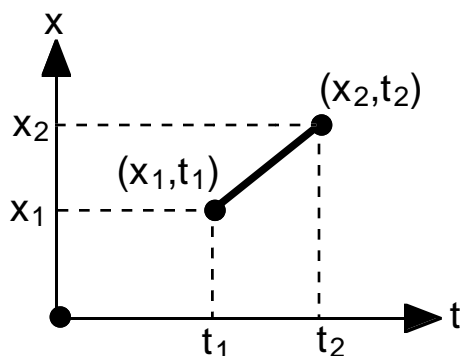

Figure 3. General straight segment of a worldline

The derivation of the Action for this segment is a simple extension of Case A. The kinetic energy depends only on the *slope* of the straight segment of worldline, so *where* the segment is on the *xt* diagram does not matter as far as the *KE* contribution to the Action is concerned. The result for Case A can simply be applied to the general case. The only difference is that the time lapse over the segment becomes $(t_2 - t_1)$, as shown in equation (15) below.

As before, the potential energy term is almost as simple—but not quite. We can choose the zero of classical potential energy anywhere. In the case of a uniform gravitational field, we can choose the zero of potential energy to be at the ground, at the second floor, or at the top of a table on the second floor. Different choices simply add or subtract a constant potential, without changing any predictions about the resulting motion. Changing the origin of *x* from Figure 2 to Figure 3 increases the potential at all points of the segment of the worldline. But the average *PE* is still calculated using the average value of the height, which this time is given by the expression: $(x_1 + x_2)/2$.

These simple translations change equations (12) and (13) to equations (15) and (16):

$$\text{Action} = S = \int_{t_1}^{t_2} (KE - PE)\, dt = \left(KE - PE_{avg}\right)(t_2 - t_1) \qquad \text{(straight segment)} \qquad (15)$$

or

$$\text{Action} = S = \frac{1}{2}m\left(\frac{x_2 - x_1}{t_2 - t_1}\right)^2 (t_2 - t_1) - mg\frac{(x_2 + x_1)}{2}(t_2 - t_1) \qquad \text{(straight segment)} \quad (16)$$

The form of equation (16) makes it fairly clear where each term comes from. However, for computational purposes, you may want to simplify this algebraically a bit:

$$\text{Action} = S = \frac{1}{2}m\left[\frac{(x_2 - x_1)^2}{t_2 - t_1} - g(x_2 + x_1)(t_2 - t_1)\right] \qquad \text{(straight segment)} \qquad (17)$$

The software program ACTION uses equation (17) with $m = 1$ kilogram to compute the action along each segment of the worldline that you construct between an initial event and a final event.

IV. CASE B: More Time at Altitude
Now it is your turn. Case A is a very unrealistic. Take the next step toward reality. Find an expression for the total Action along the *slightly* more realistic two-segment path shown for Case B in Figure 1, repeated as Figure 4.
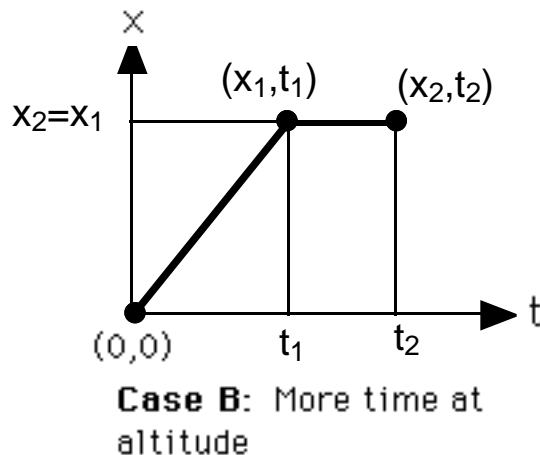

Figure 4. Two-segment worldline joining same initial and final events as Case A in Figure 3.

On the way to your solution, you may answer the following questions in sequence— or do it your own way and write up your results.

      Q3. Adapt the result for Case A to the shorter time t_1 thus finding the value of the Action for the first segment, the upward part of the

worldline in Case B. What is this algebraic expression for the Action along this first, rising segment? Do NOT turn in this complicated expression. Instead, use a hand calculator to find the value along this first segment using the values given in equation (14) plus the value:

$$t_1 = 5 \text{ seconds} \tag{18}$$

Turn in your numerical result.

Q4. Check your numerical result of question Q3 using the ACTION program to find the value of S along the first, rising segment for Case B. (In this case there is only a left-hand point at (0,0) and a right-hand point at . . .?) Report your numerical value from the program. **Comment:** You will find a large variation in the result, depending on exact placement of the dots at each end. This is because kinetic energy and potential energy are nearly equal along this segment, so ($KE - PE$) varies widely as KE varies a little bit. You can minimize this variation by choosing Digital Entry from the Endpoints menu.

Q5. From the general result for a straight segment, equation (17), use a hand calculator to find a numerical value for the Action along the second, horizontal segment of worldline in Figure 4.

Q6. Check your numerical result of question Q5 using the ACTION program to find the value of S along the second, horizontal segment for Case B. (In this case there is only a left-hand point at (x_1, t_1) and a right-hand point at . . . ?) Report your numerical value from the program.

Q7. Now start again, and use the ACTION program to draw the complete two-segment worldline shown in Figure 4 for the points described in equations (14) and (18). (Hint: Choose the endpoints first.) Read off the TOTAL action for this worldline and compare this total with the numbers derived for the individual segments in questions Q3 through Q6.

Q8. Now click on CHANGE TO 'MOVE DOTS' and move the middle event up and down to determine the minimum value of the Action for this two-segment worldline. Report the value of this minimum Action and compare it to the value of the Action for the one-segment worldline analyzed in Section II.

V. An Even More General Case

This is getting tedious. Computers were made to do tedious work. The software program ACTION helps you to compute the Action between two events along a worldline with as many segments as you want. Section III above derives the algebraic expression that the computer uses for each segment of the worldline.

Q9. Set up the Action program with the GRAVITY potential and initial and final points the same as for Cases A and B [initial point at (0,0), final point at 25 meters and 7.5 seconds]. Add TWO intermediate points equally spaced in time at $t = 2.5$ seconds and 5 seconds. Click on CHANGE TO 'MOVE DOTS', then *drag* these intermediate points UP and DOWN (not left to right) to find the minimum value of the Action. Report this numerical value.

Q10. Now click on CHANGE TO 'ADD DOTS' and click on a bunch more intermediate dots along this worldline. Now click on CHANGE TO 'MOVE DOTS' and use the HUNT MINIMUM menu at the top of the screen to Hunt Vertical for the worldline of minimum Action. Report the value of this minimum Action.

Q11. With many dots in place in Q10, is energy (approximately) conserved along this worldline? Drag one of the dots a long way up or down. What does this do to the value of the total energy for the segments on either side of the moved dot? **Comment**: Equal values of total energy along each segment is what we mean by the **conservation of energy**. You have demonstrated that conservation of energy is a natural consequence of the Principle of Least Action.

AN ASIDE: The Principle of Least Action is very powerful. It tells us how a particle moves in a potential and leads to the conservation of energy. In contrast, the conservation of energy does not tell us how a particle moves. A candidate motion of a particle may satisfy the conservation of energy and still be incorrect. Consider the following.

The pitcher throws a ball toward the batter. How does the ball move? One (incorrect) possibility is that the ball stays at the same height above the ground and moves with constant speed along a straight line from pitcher to batter. For such a candidate motion, kinetic energy *KE* is constant and potential energy *PE* is constant; therefore total energy $E = KE + PE$ is constant—energy is conserved. But this is NOT the way a pitched ball moves. The actual motion is predicted by varying the worldline to find the worldline of minimum Action. For this worldline, too, energy is conserved. But the argument does not work the other way: conservation of energy does not necessarily predict the correct motion. Least Action does.

APPENDIX: INTRODUCTION TO THE ACTION PROGRAM

The following exercises are intended to introduce you to the ACTION software.

For these exercises there is no gravitational field. The particle moves in an inertial frame (as in a space laboratory orbiting Earth).

> NOTE #1: Most of the menus described below appear at the top of the screen only when you press the NEW CASE button.
> NOTE #2: Use the MACRO default under the DEFAULT. This is the one for classical Action.
> NOTE #3: The default direction for the time axis is horizontal, the same as in Feynman's lecture on the subject. If you need to change this, use the TIME AXIS menu.

Use the ACTION software to read off the answers to the following questions for a one-kilogram mass.

Place the endpoints at (space = 0, time = 0) and (space = 100 meters, time = 10 seconds). NOTE: The space and time axes are labeled in a funny way.

Z1. What is the value of the Action S along the straight worldline between these two events? (See display at the bottom of the screen.)

Z2. What is the KE of the particle along this worldline? (See display at the right.)

Z3. What is the value of the Action S along a two-segment worldline that begins at point (space = 0, time = 0)
moves to point (space = 75 meters, time = 5 seconds)
and ends at point (space = 100 meters, time = 10 seconds)?
HINT: Place the endpoints first.

Z4. Is energy (approximately) conserved along this two-segment worldline?

Now click on the button CHANGE TO 'MOVE DOTS' and drag the intermediate

point up and down to find the minimum Action S_{min} (bottom of the screen). Click on buttons to allow you to drag left-right as well as up-down. NOTE: Written at the bottom of the screen is both the value of the Action S and also the minimum value

S_{min} recorded for past positions of the intermediate point or points. This will help you find the minimum value as you drag the point(s) in various directions.

Z5. What is the value of this minimum Action S_{min}?

Z6. Is energy (approximately) conserved along this two-segment worldline of minimum Action?

Click on CHANGE TO 'ADD DOTS' and click on a bunch more intermediate dots along this worldline. Now click on CHANGE TO 'MOVE DOTS' and use the HUNT MINIMUM menu at the top of the screen to search automatically for the worldline of minimum Action.

Z7. What is the value of the minimum Action for this many-segment line?

Z8. Is energy (approximately) conserved along this worldline of minimum Action?

RECESS!
Play with the Menus at the top of the screen, the ones that appear when you start a NEW CASE. In particular, learn how to change the potential and how to place the endpoints digitally. (Note that digital placement of endpoints may include the extremes, such as 0 and 100 -- and that the program will accept second-decimal-place accuracy in endpoints, such as 12.34.)

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