SPECIAL TOPIC

LORENTZ TRANSFORMATION

L.1 LORENTZ TRANSFORMATION: USEFUL OR NOT?

related events or lonely events?

Events, and the intervals between events, define the layout of the physical world. No latticework of clocks there! Only events and the relation between event and event as expressed in the interval. That's spacetime physics, lean and spare, as it offers itself to us to meet the needs of industry, science, and understanding.

Events and intervals only: Spacetime lean and spare

There's another way to express the same information and use it for the same purposes: Set up a free-float latticework of recording clocks, or the essential rudiments of such a latticework. The space and time coordinates of that Lorentz frame map each event as a lonesome individual, with no mention of any connection, any spacetime interval, to any other event.

Or isolated events described using latticework

This lattice-based method for doing spacetime physics has the advantage that it can be mechanized and applied to event after event, wholesale. These regimented space and time coordinates then acquire full usefulness only when we can translate them from the clock-lattice frame used by one analyst to the clock-lattice frame used by another.

This scheme of translation has acquired the name "Lorentz transformation." Its usefulness depends on the user. Some never need it because they deal always with intervals. Others use it frequently because it regiments records and standardizes analysis. For their needs we insert this Special Topic on the Lorentz transformation. The reader may wish to read it now, or skip it altogether, or defer it until after Chapter 4, 5, or 6. The later the better, in our opinion.

Lorentz transformation: Translate event description from lattice to lattice

L.2 FASTER THAN LIGHT?

a reason to examine the Lorentz transformation

No object travels faster than light.

So YOU say, but watch ME: I travel in a rocket that you observe to move at 4/5 light speed. Out the front of my rocket I fire a bullet that I observe to fly forward at 4/5 light speed. Then you measure this bullet to streak forward at $4/5 + 4/5 = 8/5 = 1.6$ light speed, which is greater than the speed of light. There!

No!

Why not? Is it not true that $4/5 + 4/5 = 1.6$?

Velocities do not add

As a mathematical abstraction: always true. As a description of the world: only sometimes true! Example 1: Add 4/5 liter of alcohol to 4/5 liter of water. The result? Less than $8/5 = 1.6$ liter of liquid! Why? Molecules of water interpenetrate molecules of alcohol to yield a combined volume less that the sum of the separate volumes. Example 2: Add the speed you measure for the bullet (4/5) to the speed I measure for your rocket (4/5). The result? The speed I measure for the bullet is $40/41 = 0.9756$. This remains less than the speed of light.

Why? And where did you get that number 40/41 for the bullet speed you measure?

I got the number from the Lorentz transformation, the subject of this Special Topic. The Lorentz transformation embodies a central feature of relativity: Space and time separations typically do not have the same values as observed in different frames.

Space and time separations between what?

Between events.

What events are we talking about here?

Event 1: You fire the bullet out the front of your rocket. Event 2: The bullet strikes a target ahead of you.

What do these events have to do with speed? We are arguing about speed!

Events define velocities

Let the bullet hit the target four meters in front of you, as measured in your rocket. Then the space separation between event 1 and event 2 is 4 meters. Suppose the time of flight is 5 meters as measured by your clocks, the time separation between the two events. Then your bullet speed measurement is (4 meters of distance)/(5 meters of time) $= 4/5$, as you said.

And what do YOU measure for the space and time separations in your laboratory frame?

For that we need the **Lorentz coordinate transformation equations.**

Phooey! I know how to reckon spacetime separations in different frames. We have been doing it for several chapters! From measurements in one frame we figure the spacetime interval, which has the same value in all frames. End of story.

No, not the end of the story, but at least its beginning. True, the invariant interval has the same value as derived from measurements in every frame. That allows you to predict the time between firing and impact as measured by the passenger riding on the bullet — and measured directly by the bullet passenger alone.

Interval: Only a start in reckoning spacetime separations in different frames

Predict how?

You know your space separation $x' = 4$ meters (primes for rocket measurements), and your time separation, $t' = 5$ meters. You know the space separation for the bullet rider, $x'' = 0$ (double primes for bullet measurements), since she is present at both the firing and the impact. From this you can use invariance of the interval to determine the wristwatch time between these events for the bullet rider:

$$(t'')^2 - (x'')^2 = (t')^2 - (x')^2$$

or

$$(t'')^2 - (0)^2 = (5 \text{ meters})^2 - (4 \text{ meters})^2 = (3 \text{ meters})^2$$

so that $t'' = 3$ meters. This is the proper time, agreed on by all observers but measured directly only on the wristwatch of the bullet rider.

Fine. Can't we use the same procedure to determine the space and time separations between these events in your laboratory frame, and thus the bullet speed for you?

Unfortunately not. We do reckon the same value for the interval. Use unprimed symbols for laboratory measurements. Then $t^2 - x^2 = (3 \text{ meters})^2$. That, however, is not sufficient to determine x or t separately. Therefore we cannot yet find their ratio x/t, which determines the bullet's speed in our frame.

Need more to compare velocities in different frames

So how can we reckon these x and t separations in your laboratory frame, thereby allowing us to predict the bullet speed you measure?

Use the Lorentz transformation. This transformation reports that our laboratory space separation between firing and impact is $x = 40/3$ meters and the time separation is slightly greater: $t = 41/3$ meters. Then bullet speed in my laboratory frame is predicted to be $v = x/t = 40/41 = 0.9756$. The results of our analysis in three reference frames are laid out in Table L-1.

Compare velocities using Lorentz transformation

Is the Lorentz transformation generally useful, beyond the specific task of reckoning speeds as measured in different frames?

Oh yes! Generally, we insert into the Lorentz transformation the coordinates x', t' of an event determined in the rocket frame. The Lorentz transformation then grinds and whirs, finally spitting out the coordinates x, t of the same event measured in the laboratory frame. Following are the Lorentz transformation equations. Here v_{rel} is the relative velocity between rocket and laboratory frames. For our convenience we lay the positive x-axis along the direction of motion of the rocket as observed in the laboratory frame and choose a common reference event for the zero of time and space for both frames.

TABLE L-1

HOW FAST THE BULLET?

	Bullet fired (coordinates of this event)	Bullet hits (coordinates of this event)	Speed of bullet (computed from frame coordinates)
Rocket frame (moves at $v_{rel} = 4/5$ as measured in laboratory)	$x' = 0$ $t' = 0$	$x' = 4$ meters $t' = 5$ meters	as measured in rocket frame: $v' = 4/5 = 0.8$
Bullet frame (moves at $v' = 4/5$ as measured in rocket)	$x'' = 0$ $t'' = 0$	$x'' = 0$ $t'' = 3$ meters (from invariance of the interval)	as measured in bullet frame: $v'' = 0$
Laboratory frame	$x = 0$ $t = 0$	$x = 40/3$ meters $t = 41/3$ meters (from Lorentz transformation)	as measured in laboratory frame: $v = 40/41 = 0.9756$

Lorentz transformation previewed

$$x = \frac{x' + v_{rel}\, t'}{(1 - v_{rel}^2)^{1/2}}$$

$$t = \frac{v_{rel}\, x' + t'}{(1 - v_{rel}^2)^{1/2}}$$

$$y = y' \qquad \text{and} \qquad z = z'$$

Check for yourself that for the impact event of bullet with target (rocket coordinates: $x' = 4$ meters, $t' = 5$ meters; rocket speed in laboratory frame: $v_{rel} = 4/5$) one obtains laboratory coordinates $x = 40/3$ meters and $t = 41/3$ meters. Hence $v = x/t = 40/41 = 0.9756$.

You say the Lorentz transformation is general. If it is so important, then why is this a special topic rather than a regular chapter?

Lorentz transformation: Useful but not fundamental

The Lorentz transformation is powerful; it brings the technical ability to transform coordinates from frame to frame. It helps us predict how to add velocities, as outlined here. It describes the Doppler shift for light (see the exercises for this chapter). On the other hand, the Lorentz transformation is not fundamental; it does not expose deep new features of spacetime. But no matter! Physics has to get on with the world's work. One uses the method of describing separation best suited to the job at hand. On some occasions the useful fact to give about a luxury yacht is the 50-meter distance between bow and stern, a distance independent of the direction in which the yacht is headed. On another occasion it may be much more important to know that the bow is 30 meters east of the stern and 40 meters north of it as observed by its captain, who uses North-Star north.

What does the Lorentz transformation rest on? On what foundations is it based?

Two foundations of Lorentz transformation

On two foundations: (1) The equations must be linear. That is, space and time coordinates enter the equations to the first power, not squared or cubed. This results from the requirement that you may choose any event as the zero of space and time.

(2) The spacetime interval between two events must have the same value when computed from laboratory coordinate separations as when reckoned from rocket coordinate separations.

All right, I'll reserve judgment on the validity of what you claim, but show me the derivation itself.

Read on!

L.3 FIRST STEPS

invariance of the interval gets us started

Recall that the coordinates y and z transverse to the direction of relative motion between rocket and laboratory have the same values in both frames (Section 3.6):

$$\begin{aligned} y &= y' \\ z &= z' \end{aligned} \tag{L-1}$$

where primes denote rocket coordinates. A second step makes use of the difference in observed clock rates when the clock is at rest or in motion (Section 1.3 and Box 3-3). Think of a sparkplug at rest at the origin of a rocket frame that moves with speed v_{rel} relative to the laboratory. The sparkplug emits a spark at time t' as measured in the rocket frame. The sparkplug is at the rocket origin, so the spark occurs at $x' = 0$.

Derive difference in clock rates

Where and when (x and t) does this spark occur in the laboratory? That depends on how fast, v_{rel}, the rocket moves with respect to the laboratory. The spark must occur at the location of the sparkplug, whose position in the laboratory frame is given by

$$x = v_{\mathrm{rel}} t$$

Now the invariance of the interval gives us a relation between t and t',

$$(t')^2 - (x')^2 = (t')^2 - (0)^2 = t^2 - x^2 = t^2 - (v_{\mathrm{rel}} t)^2 = t^2(1 - v_{\mathrm{rel}}^2)$$

from which

$$t' = t\,(1 - v_{\mathrm{rel}}^2)^{1/2}$$

or

$$t = \frac{t'}{(1 - v_{\mathrm{rel}}^2)^{1/2}} \qquad [\text{when } x' = 0] \tag{L-2}$$

The awkward expression $1/(1 - v_{\mathrm{rel}}^2)^{1/2}$ occurs often in what follows. For simplicity, this expression is given the symbol Greek lower-case gamma: γ.

$$\gamma \equiv \frac{1}{(1 - v_{\mathrm{rel}}^2)^{1/2}}$$

Time stretch factor defined

Because it gives the ratio of observed clock rates, γ is sometimes called the **time stretch factor** (Section 5.8). Strictly speaking, we should use the symbol γ_{rel}, since the value of γ is determined by v_{rel}. For simplicity, however, we omit the subscript in the hope that this will cause no confusion. With this substitution, equation (L-2) becomes

$$t = \gamma t' \qquad [\text{when } x' = 0] \tag{L-3}$$

Substitute this into the equation $x = v_{rel}\, t$ above to find laboratory position in terms of rocket measurements:

$$x = v_{rel}\gamma t' \qquad \text{[when } x' = 0\text{]} \qquad \text{(L-4)}$$

Equations (L-1), (L-3), and (L-4) give the first answer to the question, "If we know the space and time coordinates of an event in one free-float frame, what are its space and time coordinates in some other overlapping free-float frame?" These equations are limited, however, since they apply only to a particular situation: one in which both events occur at the same place ($x' = 0$) in the rocket.

L.4 FORM OF THE LORENTZ TRANSFORMATION

any event can be reference event? then transformation is linear

What general form does the Lorentz transformation have? It has the form that mathematicians call a **linear transformation.** This means that laboratory coordinates x and t are related to linear (first) power of rocket coordinates x' and t' by equations of the form

Lorentz transformation: Linear equations

$$\begin{aligned} t &= Bx' + Dt' \\ x &= Gx' + Ht' \end{aligned} \qquad \text{(L-5)}$$

where our task is to find expressions for the coefficients B, D, G, and H that do not depend on either the laboratory or the rocket coordinates of a particular event, though they do depend on the relative speed v_{rel}.

Why must these transformations be linear? Because we are free to choose any event as our reference event, the common origin $x = y = z = t = 0$ in all reference frames. Let our rocket sparkplug emit the flashes at $t' = 1$ and 2 and 3 meters. These are equally spaced in rocket time. According to equation (L-3) these three events occur at laboratory times $t = 1\gamma$ and 2γ and 3γ meters of time. These are equally spaced in laboratory time. Moving the reference event to the first of these events still leaves them equally spaced in time for both observers: $t' = 0$ and 1 and 2 meters in the rocket and $t = 0$ and 1γ and 2γ in the laboratory.

In contrast, suppose that equation (L-3) were not linear, reading instead $t = Kt'^2$, where K is some constant. Rocket times $t' = 1$ and 2 and 3 meters result in laboratory times $t = 1K$ and $4K$ and $9K$ meters. These are not equally spaced in time for the laboratory observer. Moving the reference event to the first event would result in rocket times $t' = 0$ and 1 and 2 meters as before, but in this case laboratory times $t = 0$ and $1K$ and $4K$ meters, with a completely different spacing. But the choice of reference event is arbitrary: Any event is as qualified to be reference event as any other. A clock that runs steadily as observed in one frame must run steadily in the other, independent of the choice of reference event. We conclude that the relation between t and t' must be a linear one. A similar argument requires that events equally separated in space in the rocket must also be equally separated in space as measured in the laboratory. Hence the Lorentz transformation must be linear in both space and time coordinates.

Arbitrary event as reference event? Then Lorentz transformation must be linear.

L.5 COMPLETING THE DERIVATION

invariance of the interval completes the story

Equations (L-3) and (L-4) provide coefficients D and H called for in equation (L-5):

$$\begin{aligned} t &= Bx' + \gamma t' \\ x &= Gx' + v_{\text{rel}}\gamma t' \end{aligned} \tag{L-6}$$

About the two constants B and G we know nothing, for an elementary reason. All events so far considered occured at point $x' = 0$ in the rocket. Therefore the two coefficients B and G could have any finite values whatever without affecting the numerical results of the calculation. To determine B and G we turn our attention from an $x' = 0$ event to a more general event, one that occurs at a point with arbitrary rocket coordinates x' and t'. Then we demand that the spacetime interval have the same numerical value in laboratory and rocket frames for any event whatever:

Demanding invariance of interval . . .

$$t^2 - x^2 = t'^2 - x'^2$$

Substitute expressions for t and x from equation (L-6):

$$(Bx' + \gamma t')^2 - (Gx' + v_{\text{rel}}\gamma t')^2 = t'^2 - x'^2$$

On the left side, multiply out the squares. This leads to the rather cumbersome result

$$B^2 x'^2 + 2B\gamma x't' + \gamma^2 t'^2 - G^2 x'^2 - 2Gv_{\text{rel}}\gamma x't' - v_{\text{rel}}^2\gamma^2 t'^2 = t'^2 - x'^2$$

Group together coefficients of t'^2, coefficients of x'^2, and coefficients of the cross-term $x't'$ to obtain

$$\gamma^2(1 - v_{\text{rel}}^2)t'^2 + 2\gamma(B - v_{\text{rel}}G)\, x't' - (G^2 - B^2)x'^2 = t'^2 - x'^2 \tag{L-7}$$

Now, t' and x' can each take on any value whatsoever, since they represent the coordinates of an arbitrary event. Under these circumstances, it is impossible to satisfy equation (L-7) with a single choice of values of B and G unless they are chosen in a very special way. The quantities B and G must first be such as to make the coefficient of $x't'$ on the left side of equation (L-7) vanish as it does on the right:

. . . between any pair of events whatsoever . . .

$$2\gamma(B - v_{\text{rel}}G) = 0$$

But γ can never equal zero. The value of $\gamma = 1/(1 - v_{\text{rel}}^2)^{1/2}$ equals unity when $v_{\text{rel}} = 0$ and is greater than this for any other values of v_{rel}. Hence the left side of this equation can be zero only if

$$(B - v_{\text{rel}}G) = 0 \qquad \text{or} \qquad B = v_{\text{rel}}G \tag{L-8}$$

Second, B and G must be such as to make the coefficient of x'^2 equal on the left and right of equation (L-7); hence

$$G^2 - B^2 = 1 \tag{L-9}$$

Substitute B from equation (L-8) into equation (L-9):

. . . leads to completed form of Lorentz transformation.

$$G^2 - (v_{\text{rel}}G)^2 = 1 \qquad \text{or} \qquad G^2(1 - v_{\text{rel}}^2) = 1$$

Divide through by $(1 - v_{\rm rel}^2)$ and take the square root of both sides:

$$G = \frac{1}{(1 - v_{\rm rel}^2)^{1/2}}$$

But the right side is just the definition of the time stretch factor γ, so that

$$G = \gamma$$

Substitute this into equation (L-8) to find B:

$$B = v_{\rm rel}\gamma$$

These results plus equations (L-1) and (L-6) yield the Lorentz transformation equations:

The Lorentz transformation

$$\begin{aligned} t &= v_{\rm rel}\gamma x' + \gamma t' \\ x &= \gamma x' + v_{\rm rel}\gamma t' \\ y &= y' \\ z &= z' \end{aligned} \qquad \text{(L-10a)}$$

or, substituting for the value of gamma, $\gamma = 1/(1 - v_{\rm rel}^2)^{1/2}$:

$$\begin{aligned} t &= \frac{v_{\rm rel}x' + t'}{(1 - v_{\rm rel}^2)^{1/2}} \\ x &= \frac{x' + v_{\rm rel}t'}{(1 - v_{\rm rel}^2)^{1/2}} \\ y &= y' \qquad \text{and} \qquad z = z' \end{aligned} \qquad \text{(L-10b)}$$

In summary, the Lorentz transformation equations rest fundamentally on the required linearity of the transformation and on the invariance of the spacetime interval. Invariance of the interval was used twice in the derivation. First, we examined a pair of events both of which occur at the same fixed location in the rocket, so that rocket time between these events—proper time, wristwatch time—equals the space-time interval between them (Section L.3). Second, we demanded that the interval also be invariant between every possible event and the reference event (the present section).

L.6 INVERSE LORENTZ TRANSFORMATION

from laboratory event coordinates, reckon rocket coordinates

Equations (L-10) provide laboratory coordinates of an event when one knows the rocket coordinates of the same event. But suppose that one already knows the laboratory coordinates of the event and wishes to predict the coordinates of the event measured by the rocket observer. What equations should be used for this purpose?

An algebraic manipulation of equations (L-10) provides the answer. The first two of these equations can be thought of as two equations in the two unknowns x' and t'. Solve for these unknowns in terms of the now-knowns x and t. To do this, multiply both sides of the second equation by $v_{\rm rel}$ and subtract corresponding sides of the resulting second equation from the first. Terms in x' cancel to yield

$$t - v_{\text{rel}}x = \gamma t' - v_{\text{rel}}^2\gamma t' = \gamma(1 - v_{\text{rel}}^2)t' = \frac{\gamma}{\gamma^2}t' = \frac{t'}{\gamma}$$

Long derivation of inverse Lorentz transformation

Here we have used the definition $\gamma^2 = 1/(1 - v_{\text{rel}}^2)$. The equation for t' can then be written

$$t' = -v_{\text{rel}}\gamma x + \gamma t$$

A similar procedure leads to the equation for x'. Multiply the first of equations (L-10) by v_{rel} and subtract corresponding sides of the first equation from the second — try it! The y and z components are respectively equal in both frames, as before. Then the **inverse Lorentz transformation equations** become

$$\begin{aligned} t' &= -v_{\text{rel}}\gamma x + \gamma t \\ x' &= \gamma x - v_{\text{rel}}\gamma t \\ y' &= y \\ z' &= z \end{aligned} \qquad \text{(L-11a)}$$

Inverse Lorentz transformation

Or, substituting again for gamma, $\gamma = 1/(1 - v_{\text{rel}}^2)^{1/2}$:

$$\begin{aligned} t' &= \frac{-v_{\text{rel}}x + t}{(1 - v_{\text{rel}}^2)^{1/2}} \\ x' &= \frac{x - v_{\text{rel}}t}{(1 - v_{\text{rel}}^2)^{1/2}} \\ y' &= y \qquad \text{and} \qquad z' = z \end{aligned} \qquad \text{(L-11b)}$$

Equations (L-11) transform coordinates of an event known in the laboratory frame to coordinates in the rocket frame.

A simple but powerful *argument from symmetry* leads to the same result. The symmetry argument is based on the relative velocity between laboratory and rocket frames. With respect to the laboratory, the rocket by convention moves with known speed in the *positive* x-direction. With respect to the rocket, the laboratory moves with the same speed but in the opposite direction, the *negative* x-direction. This convention about positive and negative directions — not a law of physics! — is the only difference between laboratory and rocket frames that can be observed from either frame. Lorentz transformation equations must reflect this single difference. In consequence, the "inverse" (laboratory-to-rocket) transformation can be obtained from the "direct" (rocket-to-laboratory) transformation by changing the sign of relative velocity, v_{rel}, in the equations and interchanging laboratory and rocket labels (primed and unprimed coordinates). Carrying out this operation on the Lorentz transformation equations (L-10) yields the inverse transformation equations (L-11).

Short derivation of inverse Lorentz transformation

L.7 ADDITION OF VELOCITIES

add light velocity to light velocity: get light velocity!

The Lorentz transformation permits us to answer decisively the apparent contradiction to special relativity outlined in Section L.2, namely the apparent addition of velocities to yield a resultant velocity greater than that of light.

Return to velocity addition paradox

I travel in a rocket that you observe to move at 4/5 light speed. Out the front of my rocket I fire a bullet that I observe to fly forward at 4/5 light speed. Then you measure this bullet to streak forward at 4/5 + 4/5 = 8/5 = 1.6 light speed, which is greater than the speed of light. There!

SAMPLE PROBLEM L-1

TRANSFORMING OVER AND BACK

A rocket moves with speed $v_{\text{rel}} = 0.866$ (so $\gamma = 2$) along the x-direction in the laboratory. In the rocket frame an event occurs at coordinates $x' =$ 10 meters, $y' = 7$ meters, $z' = 3$ meters, and $t' =$ 20 meters of light-travel time with respect to the reference event.

a. What are the coordinates of the event as observed in the laboratory?

b. Transform the laboratory coordinates back to the rocket frame to verify that the resulting coordinates are those given above.

SOLUTION

a. We already know from Section 3.6 — as well as from the Lorentz transformation, equation (L-10) — that coordinates transverse to direction of relative motion are equal in laboratory and in rocket. Therefore we know immediately that

$$
\begin{aligned}
y &= y' = 7 \text{ meters}\\
z &= z' = 3 \text{ meters}
\end{aligned}
$$

The x and t coordinates of the event as observed in the laboratory make use of the first two equations (L-10):

$$
\begin{aligned}
t &= v_{\text{rel}}\gamma x' + \gamma t' = (0.866)(2)(10 \text{ meters}) + (2)(20 \text{ meters})\\
&= 17.32 + 40 = 57.32 \text{ meters}
\end{aligned}
$$

and

$$
\begin{aligned}
x &= \gamma x' + v_{\text{rel}}\gamma t' = 2(10 \text{ meters}) + (0.866)(2)(20 \text{ meters})\\
&= 20 + 34.64 = 54.64 \text{ meters}
\end{aligned}
$$

So the coordinates of the event in the laboratory are $t = 57.32$ meters, $x = 54.64$ meters, $y = 7$ meters, and $z = 3$ meters.

b. Use equation (L-11) to transform back from laboratory to rocket coordinates.

$$
\begin{aligned}
t' &= -v_{\text{rel}}\gamma x + \gamma t = -(0.866)(2)(54.64 \text{ meters}) + (2)(57.32 \text{ meters})\\
&= -94.64 + 114.64 = 20.00 \text{ meters}
\end{aligned}
$$

and

$$
\begin{aligned}
x' &= \gamma x - v_{\text{rel}}\gamma t = 2(54.64 \text{ meters}) - (0.866)(2)(57.32 \text{ meters})\\
&= 109.28 - 99.28 = 10.00 \text{ meters}
\end{aligned}
$$

as given in the original statement of the problem.

To analyze this experiment, convert statements about the bullet to statements about events, since event coordinates are what the Lorentz transformation transforms. Event 1 is the firing of the gun, event 2 the arrival of the bullet at the target. The Lorentz transformation equations can give locations x_1, t_1 and x_2, t_2 of these events in the laboratory frame from their known locations x'_1, t'_1 and x'_2, t'_2 in the rocket frame. In particular:

$$x_2 = \gamma x_2' + v_{\text{rel}}\gamma t'_2$$
$$x_1 = \gamma x_1' + v_{\text{rel}}\gamma t_1'$$

Subtract corresponding sides of these two equations:

$$(x_2 - x_1) = \gamma\,(x'_2 - x'_1) + v_{\text{rel}}\gamma(t'_2 - t'_1)$$

We are interested in the *differences* between the coordinates of the two emissions. Indicate these differences with the Greek uppercase delta, Δ, for example Δx. Then this x-equation and the corresponding t-equation become

Incremental event separations define velocities

$$\Delta x = \gamma\Delta x' + v_{\text{rel}}\gamma\Delta t'$$
$$\Delta t = v_{\text{rel}}\gamma\Delta x' + \gamma\Delta t' \qquad \text{(L-12)}$$

The subscript "rel" distinguishes *relative* speed between laboratory and rocket frames from other speeds, such as particle speeds in one frame or the other.

Bullet speed in any frame is simply space separation between two events on its trajectory measured in that frame divided by time between them, observed in the same frame. In the special case chosen, only the x-coordinate needs to be considered, since the bullet moves along the direction of relative motion. Divide the two sides of the first equation (L-12) by the corresponding sides of the second equation to obtain laboratory speed:

$$\frac{\Delta x}{\Delta t} = \frac{\gamma\Delta x' + v_{\text{rel}}\gamma\Delta t'}{v_{\text{rel}}\gamma\Delta x' + \gamma\Delta t'}$$

Then the time stretch factor γ cancels from the numerator and denominator on the right. Divide every term in numerator and denominator on the right by $\Delta t'$.

$$\frac{\Delta x}{\Delta t} = \frac{(\Delta x'/\Delta t') + v_{\text{rel}}}{v_{\text{rel}}(\Delta x'/\Delta t') + 1}$$

Now, $\Delta x'/\Delta t'$ is just distance covered per unit time by the particle as observed in the rocket, its speed—call it v', with a prime. And $\Delta x/\Delta t$ is particle speed in the laboratory—call it simply v. Then (reversing order of terms in the denominator to give the result its usual form) the equation becomes

Law of Addition of Velocities

$$v = \frac{v' + v_{\text{rel}}}{1 + v'v_{\text{rel}}} \qquad \text{(L-13)}$$

This is called the **Law of Addition of Velocities** in one dimension. A better name is the **Law of Combination of Velocities,** since velocities do not "add" in the usual sense. Using the Law of Combination of Velocities, we can predict bullet speed in the laboratory. The bullet travels at $v' = 4/5$ with respect to the rocket and the rocket moves at $v_{\text{rel}} = 4/5$ with respect to the laboratory. Therefore, speed v of the bullet

(continued on page 110)

SAMPLE PROBLEM L-2

"ET TU, SPACETIME!"

Julius Caesar was murdered on March 15 in the year 44 B.C. at the age of 55 approximately 2000 years ago. Is there some way we can use the laws of relativity to save his life?

Let Caesar's death be the reference event, labeled O: $x_o = 0$, $t_o = 0$. Event A is you reading this exercise. In the Earth frame the coordinates of event A are $x_A = 0$ light-years, $t_A = 2000$ years. Simultaneous with event A in your frame, Starship Enterprise cruising the Andromeda galaxy sets off a firecracker: event B. The Enterprise moves along a straight line in space that connects it with Earth. Andromeda is 2 million light-years distant in our frame. Compared with this distance, you can neglect the orbit of Earth around Sun. Therefore, in our frame, event B has the coordinates $x_B = 2 \times 10^6$ light-years, $t_B = 2000$ years. Take Caesar's murder to be the reference event for the Enterprise too ($x_o' = 0$, $t_o' = 0$).

a. How fast must the Enterprise be going in the Earth frame in order that Caesar's murder is happening NOW (that is, $t_B' = 0$) in the Enterprise rest frame? Under these circumstances is the Enterprise moving toward or away from Earth?

b. If you are acquainted with the spacetime diagram (Chapter 5), draw a spacetime diagram for the Earth frame that displays event O (Caesar's death), event A (you reading this exercise), event B (firecracker exploding in Andromeda), your line of NOW simultaneity, the position of the Enterprise, the worldline of the Enterprise, and the Enterprise NOW line of simultaneity. The spacetime diagram need not be drawn to scale.

c. In the Enterprise frame, what are the x and t coordinates of the firecracker explosion?

d. Can the Enterprise firecracker explosion warn Caesar, thus changing the course of Earth history? Justify your answer.

SOLUTION

a. From the statement of the problem,

$$\begin{array}{lll} x_o = x_o' = 0 & x_A = 0 & x_B = 2 \times 10^6 \text{ light-years} \\ t_o = t_o' = 0 & t_A = 2000 \text{ years} & t_B = 2000 \text{ years} \end{array}$$

We want the speed v_{rel} of the Enterprise such that $t_B' = 0$. The first two Lorentz transformation equations (L-10) with $t_B' = 0$ become

$$t_B = v_{rel}\,\gamma x_B'$$
$$x_B = \gamma x_B'$$

We do not yet know the value of x_B'. Solve for v_{rel} by dividing the two sides of the first equation by the respective sides of the second equation. The unknown x_B' drops out (along with γ), and we are left with v_{rel} in terms of the known quantities t_B and x_B:

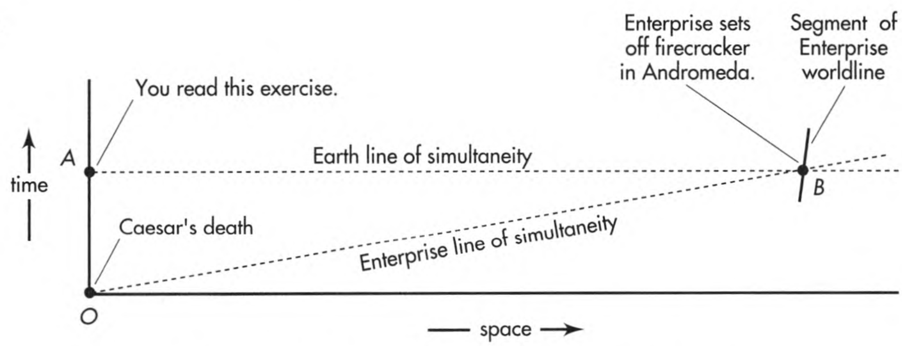
$$v_{rel} = \frac{t_B}{x_B} = \frac{2 \times 10^3 \text{ years}}{2 \times 10^6 \text{ years}} = 10^{-3} = 0.001$$

This is the desired speed v_{rel} between Earth and Enterprise frames. This velocity is a positive quantity, so the Enterprise moves in the positive x-direction, namely away from Earth.

Surprised to see a speed given as the ratio of a time separation to a space separation: t_B/x_B? Then realize that x_B and t_B are not displacements of any particle. Nothing can travel the distance x_B in the time t_B, as discussed in **d**. The goal here is to find a frame in which Caesar's death and the firecracker explosion are simultaneous. For this limited purpose the rocket speed $v_{rel} = t_B/x_B$ is correct.

Why is the relative velocity v_{rel} so small compared with the speed of light? Because of the large denominator x_B in the equation that leads to this value. Consider the string of Earth clocks stretching toward Andromeda when all Earth clocks read zero time (Caesar's death). Enterprise clocks read (from equations L-11 with $t = 0$) as follows: $t' = -v_{rel}\gamma x$. This is an example of the relativity of simultaneity (Section 3-4). The farther the x-distance from Earth, the earlier will Enterprise clock read. With $x = 2$ million light-years, the relative speed v_{rel} does not have to be large to carry Enterprise time back 2000 years for Earth.

b.



Earth spacetime diagram, showing events O, A, and *B. Not to scale.*

c. We need the value of gamma, γ, for the inverse Lorentz transformation equation (L-11). This value is very close to unity, and from it come t_B' and x_B'.

$$\gamma = \frac{1}{[1 - v_{rel}^2]^{1/2}} = \frac{1}{[1 - (10^{-3})^2]^{1/2}} = \frac{1}{[1 - 10^{-6}]^{1/2}} \approx 1 + \frac{10^{-6}}{2}$$

$$t_B' = -v_{rel}\gamma x_B + \gamma t_B = \gamma(-10^{-3} \times 2 \times 10^6 + 2 \times 10^3)$$

$$= \gamma(-2 \times 10^3 + 2 \times 10^3) = 0 \text{ years}$$

$$x_B' = \gamma x_B - v_{rel}\gamma t_B = \gamma(2 \times 10^6 - 10^{-3} \times 2 \times 10^3) = 2\gamma(1 - 10^{-6})\, 10^6$$

$$= 2\left(1 + \frac{10^{-6}}{2}\right)(1 - 10^{-6})10^6 = 2\left(1 - \frac{10^{-6}}{2} - \frac{10^{-12}}{2}\right)10^6$$

$$\approx 1.999999 \times 10^6 \text{ light-years.}$$

We *chose* the relative velocity so that the time of the firecracker explosion as observed in the rocket is the same as the time of Caesar's death, namely $t_B' = 0$. The x-coordinate of this explosion is not much different in the two frames because their relative velocity is so small.

d. There exists a frame—the rest frame of the Enterprise—in which Caesar's death and the firecracker explosion occur at the same time. In this frame a signal connecting the two events would have to travel at infinite speed. But this is impossible. Therefore the Enterprise cannot warn Caesar; his death is final. Sorry. (Note: In the language of Chapter 6, the relation between the two events is spacelike, and spacelike events cannot have a cause–effect relationship.)

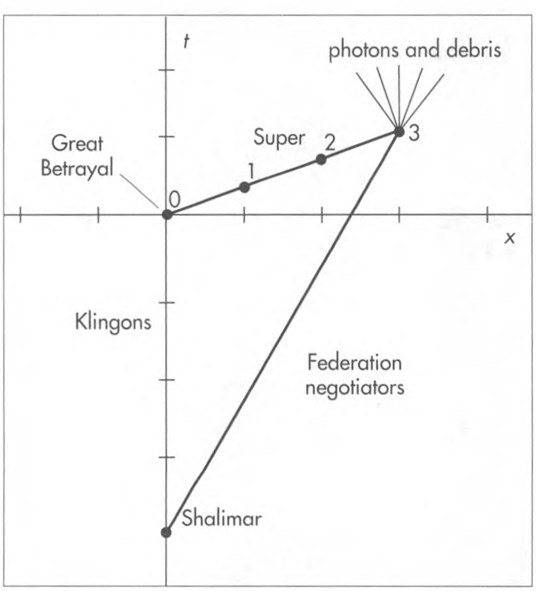
BOX L-1

WHY NO THING TRAVELS FASTER THAN LIGHT

A material object traveling faster than light? No! If one did, we could violate the normal order of cause and effect in a million testable ways, totally contrary to all experience. Here we investigate one example, making use of Lorentz transformation equations.

The Peace Treaty of Shalimar was signed four years before the Great Betrayal. So pivotal an event was the Great Betrayal that it was taken as zero of space and time.

By the Treaty of Shalimar, the murderous Klingons agreed to stop attacking Federation outposts in return for access to the Federation Technical Database. Federation negotiators left immediately after signing the Shalimar Treaty in a ship moving at 0.6 light speed.



Klingon ("laboratory") spacetime diagram. *The Klingon worldline is the vertical time axis. The Treaty of Shalimar is followed four years later by the Great Betrayal (event 0) at which Klingons launch the Super, which moves at three times light speed. Traveling from left to right, the Super passes one Federation colony (event 1) and then another (event 2). Finally the Super destroys the retreating ship of Federation negotiators (event 3).*

Within four years the Klingons used the Federation Technical Database to develop a faster-than-light projectile, the slaughtering Super. On that dark day of Great Betrayal (reference event 0), the Klingons launched the Super at three times light speed toward the retreating Federation ship.

Two Federation space colonies lay between the Klingons and the point of impact of the Super with the Federation ship. A lonely lookout at the first colony witnessed with awe the blinding passage of the Super (event 1). Later many citizens of the second colony gaped as the Super demolished one of their communication structures (event 2) and zoomed on. Both colonies desperately sent warnings toward the Federation ship, but to no avail since the Super outran the radio signals.

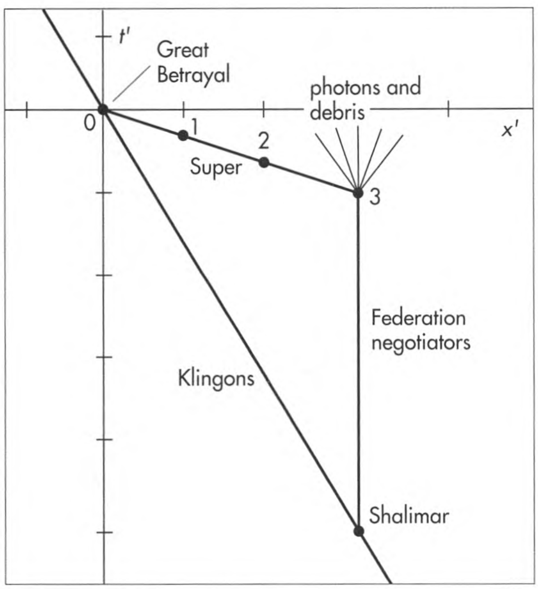
Finally, at event 3, the Super overtook and destroyed the Federation ship. All Federation negotiators were lost in a terrible flash of light and scattering of debris. A long dark period of renewed warfare began.

But wait! Look again at events of the Great Betrayal, this time from the point of view of the Federation rocket ship. Where and when does the Great Betrayal occur in this frame? The Great Betrayal is the "hinge of history," the reference event, the zero of space and time coordinates for all laboratory and rocket frames.

Where and when does the Super explode (event 3) in this rocket frame? In the Klingon "laboratory" frame, event 3 has coordinates $x_3 = 3$ light-years and $t_3 = 1$ year. Use the inverse Lorentz transformation equations to find the location of event 3 in the rocket frame of the Federation negotiators. Calculate the time stretch factor γ using speed of the Federation rocket, $v_{rel} = 0.6$, with respect to the Klingon frame:

$$\gamma = \frac{1}{[1 - v_{rel}^2]^{1/2}} = \frac{1}{[1 - (0.6)^2]^{1/2}} = \frac{1}{[1 - 0.36]^{1/2}}$$

$$= \frac{1}{[0.64]^{1/2}} = \frac{1}{0.8} = 1.25$$



"Rocket" spacetime diagram of departing Federation negotiators. In this frame their destruction comes first (event 3), followed by the passage of the Super from right to left past Federation colonies in reverse order (event 2 followed by event 1). Finally, the Super enters the Klingon launcher without doing further damage (event 0). The Great Betrayal has become the Great Confusion of Cause and Effect.

Substitute these values into equations (L-11) to reckon the rocket coordinates of event 3:

$$\begin{aligned} t'_3 &= -v_{rel}\gamma x_3 + \gamma t_3 \\ &= -(0.6)(1.25)(3 \text{ years}) + (1.25)(1 \text{ year}) \\ &= -2.25 \text{ years} + 1.25 \text{ years} = -1 \text{ year} \\ x'_3 &= \gamma x_3 - v_{rel}\gamma t_3 \\ &= (1.25)(3 \text{ years}) - (0.6)(1.25)(1 \text{ year}) \\ &= 3.75 \text{ years} - 0.75 \text{ year} = 3 \text{ years} \end{aligned}$$

Event 3 is plotted in the rocket diagram and the worldline of the Super drawn by connecting event 3 with the launching of the Super at event 0. Notice that this worldline slopes downward to the right. More about the significance of this in a minute.

In a similar manner find the rocket coordinates of the treaty signing at Shalimar (subscript Sh), which has laboratory coordinates $x_{Sh} = 0$ and $t_{Sh} = -4$ years:

$$\begin{aligned} t'_{Sh} &= -v_{rel}\gamma x_{Sh} + \gamma t_{Sh} \\ &= -(0.6)(1.25)(0 \text{ years}) + (1.25)(-4 \text{ years}) \\ &= -5 \text{ years} \\ x'_{Sh} &= \gamma x_{Sh} - v_{rel}\gamma t_{Sh} \\ &= (1.25)(0 \text{ years}) - (0.6)(1.25)(-4 \text{ years}) \\ &= +3 \text{ years} \end{aligned}$$

In the Federation (rocket) spacetime diagram, the worldline of Federation negotiators extends from treaty signing at Shalimar vertically to explosion of the Super (event 3). The worldline of the Klingons extends from Shalimar diagonally through the launch of the Super at event 0.

In the Federation spacetime diagram, the worldline for the Super tilts downward to the right. In this frame deaths of Federation negotiators (event 3) occur at a time $t'_3 =$ *minus 1 year,* that is, *before* the treacherous Klingons launch the Super at the event of Great Betrayal (reference event 0). From the diagram one would say that the Super moves with three times light speed *from* Federation ship *toward* the Klingons. This seems to be verified by the fact that in this frame the Super passes Federation colonies in reverse order, event 2 followed by event 1, going in the opposite direction. Yet Federation negotiators have created no such terrible weapon and in fact are destroyed by it at the moment they are supposed to launch it, as proved by the flying photons and debris. More: Klingons suffer no damage from the mighty impact of the slaughtering Super (event 0). Rather, in this frame it enters their launching cannon mild as a lamb.

What have we here? A confusion of cause and effect, a confusion that cannot be straightened out as long as we assume that the Super—or any other material object—travels faster than light in a vacuum.

Why does no signal and no object travel faster than light in a vacuum? Because if either signal or object did so, the entire network of cause and effect would be destroyed, and science as we know it would not be possible.

relative to the laboratory comes from the expression

Velocity addition paradox resolved

$$v = \frac{4/5 + 4/5}{1 + (4/5)(4/5)} = \frac{8/5}{1 + 16/25} = \frac{8/5}{41/25} = \frac{40}{41}$$

Thus the bullet moves in the laboratory at a speed less than light speed.

As a limiting case, suppose that the "bullet" shot out from the front of the rocket is, in fact, a pulse of light. Guess: What is the speed of this light pulse in the laboratory? Here is the calculated answer. Light moves with respect to the rocket at speed $v' = 1$ while the rocket continues along at a speed $v_{rel} = 4/5$ with respect to the laboratory. The light then moves with respect to the laboratory at speed v:

Light speed is invariant, as expected.

$$v = \frac{1 + 4/5}{1 + (1)(4/5)} = \frac{9/5}{9/5} = 1$$

So light moves with the same speed in both frames, as required by the Principle of Relativity. Question: Is this true also when a light pulse is shot out of the *rear* of the rocket?

SAMPLE PROBLEM L-3

THE FIRING MESON

A K^o (pronounced "K-naught") meson at rest in a rocket frame decays into π^+ ("pi plus") meson and a π^- ("pi minus") meson, each having a speed of $v' = 0.85$ with respect to the rocket. Now consider this decay as observed in a laboratory with respect to which the K^o meson travels at a speed of $v_{rel} = 0.9$. What is the greatest speed that one of the π mesons can have with respect to the laboratory? What is the least speed?

SOLUTION

Let the speeding K^o-meson move in the positive x-direction in the laboratory. In the rocket frame, daughter π-mesons come off in opposite directions. Their common line of motion can, however, be oriented arbitrarily in this frame. The maximum speed of a daughter π-meson in the laboratory results when it is emitted in the forward x-direction. For such a meson, the law of addition of velocities gives

$$v_{max} = \frac{v' + v_{rel}}{1 + v'v_{rel}} = \frac{0.85 + 0.9}{1 + (0.85)(0.9)} = \frac{1.75}{1.765} = 0.9915$$

Thus adding a speed of 0.85 to a speed of 0.9 does not yield a resulting speed greater than 1, light speed.

The slowest laboratory speed for a daughter meson occurs when it is emitted in the negative x-direction in the rocket frame. In this case the velocity of the daughter meson is negative and the law of addition of velocities becomes a law of subtraction of velocities:

$$v_{min} = \frac{-v' + v_{rel}}{1 - v'v_{rel}} = \frac{-0.85 + 0.9}{1 - (0.85)(0.9)} = \frac{0.05}{0.235} = 0.2128$$

Although the minimum-speed meson moves to the left in the rocket, it moves to the right in the laboratory because of the very great speed of the original K^o-meson in the laboratory.

L.8 SUMMARY

Lorentz transformation deals with coordinates, not invariant quantities

Given the space and time coordinates of an event with respect to the reference event in one free-float frame, the **Lorentz coordinate transformation equations** tell us the coordinates of the same event in an overlapping free-float frame in relative motion with respect to the first. The equations that transform rocket coordinates (primed coordinates) to laboratory coordinates (unprimed coordinates) have the form

$$\begin{aligned} t &= \frac{v_{rel}x' + t'}{(1 - v_{rel}^2)^{1/2}} \\ x &= \frac{x' + v_{rel}t'}{(1 - v_{rel}^2)^{1/2}} \\ y &= y' \qquad \text{and} \qquad z = z' \end{aligned} \tag{L-10b}$$

where v_{rel} stands for relative speed of the two frames (rocket moving in the positive x-direction in the laboratory). The **inverse Lorentz transformation equations** transform laboratory coordinates to rocket coordinates:

$$\begin{aligned} t' &= \frac{-v_{rel}x + t}{(1 - v_{rel}^2)^{1/2}} \\ x' &= \frac{x - v_{rel}t}{(1 - v_{rel}^2)^{1/2}} \\ y' &= y \qquad \text{and} \qquad z' = z \end{aligned} \tag{L-11b}$$

in which v_{rel} is treated as a positive quantity. In both these sets of equations, coordinates of events are measured with respect to a reference event. It is really only the *difference* in coordinates between events that matter, for example $x_2 - x_1 = \Delta x$ for any two events 1 and 2, not the coordinates themselves. This is important in deriving the Law of Addition of Velocities.

The **Law of Addition of Velocities** or **Law of Combination of Velocities** in one dimension follows from the Lorentz transformation equations. This law tells us the velocity v of a particle in the laboratory frame if we know its velocity v' with respect to the rocket and relative speed v_{rel} between rocket and laboratory,

$$v = \frac{v' + v_{rel}}{1 + v'v_{rel}} \tag{L-13}$$

REFERENCE

Sample Problem L-3, The Firing Meson, was adapted from A. P. French, *Special Relativity* (W.W. Norton, New York, 1968), page 159.

SPECIAL TOPIC EXERCISES

PRACTICE

L-1 a super-speed super?

Take two more steps in the parable of the Great Betrayal (Box L-1).

a Find the speed of a new rocket frame moving relative to the Klingon frame such that the Super travels at 6 times the speed of light in this new frame. Hint: Examine the coordinates x' and t' of event 3 in the new frame. The ratio of these two, x'/t', is the speed of the Super in this frame. We know the coordinates of event 3 in the Klingon frame. Therefore . . .

b Find the speed of yet another rocket frame, relative to the Klingon frame, such that the Super travels with infinite speed in this frame. Hint: What does infinite speed imply about the time t' between events 0 and 3 in this new frame?

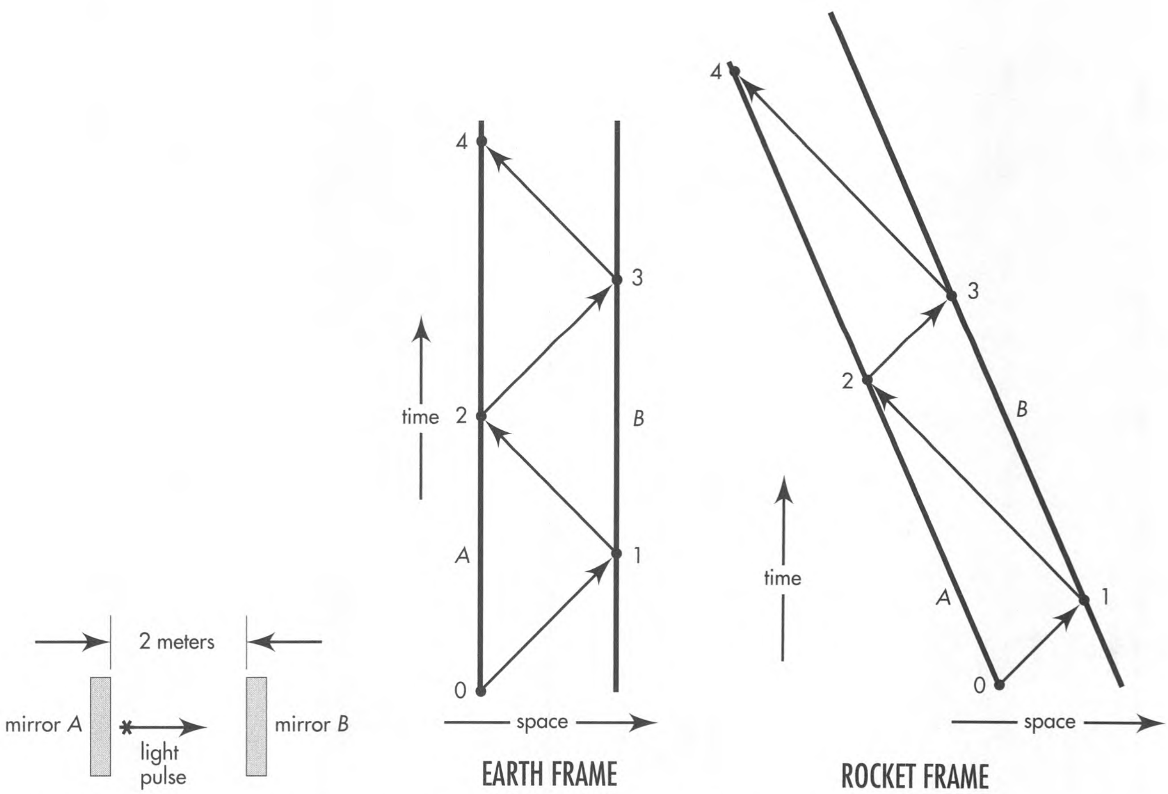
L-2 a bad clock

Note: This exercise uses spacetime diagrams, introduced in Chapter 5.

A pulse of light is reflected back and forth between mirrors A and B separated by 2 meters of distance in the x-direction in the Earth frame, as shown in the figure (left). A swindler tells us that this device constitutes a clock that "ticks" every time the pulse arrives at either mirror.

The swindler claims that events 1 through 6 are sequential "ticks" of this clock (center). However, we notice that the ticking of the clock is uneven in a rocket frame moving with speed v_{rel} in the Earth frame (right). For example, there is less time between events 0 and 1 than between events 1 and 2 as measured in the rocket frame.

a What is the physical basis for the "bad" behavior of this clock? Use the Lorentz transformation



EXERCISE L-2. *Left: Horizontal light-pulse clock as observed in the Earth frame.* ***Center:*** *Spacetime diagram showing worldlines of mirrors* A *and* B *and the "uniformly ticking" light pulse as observed in the Earth frame.* ***Right:*** *Time lapses between sequential ticks of the light-pulse clock are not uniform as observed in the rocket frame.*

equations to account for the uneven ticking of this clock in the rocket frame.

b Use some of the same events 0 through 4 to define a "good" clock that ticks evenly in both the laboratory frame and the rocket frame. From the spacetime diagrams, show qualitatively that your good clock "runs slow" as observed from the rocket frame—as it must, since the clock is in motion with respect to the rocket frame.

c Explain why the clock of Figure 1-3 in the text is a "good" clock.

L-3 the Galilean transformation

a Use everyday, nonrelativistic Newtonian arguments to derive transformation equations between reference frames moving at low relative velocities. Show that the result is

$$x' = x - v_{conv}\, t_{sec} \qquad \text{(Newtonian: } v_{conv} << c\text{)} \qquad (1)$$

$$t'_{sec} = t_{sec} \qquad \text{(Newtonian: } v_{conv} << c\text{)} \qquad (2)$$

where t_{sec} is time measured in seconds and v_{conv} is speed in conventional units (meters/second for example). List the assumptions you make in your derivation.

b Convert equations (1) and (2) to measure time t in meters and unitless measure of relative velocity, $v_{rel} = v_{con}/c$. Show the results are:

$$x' = x - v_{rel}\, t \qquad \text{(Newtonian: } v << 1\text{)} \qquad (3)$$

$$t' = t \qquad \text{(Newtonian: } v << 1\text{)} \qquad (4)$$

Do the new units make these equations correct at high relative velocity between frames?

c Use the first two terms in the binomial expansion to find a low-velocity approximation for γ in the Lorentz transformation.

$$\gamma = \frac{1}{(1 - v_{rel}^2)^{1/2}} = (1 - v_{rel}^2)^{-1/2} \approx 1 + \frac{v_{rel}^2}{2}$$

Show that this expression differs from unity by less than one percent provided v is less than $1/7$. A sports car can accelerate uniformly from rest to 60 miles/hour (about 27 meters/second) in 7 seconds. Roughly how many days would it take for the sports car to reach $v = 1/7$ at the same constant acceleration?

d Set $\gamma = 1$ in the Lorentz transformation equations. Show that the resulting "low-velocity Lorentz transformation" is

$$x' = x - v_{rel}\, t \qquad \text{(Lorentz: } v << 1\text{)} \qquad (5)$$

$$t' = -v_{rel}\, x + t \qquad \text{(Lorentz: } v << 1\text{)} \qquad (6)$$

What is the difference between the time transformations for the "Newtonian low-velocity limit" of equation (4) and the "Lorentz low-velocity limit" of equation (6)? How can they both be correct? The term $-v_{rel}x$ does not depend on any time lapse, but only on the separation x of the event from the laboratory origin. This term is due to the difference of synchronization of clocks in the two frames.

e In each of the following cases a laboratory clock (measuring t) at a distance x from the origin as measured in the laboratory frame is compared with a passing rocket clock (measuring t'). Say whether or not the time difference $t - t' = v_{rel}x$ can be detected using wristwatches (accuracy of 10^{-1} second $= 3 \times 10^7$ meters of light-travel time) and using modern electronic clocks (accuracy of 10^{-9} second $= 0.3$ meter of time).

(1) Sports car traveling at 100 kilometers/hour (roughly 30 meters/second) located 1000 kilometers down the road from the origin as measured in the Earth frame.

(2) Moon probe traveling at 30,000 kilometers/hour passing Moon, 3.8×10^5 kilometers from the origin on Earth as measured in the Earth frame.

(3) Distance from origin on Earth at which space probe traveling at 30,000 kilometers/hour leads to detectable time difference between rocket wristwatch and adjacent Earth-linked latticework clock. Compare with Earth–Sun distance of 1.5×10^{11} meters.

f Summarize in a sentence or two the conditions under which the regular Galilean transformation equations (3) and (4) will lead to correct predictions.

L-4 limits of Newtonian mechanics

Use the particle speed $v_{crit} = 1/7$ (Exercise L-3) as an approximate maximum limit for the validity of Newtonian mechanics. Determine whether or not Newtonian mechanics is adequate to analyze motion in each of the following cases, following the example.

Example: Satellite circling Earth at 30,000 kilometers/hour = 18,000 miles/hour. **Answer:** Light moves at a speed $v_{conv} = (3 \times 10^5$ kilometers/second) $\times$ (3600 seconds/hour) $= 1.08 \times 10^9$ kilometers/hour. Therefore the speed of the satellite in meters/meter is $v = v_{conv}/c = 2.8 \times 10^{-5}$. This is much less than $v_{crit} = 1/7$, so the Newtonian description of satellite motion is adequate.

a Earth circling Sun at an orbital speed of 30 kilometers/second.

b Electron circling a proton in the orbit of smallest radius in a hydrogen atom. **Discussion:** The classical speed of the electron in the inner orbit of an atom of atomic number Z, where Z is the number of protons in the nucleus, is given, for low velocities, by the expression $v = Z/137$. For hydrogen, $Z = 1$.

c Electron in the inner orbit of the gold atom, for which $Z = 79$.

d Electron after acceleration from rest through a voltage of 5000 volts in a black-and-white television picture tube. **Discussion:** We say that this electron has a kinetic energy of 5000 electron-volts. One electron-volt is equal to 1.6×10^{-19} joule. Try using the Newtonian expression for kinetic energy.

e Electron after acceleration from rest through a voltage of 25,000 volts in a color television picture tube.

f A proton or neutron moving with a kinetic energy of 10 MeV (million electron-volts) in a nucleus.

PROBLEMS

L-5 Doppler shift

A sparkplug at rest in the rocket emits light with a frequency f' pulses or waves per second. What is the frequency f of this light as observed in the laboratory? Let this train of waves (or pulses) of light travel in the positive x-direction with speed c, so that in the course of one meter of light-travel time, f/c of these pulses pass the origin of the laboratory frame. It is understood that the *zero*th or "fiducial" crest or pulse passes the origin at the zero of time—and that the origin of the rocket frame passes the origin of the laboratory frame at this same time.

a Show that the x-coordinate of the nth pulse or wave crest is related to the time of observation t (in meters) by the equation

$$n = (f/c)(t - x)$$

b The same argument, applied in the rocket frame, leads to the relation

$$n = (f'/c)(t' - x')$$

Express this rocket formula in laboratory coordinates x and t using the Lorentz transformation. Equate the resulting expression for f' to the laboratory formula for f in terms of x and t to derive the simple formula for f in terms of f' and v_{rel}, the relative speed of laboratory and rocket frames.

$$f = \left(\frac{1 + v_{rel}}{1 - v_{rel}}\right)^{1/2} f' \qquad \text{[wave moves in positive x-direction]}$$

c Now observe a wave moving along the negative x-direction from the same source at rest in the rocket frame. Show that the frequency of the wave observed in the laboratory frame is

$$f = \left(\frac{1 - v_{rel}}{1 + v_{rel}}\right)^{1/2} f' \qquad \text{[wave moves in negative x-direction]}$$

d Astronomers define the **redshift** z of light from a receding astronomical object by the formula

$$z = \frac{f_{emit} - f_{obs}}{f_{obs}}$$

Here f_{emit} is the frequency of the light measured in the frame in which the emitter is at rest and f_{obs} the frequency observed in another frame in which the emitter moves directly away from the observer.

The most distant quasar reported as of 1991 has a redshift $z = 4.897$. With what fraction of the speed of light is this quasar receding from us?

Reference: D. P. Schneider, M. Schmidt, and J. E. Gunn, *Astronomical Journal,* Volume 102, pages 837–840 (1991).

L-6 transformation of angles

a A meter stick lies at rest in the rocket frame and makes an angle ϕ' with the x'-axis. Laboratory observers measure the x- and y-projections of the stick as it streaks past. What values do they measure for these projections, compared with the x'- and y'-projections measured by rocket observers? Therefore what angle ϕ does the same meter stick make with the x-axis of the laboratory frame? What is the length of the "meter stick" as observed in the laboratory frame?

b Make the courageous assumption that the directions of electric-field lines around a point charge transform in the same way as the directions of meter sticks that lie along these lines. (Electric field lines around a point charge are assumed to be infinite in length, so the length transformation of part **a** does not apply.) Draw qualitatively the electric-field lines due to an isolated positive point charge at rest in the rocket frame as observed in (1) the rocket frame and (2) the laboratory frame. What conclusions follow concerning the time variation of electric forces on nearby charges at rest in the laboratory frame?

L-7 transformation of y-velocity

A particle moves with uniform speed $v'_y = \Delta y'/\Delta t'$ along the y'-axis of the rocket frame. Transform $\Delta y'$ and $\Delta t'$ to laboratory displacements Δx, Δy, and Δt using the Lorentz transformation equations. Show that the x-component and the y-component of the velocity of this particle in the laboratory frame are given by the expressions

$$v_x = v_{\text{rel}}$$

$$v_y = v_y'(1 - v_{\text{rel}}^2)^{1/2}$$

L-8 transformation of velocity direction

A particle moves with velocity v' in the $x'y'$ plane of the rocket frame in a direction that makes an angle ϕ' with the x'-axis. Find the angle ϕ that the velocity vector of this particle makes with the x-axis of the laboratory frame. (Hint: Transform space and time displacements rather than velocities.) Why does this angle differ from that found in Exercise L-6 on transformation of angles? Contrast the two results when the relative velocity between the rocket and laboratory frames is very great.

L-9 the headlight effect

A flash of light is emitted at an angle ϕ' with respect to the x'-axis of the rocket frame.

a Show that the angle ϕ the direction of motion of this flash makes with respect to the x-axis of the laboratory frame is given by the equation

$$\cos \phi = \frac{\cos \phi' + v_{\text{rel}}}{1 + v_{\text{rel}} \cos \phi'}$$

b Show that your answer to Exercise L-8 gives the same result when the velocity v' is given the value unity.

c A particle at rest in the rocket frame emits light uniformly in all directions. Consider the 50 percent of this light that goes into the forward hemisphere in the rocket frame. Show that in the laboratory frame this light is concentrated in a narrow forward cone of half-angle ϕ_o whose axis lies along the direction of motion of the particle. The half-angle ϕ_o is the solution to the following equation:

$$\cos \phi_o = v_{\text{rel}}$$

This result is called the **headlight effect.**

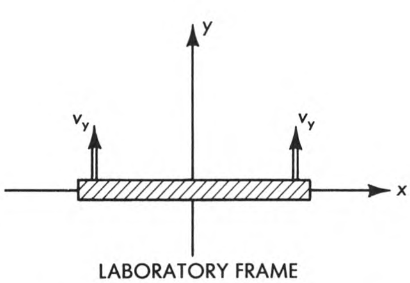
L-10 the tilted meter stick

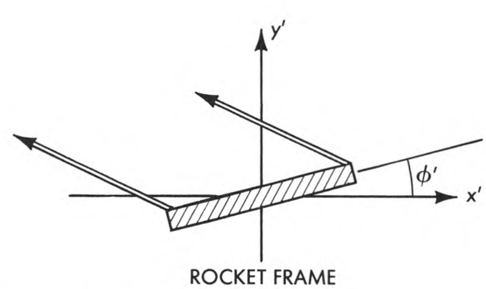
Note: This exercise uses the results of Exercise L-7.

A meter stick lying parallel to the x-axis moves in the y-direction in the laboratory frame with speed v_y as shown in the figure (left).

a In the rocket frame the stick is tilted upward in the positive x'-direction as shown in the figure (right). Explain why this is, first without using equations.

b Let the center of the meter stick pass the point $x = y = x' = y' = 0$ at time $t = t' = 0$. Calculate the angle ϕ' at which the meter stick is inclined to the x'-axis as observed in the rocket frame. **Discussion:** Where and when does the right end of the meter stick cross the x-axis as observed in the laboratory frame? Where and when does this event of right-end crossing occur as measured in the rocket frame? What is the direction and magnitude of the velocity of the meter stick in the rocket frame (Exercise L-7)? Therefore where is the right end of the meter stick at $t' = 0$, when the center is at the origin? Therefore . . .





EXERCISE L-10. ***Left:*** *Meter stick moving transverse to its length as observed in the laboratory frame.* ***Right:*** *Meter stick as observed in rocket frame.*

L-11 the rising manhole

Note: This exercise uses the results of Exercise L-10.

A meter stick lies along the x-axis of the laboratory frame and approaches the origin with velocity v_{rel}. A very thin plate parallel to the xz laboratory plane moves upward in the y-direction with speed v_y as shown in the figure. The plate has a circular hole with a diameter of one meter centered on the y-axis. The center of the meter stick arrives at the laboratory origin at the same time in the laboratory frame as the rising plate arrives at the plane $y = 0$. Since the meter stick is Lorentz-contracted in the laboratory frame it will easily pass through the hole in the rising plate. Therefore there will be no collision between meter stick and plate as each continues its motion. However, someone who objects to this conclusion can make the following argument: "In the rocket frame in which the meter stick is at rest the meter stick is not contracted, while in this frame the hole in the plate is Lorentz-contracted. Hence the full-length meter stick cannot possibly pass through the contracted hole in the plate. Therefore there must be a collision between the meter stick and the plate." Resolve this paradox using your answer to Exercise L-10. Answer unequivocally the question, Will there be a collision between the meter stick and the plate?

Reference: R. Shaw, *American Journal of Physics,* Volume 30, page 72 (1962).

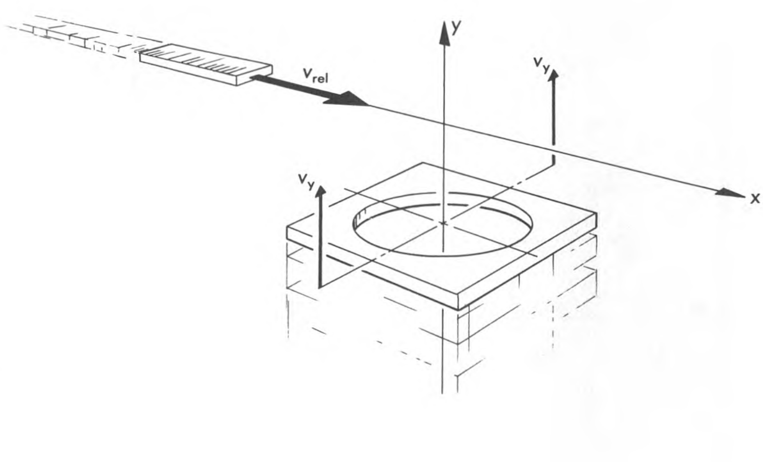
L-12 paradox of the skateboard and the grid

A girl on a skateboard moves very fast, so fast that the relativistic length contraction makes the skateboard very short. On the sidewalk she has to pass over a grid. A man standing at the grid fully expects the fast short skateboard to fall through the holes in the grid. Yet to the fast girl her skateboard has its usual length and it is the grid that has the relativistic contraction. To her the holes in the grid are much narrower than to the stationary man, and she certainly does not expect her skateboard to fall through them. Which person is correct? The answer hinges on the relativity of rigidity.

Idealize the problem as a one-meter rod sliding lengthwise over a flat table. In its path is a hole one meter wide. If the Lorentz contraction factor is ten, then in the table (laboratory) frame the rod is 10 centimeters long and will easily drop into the one-meter-wide hole. Assume that in the laboratory frame the meter stick moves fast enough so that it remains essentially horizontal as it descends into the hole (no "tipping" in the laboratory frame). Write an equation in the laboratory frame for the motion of the bottom edge of the meter stick assuming that $t = t' = 0$ at the instant that the back end of the meter stick leaves the edge of the hole. For small vertical velocities the rod will fall with the usual acceleration g. Note that in the laboratory frame we have assumed that every point along the length of the meter stick begins to fall simultaneously.

In the meter stick (rocket) frame the rod is one meter long whereas the hole is Lorentz-contracted to a 10-centimeter width so that the rod cannot possibly fit into the hole. Moreover, in the rocket frame different parts along the length of the meter stick begin to drop at different times, due to the relativity of simultaneity. Transform the laboratory equations into the rocket frame. Show that the front and back of the rod will begin to descend at different times in this frame. The rod will "droop" over the edge of the hole in the rocket frame—that is, it will not be rigid. Will the rod ultimately descend into the hole in both frames? Is the rod *really* rigid or nonrigid during the experiment? Is it possible to derive any physical characteristics of the rod (for example its flexibility or compressibility) from the description of its motion provided by relativity?

Reference: W. Rindler, *American Journal of Physics,* Volume 29, page 365–366 (1961).



EXERCISE L-11. *Will the "meter stick" pass through the "one-meter-diameter" hole without collision?*

L-13 paradox of the identically accelerated twins

Note: This exercise uses spacetime diagrams, introduced in Chapter 5.

Two fraternal twins, Dick and Jane, own identical spaceships each containing the same amount of fuel. Jane's ship is initially positioned a distance to the right of Dick's in the Earth frame. On their twentieth birthday they blast off at the same instant in the Earth frame and undergo identical accelerations to the right as measured by Mom and Dad, who remain at home on Earth. Mom and Dad further observe that the twins run out of fuel at the same time and move thereafter at the same speed v. Mom and Dad also measure the distance between Dick and Jane to be the same at the end of the trip as at the beginning.

Dick and Jane compare the ships' logs of their accelerations and find the entries to be identical. However when both have ceased accelerating, Dick and Jane, in their new rest frame, discover that Jane is older than Dick! How can this be, since they have an identical history of accelerations?

a Analyze a simpler trip, in which each spaceship increases speed not continuously but by impulses, as shown in the first spacetime diagram and the event table. How far apart are Dick and Jane at the beginning of their trip, as observed in the Earth frame? How far apart are they at the end of their accelerations? What is the final speed v (not the average speed) of the two spaceships? How much does each astronaut age along the worldline shown in the diagram? (The answer is not the Earth time of 12 years.)

b The second spacetime diagram shows the two worldlines as recorded in a rocket frame moving with the final velocity of the two astronauts. Copy the figure. On your copy extend the worldlines of Dick and Jane after each has ceased accelerating. Label your figure to show that Jane ceased accelerating before Dick as observed in this frame. Will Dick age the same between events 0 and 3 in this frame as he aged in the Earth frame? Will Jane age the same between events 4 and 7 in this frame as she aged in the Earth frame?

c Now use the Lorentz transformation to find the space and time coordinates of one or two critical events in this final rest frame of the twins in order to answer the following questions

(1) How many years earlier than Dick did Jane cease accelerating?

(2) What is Dick's age at event 3? (not the rocket time t' of this event!)

(3) What is Jane's age at event 7?

(4) What is Jane's age at the same time (in this frame) as event 3?

(5) What are the ages of Dick and Jane 20 years after event 3, assuming that neither moves again with respect to this frame?

(6) How far apart in space are Dick and Jane when both have ceased accelerating?

(7) Compare this separation with their initial (and final!) separation measured by Mom and Dad in the Earth frame.

d Extend your results to the general case in which Mom and Dad on Earth observe a period of identical *continuous* accelerations of the two twins.

(1) At the two start-acceleration events (the two events at which the twins start their rockets), the twins are the same age as observed in the Earth frame. Are they the same age at these events as observed in every rocket frame?

(2) At the two cease-acceleration events (the two events at which the rockets run out of fuel), are the twins the same age as observed in the Earth frame? Are they the same age at these events as observed in every rocket frame?

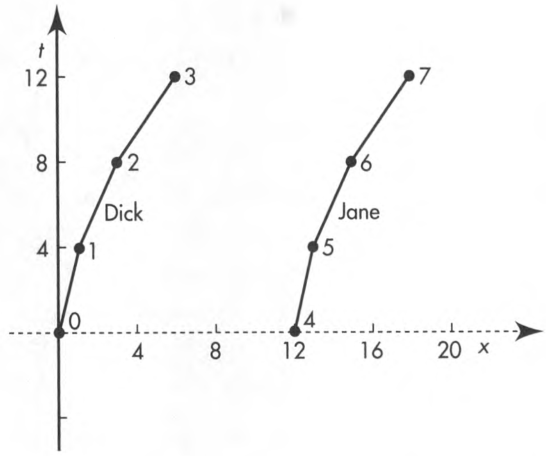
(3) The two cease-acceleration events are simultaneous in the Earth frame. Are they simultaneous as observed in every rocket frame? (No!) Whose cease-acceleration event occurs first as observed in the final frame in which both twins come to rest? (Recall the Train Paradox, Section 3.4.)

(4) "When Dick ceases accelerating, Jane is older than Dick." Is this statement true according to the astronauts in their final rest frame? Is the statement true according to Mom and Dad in the Earth frame?

(5) Criticize the lack of clarity (swindle?) of the word *when* in the statement of the problem: "However when both have ceased accelerating, Dick and Jane, in their new rest frame, discover that Jane is older than Dick!"

e Suppose that Dick and Jane both accelerate to the left, so that Dick is in front of Jane, but their history is otherwise the same. Describe the outcome of this trip and compare it with the outcome of the original trip.

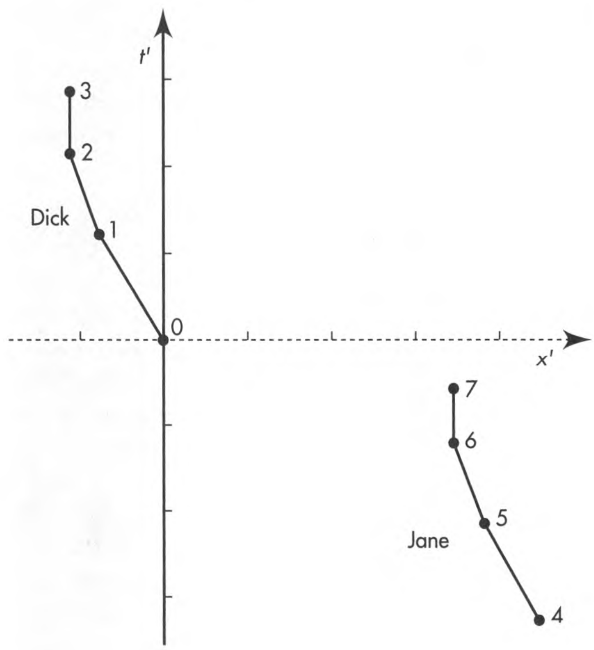
f Suppose that Dick and Jane both accelerate in a direction perpendicular to the direction of their separation. Describe the outcome of this trip and compare it with the outcome of the original trip.



Earth Frame Observations

Event number	x-position (light years)	Time (years)
0	0	0
1	1	4
2	3	8
3	6	12
4	12	0
5	13	4
6	15	8
7	18	12

EARTH FRAME



ROCKET FRAME

EXERCISE L-13. ***Top:*** *Worldlines of Dick and Jane as observed in the Earth frame of Mom and Dad.* ***Bottom:*** *Worldlines of Dick and Jane as observed in the "final" rocket frame in which both Dick and Jane come to rest after burnout.*

Discussion: Einstein postulated that physics in a uniform gravitational field is, locally and for small particle speeds, the same as physics in an accelerated frame of reference. In this exercise we have found that two accelerated clocks separated along the direction of acceleration do not remain in synchronism as observed simultaneously in their common frame. Rather, the forward clock reads a later time ("runs faster") than the rearward clock as so observed. Conclusion from Einstein's postulate: Two clocks one above the other in a uniform gravitational field do not remain in synchronism; rather the higher clock reads a later time ("runs faster") than the lower clock. General relativity also predicts this result, and experiment verifies it. (Read about the patrol plane experiment in Section 4.10.)

Reference: S. P. Boughn, *American Journal of Physics,* Volume 57, pages 791–793 (September 1989). Reference to general relativity result: Wolfgang Rindler, *Essential Relativity* (Springer, New York, 1977), pages 17 and 117.

L-14 how do rods Lorentz-contract?

Note: Calculus is used in the solution to this exercise; so is the formula for Lorentz contraction from Section 5.8.

Laboratory observers measure the length of a moving rod lying along its direction of motion in the laboratory frame. Then the rod speeds up a little. Again laboratory observers measure its length, which they find to be a little shorter than before. They call this shortening of length Lorentz contraction. How did this shortening of length come about? As happens so often in relativity, the answer lies in the relativity of simultaneity.

First, how much shortening takes place when the rod changes from speed v to speed $v + dv$? Let L_o be the proper length of the rod when measured at rest. At speed v its laboratory-measured length L will be shorter than this by the Lorentz contraction factor (Section 5.8):

$$L = (1 - v^2)^{1/2} L_o$$

a Using calculus, show that when the rod speeds up from v to a slightly greater speed $v + dv$, the change in length dL is given by the expression

$$dL = -\frac{L_o v dv}{(1 - v^2)^{1/2}}$$

The negative sign means that the change is a shortening of the rod. We want to explain this change in length.

How is the rod to be accelerated from v to $v + dv$? Fire a rocket attached to the rear of the rod? No. Why not? Because the rocket pushes only against the rear of the rod; this push is transmitted along the rod to the front at the speed of a compression wave — very slow! We want the front and back to change speed "at the same time" (exact meaning of this phrase to be determined later). How can this be done? Only by prearrangement! Saw the rod into a thousand equal pieces and tap each piece in the forward direction with a mallet "at exactly 12 noon" as read off a set of synchronized clocks. To simplify things for now, set aside all but the front and back pieces of the rod. Now tap the front and back pieces "at the same time." The change in length of the rod dL is then the change in distance between these two pieces as a result of the tapping. So much for how to accelerate the "rod."

Now the central question: What does it mean to tap the front and back pieces of the rod "at the same time"? To answer this question, ask another: What is our final goal? Answer: To account for the Lorentz contraction of a fast-moving rod of proper length L_o. More: We want a careful inspector riding on the fast-moving rod to certify that it has the same proper length L_o as it did when it was at rest in the laboratory frame. To achieve this goal, the inspector insists that the pair of accelerating taps be applied to the front and back rod pieces at the same time *in the current rest frame of the rod*. Otherwise the distance between these pieces would not remain the same in the frame of the rod; the rod would change proper length. [Notice that in Exercise L-13 the taps occur at the same time in the laboratory (Earth) frame. This leads to results different from those of the present exercise.]

b You are the inspector riding along with the front and back pieces of the rod. Consider the two events of tapping the front and back pieces. How far apart $\Delta x'$ are these events along the x-axis in your (rocket) frame? How far apart $\Delta t'$ in time are these events in your frame? Predict how far apart in time Δt these events are as measured in the laboratory frame. Use the Lorentz transformation equation (L-10):

$$\Delta t = v\,\gamma \Delta x' + \gamma \Delta t'$$

The relative velocity v_{rel} in equation (L-10) is just v, the current speed of the rod. In the laboratory frame is the tap on the rear piece earlier or later than the tap on the front piece?

Your answer to part **b** predicts how much earlier the laboratory observer measures the tap to occur on the back piece than on the front piece of the rod. Let the tap increase the speed of the back end by dv as measured in the laboratory frame. Then during laboratory time Δt the back end is moving at a speed dv faster than the front end. This relative motion will shorten the distance between the back and front ends. After time interval Δt the front end receives the identical tap, also speeds up by dv, and once again moves at the same speed as the back end.

c Show that the shortening dL predicted by this analysis is

$$dL = -dv\Delta t = -\gamma \Delta x' v dv = -v\gamma L_o dv$$
$$= -\frac{L_o v dv}{(1 - v^2)^{1/2}}$$

which is identical to the result of part **a**, which we wanted to explain. QED.

d Now start with the front and back pieces of the rod at rest in the laboratory frame and a distance L_o apart. Tap them repeatedly and identically. As they speed up, be sure these taps take place simultaneously in the rocket frame in which the two ends are currently at rest. (This requires you, the ride-along inspector, to resynchronize your rod-rest-frame clocks after each set of front-and-back taps.) Make a logically rigorous argument that after many taps, when the rod is moving at high speed relative to the laboratory, the length of the rod measured in the laboratory can be reckoned using the first equation given in this exercise.

e Now, by stages, put the rod back together. The full thousand pieces of the rod, lined up but not touching, are all tapped identically and at the same time in the current rest frame of the rod. One set of taps increases the rod's speed from v to $v + dv$ in the laboratory frame. Describe the time sequence of these thousand taps as observed in the laboratory frame. If you have studied Chapter 6 or the equivalent, answer the following questions: What kind of interval—timelike, lightlike, or spacelike—separates any pair of the thousand taps in this set? Can this pair of taps be connected by a light flash? by a compression wave moving along the rod when the pieces are glued back together? Regarding the "logic of acceleration," is there any reason why we should *not* glue these pieces back together? Done!

f During the acceleration process is the reglued rod *rigid*—unchanging in dimensions—as observed in the rod frame? As observed in the laboratory frame? Is the *rigidity* property of an object an invariant, the same for all observers in uniform relative motion? Show how an ideal rigid rod could be used to transmit signals instantaneously from one place to another. What do you conclude about the idea of a "rigid body" when applied to high-speed phenomena?

Reference: Edwin F. Taylor and A. P. French, *American Journal of Physics,* Volume 51, pages 889–893, especially the Appendix (1983).

L-15 the place where both agree

At any instant there is just one plane in which both the laboratory and the rocket clocks agree.

a By a symmetry argument, show that this plane lies perpendicular to the direction of relative motion. Using the Lorentz transformation equations, show that the velocity of this plane in the laboratory frame is equal to

$$v_{t=t'} = \frac{1}{v_{\text{rel}}}[1 - (1 - v_{\text{rel}}^2)^{1/2}]$$

b Does the expression for $v_{t=t'}$ seem strange? From our everyday experience we might expect that by symmetry the "plane of equal time" would move in the laboratory at half the speed of the rocket. Verify that indeed this is correct for the low relative velocities of our everyday experience. Use the first two terms of the binomial expansion

$$(1 + z)^n \approx 1 + nz \text{ for } |z| << 1$$

to show that for low relative velocity, $v_{t=t'} \longrightarrow v_{\text{rel}}/2$.

c What is $v_{t=t'}$ for the extreme relativistic case in which $v_{\text{rel}} \longrightarrow 1$? Show that in this case $v_{t=t'}$ is completely different from $v_{\text{rel}}/2$.

d Suppose we want to go from the laboratory frame to the rocket frame in two equal velocity jumps. Try a first jump to the plane of equal laboratory and rocket times. Now symmetry does work: Viewed from this plane the laboratory and rocket frames move apart with equal and opposite velocities, whose magnitude is given by the equation in part **a**. A second and equal velocity jump should then carry us to the rocket frame at speed v_{rel} with respect to the laboratory. Verify this directly by using the Law of Addition of Velocities (Section L.7) to show that

$$v_{\text{rel}} = \frac{v_{t=t'} + v_{t=t'}}{1 + v_{t=t'}v_{t=t'}}$$

L-16 Fizeau experiment

Light moves more slowly through a transparent material medium than through a vacuum. Let v_{medium} represent the reduced speed of light measured in the frame of the medium. Idealize to a case in which this reduced velocity is independent of the wavelength of the light. Place the medium at rest in a rocket moving at velocity v_{rel}, to the right relative to the laboratory frame, and let light travel through the medium, also to the right. Use the Law of Addition of Velocities (Section L.7) to find an expression for the velocity v of the light in the laboratory frame. Use the first two terms of the binomial expansion

$$(1 + z)^n \approx 1 + nz \text{ for } |z| << 1$$

to show that for small relative velocity v_{rel} between the rocket and laboratory frames, the velocity v of the light with respect to the laboratory frame is given approximately by the expression

$$v \approx v_{\text{medium}} + v_{\text{rel}}(1 - v_{\text{medium}}{}^2)$$

This expression has been tested by Fizeau using water flowing in opposite directions in the two arms of an interferometer similar (but not identical) to the interferometer used later by Michelson and Morley (Exercise 3-12).

Reference: H. Fizeau, *Comptes rendus,* Volume 33, pages 349–355 (1851). A fascinating discussion (in French) of some central themes in relativity theory—delivered more than fifty years before Einstein's first relativity paper.