WORLDLINES FOR THE QUANTUM PARTICLE

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I. The Boundary Between Classical And Quantum Mechanics

The first part of the following exercises investigates the QUANTUM motion of a FREE particle, a particle that moves in the absence of a potential. In particular, these exercises explore the boundary between classical and quantum mechanics. When are you FORCED to use quantum mechanics? That is, under what circumstances are your predictions WRONG unless you use quantum mechanics instead of classical mechanics? We will use worldlines for particles of different mass to explore the boundary between quantum mechanics and classical mechanics.

For a given small segment of worldline (for a given speed *v* of the particle), the rotation rate of the quantum stopwatch for a free particle is greater if the particle mass *m* is greater and smaller if *m* is smaller, according to our formula:

$$f = \frac{KE}{h} = \frac{mv^2}{2h} \quad \text{rotations per second} \tag{1}$$

The electron has the smallest mass *m* of any known stable particle. Therefore the quantum clock for the electron will have the smallest rotation rate along any given segment of worldline—and the smallest total number of rotations along any complete worldline. A particle with 10 times the mass of the electron will have 10 times the number of rotations of its quantum clock between given events along the same worldline. ("IS there a particle with 10 times the mass of the electron?" Quiet, you!)

Figure 1 recalls the OneParticle program for the electron, showing the resulting arrow when a series of paths are chosen between Emitter and Detector. Adding together head-to-tail the little stopwatch hands for the alternative paths, shown at the right, yields an S-curve called the Cornu Spiral. One can see that the little stopwatch hands for paths lying near the direct path all point in nearly the same direction, and so make the major contribution to the resulting arrow.
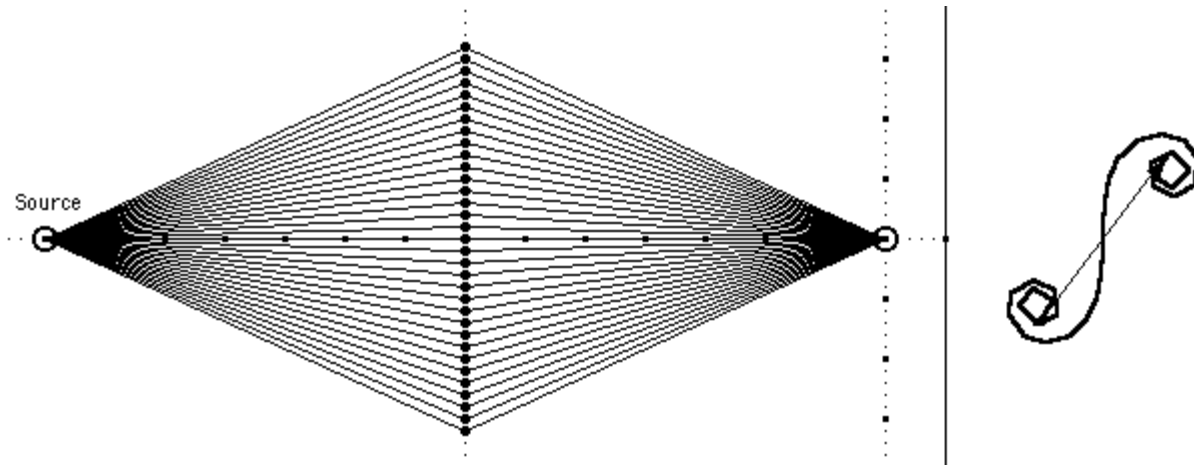
Figure 1. Reminder of the OneParticle display for the electron, in which the little stopwatch hands from a series of paths, when these stopwatch hands are added head-to-tail, yield a Cornu spiral (right panel). The stopwatch hands for the more direct paths point in nearly the same direction as one another, and so contribute the major part of the length of the resulting arrow at the detector.

Figure 2 compares the resulting arrow for a (nearly) complete Cornu Spiral with the resulting arrow for the case in which only those paths are included for which the quantum clock rotates half a rotation or less from the direction of the quantum stopwatch hand for the direct path. (Review the earlier section entitled "Quantum Stopwatch for the Free Electron," in particular Figure 2 and the questions that follow it.) In Figures 1 and 2 above and below, the little quantum stopwatch hand for the direct path happens to point straight upward at the detector. This is the direction of the middle segment in the Cornu spiral, shown by the central "sampling arrow" in the right-hand panel of Figure 2. In contrast, the little stopwatch hands at the extreme ends of the S-curve point opposite to this direction, half a rotation away from the direction of the stopwatch hand from the central segment. Yet the long resulting arrow in this right-hand panel is approximately the same length and direction as the resulting arrow in the left-hand panel, which includes contributions from a much wider range of paths.

In summary, it remains true that EVERY path from source to detector contributes a little stopwatch hand of EQUAL length at the detector. However, there exists a set of paths whose stopwatch hands lie more or less in the same direction as one another. We can use this smaller set of paths to obtain a "good-enough" prediction of the resulting arrow.
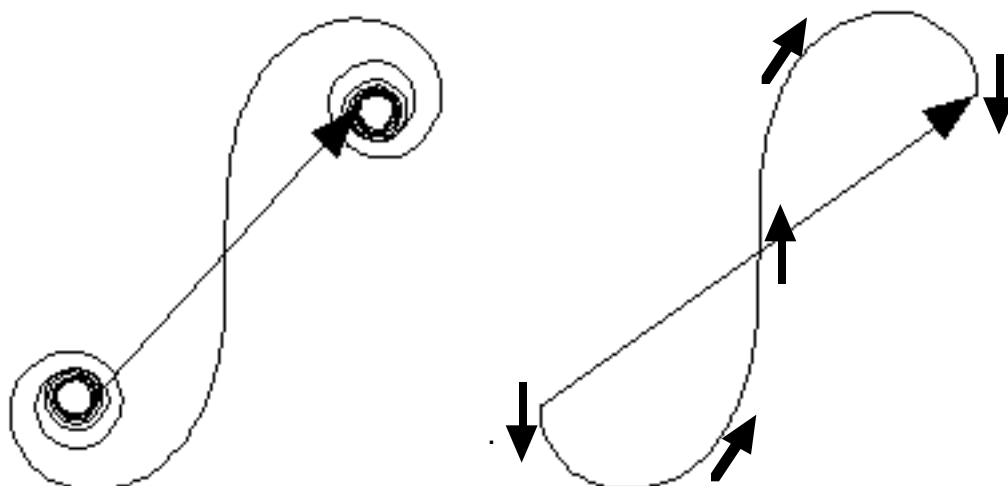
Figure 2: Addition of stopwatch hands for alternative paths, as begun in Figure 1. The resulting arrow for a (nearly) complete Cornu spiral (left) is approximated (right) by contributions from only those worldlines along which the number of rotations differs by one-half-rotation or less from that of the direct worldline. The little black arrows sample the directions of the little arrows from different paths that make up the S-curve.

II. The Action Program for the Microworld

Under the Default menu, the Action program has a MICRO default, intended to help you explore the consequences of different masses on the behavior of a free particle. You can show how a particle of small mass requires the Sum Over Paths theory of quantum mechanics to describe its motion, while the single path of classical mechanics describes the motion of a more massive particle "accurately enough."

In the MICRO default, the time axis is always vertical. This is the professional convention. Learn how to set up several alternative worldlines between emission event and detection event. Notice that in this spacetime diagram you can drag intermediate event-dots back and forth horizontally.

III. THE EFFECTIVE PENCIL OF WORLDLINES IN ZERO POTENTIAL

Using the ACTION software program, carry out the following procedure.

STEP 1. MENU CHOICES
From the DEFAULTS menu choose MICRO Default.
From the GRID menu choose LINE GRID.
From the ENDPOINTS menu choose OPPOSITE ENDS.

Now at the bottom of the screen should be the words:
"Particle mass = 10 m(electron), Potential = ZERO POTENTIAL."
If the words are different, use the menus to fix them.

STEP 2. CREATE WORLDLINES
Create (in any order) two more alternative worldlines between these emission and detection events, making a total of three.

THE FIRST worldline is the direct one already created, with no intermediate events. In zero potential it is the worldline of minimum Action.

NOTE: The second and third worldlines added below are NOT worldlines of minimum Action.

THE SECOND worldline is created as follows:
A1. Click on the button ADD WLDLN. (If you cannot see this button, click on the button CHANGE TO 'ADD DOTS'.)
A2. Click on the graph on the time line 0.5 x 10–7 second to the RIGHT of the first worldline.
A3. Click on the button CHANGE TO 'MOVE DOTS'
A4. Drag the new dot left or right so that the number of rotations (shown in the right-hand window) is one-half rotation more than for the direct path.
EXAMPLE: If the number of rotations along the direct worldline is 11.4, then you goal is to find worldlines on either side with 0.5 more rotations, namely 11.9 rotations.
Don't be too fussy about getting an *exact* additional half-revolution.
(WHY one-half rotation more? Review the argument in Section I above.)

THE THIRD worldline is created as follows:
A0. Click on the button CHANGE TO 'ADD DOTS'
A1. Click on the button ADD WLDLN.
A2. Click on the graph on the time line 0.5 x 10^{-7} second to the LEFT of the first worldline.
A3. Click on the button CHANGE TO 'MOVE DOTS'
A4. Drag the new dot left or right so that the number of rotations (shown in the right-hand window) is one-half rotation more than for the direct path.

Now look at the display you have created. *The worldlines you have drawn approximately span the pencil of worldlines that make the greatest contribution to the resulting arrow at the detector.*

STEP 3. Read and record the approximate *x*-positions (scale along the horizontal axis at the bottom) of the intermediate points of the second and third worldlines that meet this criterion. NOTE: It is important that you GET THE UNITS RIGHT—see label on horizontal scale.

STEP 4. Now repeat the procedure of Steps 1 through 3 for a particle of mass 100m(electron), and finally for a particle of mass m(electron).
NOTE 1: Choose different masses from the MASS menu when you have pressed the NEW CASE button.

NOTE 2: You may run into difficulty with the smallest mass-value.

Q1. Submit a neat, clear, table of your results, showing the mass of each particle and the positions (with correct units) of the intermediate points for the second and third worldlines found in Step 3.

Q2. Summarize in a BRIEF report your conclusions about the transition between quantum behavior and classical behavior of a free particle (a particle moving in a region of zero potential).

IV. PARTICLES OF DIFFERENT MASS IN A BINDING POTENTIAL
In this section we investigate the "spread" of minimum-action worldlines in a binding potential—a potential that confines a charged particle near to a center of electrical attraction. Does a particle of low mass "spread out" more or less than a particle of greater mass?

B1. MENU CHOICES
From the DEFAULT menu select MICRO DEFAULT
Use the POTENTIAL menu to select EXPONENTIAL WELL.
From the GRID menu select LINE GRID.
From the ENDPOINTS menu select CENTER POSITIONS.

B2. Click on one intermediate point off to one side, as shown in Figure 3.
B3. From the #DOTS menu, choose SOME DOTS.
B4. Click on the button CHANGE TO 'MOVE DOTS'
B5. From the HUNT menu select HUNT HORIZONTAL.
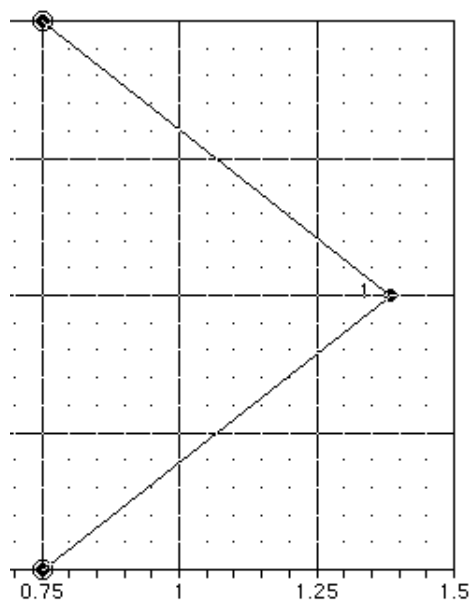B6. Let it run until the points do not move any more. Click STOP HUNT.

Figure 3. Placing an intermediate point near the right side, centered vertically.

B7. Record the number of rotations and the maximum x-value of the worldline in its bow to the right. Please, oh please, get the UNITS RIGHT—see label on horizontal scale.

B8. Press NEW CASE and from the MASS menu choose $m = m$(electron)

B9. Now repeat steps B2 through B7. You will find that the curve slams into the right hand limit.

B10. Press NEW CASE and use the COORDINATES menu to set $X_{max} = 1.5 \times 10^{-4}$ meters. For this exercise, do NOT change the time scale, which has the maximum value $T_{max} = 10^{-7}$ seconds.

B11. Now repeat steps B2 through B7, recording the number of rotations.

B12. Repeat for all the masses given under the Mass menu, adjusting X_{max} as needed from the Coordinates menu.

Q4. Make a neat table showing (a) the mass of the particle under investigation, (b) the number of rotations of the minimum-rotation worldline, and (c) the maximum *x*-value of the worldline in its bow to the right (with the correct units).

Q5. Does the width of the space that a bound particle traverses when following a path of minimum action get WIDER or NARROWER as the mass of the particle INCREASES?

V. A LONG-WINDED SUMMARY OF WHAT THIS IS ALL ABOUT.

SUMMARY OF SECTION III: THE EFFECTIVE PENCIL OF WORLDLINES IN ZERO POTENTIAL

In Section III you study the spread of quantum paths around the minimum-rotation worldline for zero potential. Are ALL quantum paths equal? Yes. Are all citizens equal? Yes (ideally), at least in the eyes of the law. But a social commentator might say that some citizens do not pull their weight in promoting social good. Why not? Because they LACK DIRECTION. They are not all pulling in the same direction.   Similarly, quantum mechanics is a perfect democracy; every single quantum path yields a little arrow of the same length. But not all quantum paths contribute significantly to the resulting arrow at the detector. Why not? Because they LACK DIRECTION. That is, their little arrows point all over the place, not in the same direction as that of the minimum-rotation path. Around the minimum- rotation worldline (and ONLY around this minimum-rotation worldline) is a "pencil" of worldlines whose little arrows differ in direction from that of the minimum-rotation arrow by half a rotation or less. THESE are the movers and shakers, the SIGNIFICANT contributors in constructing the resulting arrow at the detector.

For larger and larger mass, this pencil of significant worldlines becomes narrower and narrower. In the limit of large mass, it LOOKS like only ONE worldline remains. This marks the classical prediction -- the single worldline of minimum rotation (*S/h*) or

minimum Action *S*. Has the baseball stopped exploring all paths? NO! A complete analysis of a baseball pitched in Chicago includes worldlines that reach the catcher's mitt by way of Hong Kong. But the "Hong Kong worldlines" do not contribute much to the resulting arrow at the catcher's mitt. Why not? Because they do not lie near to the worldline of minimal action, so they point in mutually-canceling directions. But the little arrows DO add up for worldlines VERY close to the minimum-action trajectory to the catcher's mitt. Or to the bat. BLAM! A home run! Do you now see baseball in a new light?

SUMMARY OF SECTION IV: PARTICLES OF DIFFERENT MASS IN A BINDING POTENTIAL
In Section IV, the story is similar. A particle moves in a potential that does not depend on the particle's mass. If emission and detection is in the center, then why should the particle spend time away from this center position? Because the potential is greater the farther the particle is from the center. By loitering farther from the center, the particle spends time in these regions of higher potential—so that MINUS *PE* will decrease the value of the Action. This is encouraged by the Principle of Least Action.

However, the greater the particle mass, the more its kinetic energy $KE = (1/2)mv^2$ increases the value of the Action along the worldline. [Remember that Action *S* is the time-summation (integral with respect to time) of (*KE - PE*).] For the more massive particle, the action is reduced a lot when the particle goes more directly (with smaller average kinetic energy) from source to detector. The greater the mass, the less relative advantage there is for the particle to loiter far from the direct path in order to spend time in regions of higher potential (so that MINUS *PE* will decrease the Action). Hence the minimum-action orbits of more-massive particles in a binding potential "spread out" less—and the orbits of less-massive particles spread out more. For a given particle mass and given space-variation of the binding potential, one can estimate or model the "spread" of these minimum-action paths. Does this give us an estimate of the size of the atom? I believe it would. But the atom is described by quantum mechanics, in which the electron explores ALL worldlines between emission and detection, many of which stray from the worldline of minimum Action.

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