CHAPTER 5

TREKKING THROUGH SPACETIME

5.1 TIME? NO. SPACETIME MAP? YES.

no such thing as the unique time of an event!

Events are the sparkling grains of history. They define spacetime. Spacetime, yes. Time, no.

"Time" of an event has no unique meaning

"Time, no"? How come? Time here in Tokyo, at this enthronement of the successor of the Emperor Hirohito? Where is any meter to be seen that shows any such quality of location as time? Meter to measure the temperature here and now? Yes, this thermometer. Meter to measure atmospheric pressure here and now? Yes, this barometer. But look as we will, nowhere can we see any meter that we can poke into the space hereabouts to measure its "time." The time of an event? Impossible! No such thing. Time is not "meterable."

Anything with which to compare time? Yes. Odometer reading, whether miles or kilometers, on the dashboard of our car. There's no such thing as the odometer reading of Tokyo. Try every gadget one can, thrust it out into this Tokyo air, not one will register anything with the slightest claim to be called the odometer reading of these hereabouts.

Car mileage depends on car's path between places

What about looking at the dashboards of the cars in this neighborhood? Not all of them; that would be nonsense. Only the cars that were new, with odometer reading zero, at the time of Hirohito's own enthronement.

Now at last we are getting into a line of questioning that shows some prospect of clearing up what we mean by "time." We ask our companion, "What do all those day-and-year-counting wristwatches now read that were set to zero at the time of that earlier ceremony?"

"Sixty-two years, two days," is her first reply. But then we ask, "What about that team that zoomed out to the nearest eye-catching star, Alpha Centauri, and back with almost the speed of light? Didn't they get back ten years younger than we stay-at-homes?"

Wristwatch reading depends on its history of travel between events

"Yes," she agrees, "surely their wristwatches now read fifty-two years, not sixty-two. So let me draw the lesson. There is no such thing as time. There is only totalized *interval* of time, time as that interval is racked up between the enthronement of Hirohito and the enthronement of the new Emperor Akihito, between event A and event B, on a wristwatch that has undergone its own individual history of travel from A to B."

"I agree. The concept of time does not apply to location in spacetime. It applies to individual history of travel through spacetime."

"How apt the comparison with odometer reading. Each dashboard shows, not the kilometerage of Akihito, but the kilometers traveled by that particular car between the one imperial ceremony and the other."

Geographic map assigns kilometer coordinates to places

Yes, it is nonsense to attribute a kilometer reading to Tokyo. However, it is not at all nonsense to make a map showing where Tokyo lies relative to all the towns roundabout, a map in which kilometers do appear, kilometers north and south, kilometers east and west. Likewise the term "the time" of an event is totally without meaning. However, that event—and every event near it—lends itself to display on a spacetime diagram (Figure 5-1), with distance (the locator of latticework clock) running in one direction, and in another direction time (the reading printed out by that clock on the occasion of that event). Time as employed in this sense acquires meaning only because it serves as a measure on a latticework-defined map. A different latticework? A different set of clocks, different readings on those clocks, a different map—but same events, same spacetime, same tools to measure the history-dependent interval between event and event.

Spacetime map assigns space and time coordinates to events

Only on such a spacetime plot does one see at a glance the layout of all nearby events, and how one history of travel from event A to event B differs from another.

Limit attention to one space dimension plus one time dimension

One problem in making our map: Spacetime has four dimensions—three space dimensions plus time. We picture our event points most readily when they occupy a two-dimensional domain and let themselves be dotted in on a two-dimensional page. Therefore for the present we limit attention to time and one space dimension; to events, whatever their timing, that occur on one line in space. All events that do not occur on this line we ignore for now. The space location of each event on this line we plot along a **horizontal axis** on the page. The lattice-clock time at which an event occurs we plot along a **vertical axis,** from bottom to top of the page. Space and time we measure in the same unit, for example meters of distance and meters of time—or light-years of distance and years of time. We call the result a **spacetime map** or a **spacetime diagram.** Each spacetime map represents data from a particular reference frame, for example "the laboratory frame." Figure 5-1 shows such a spacetime map.

Five sample event points appear on the laboratory spacetime map of Figure 5-1, events labeled O, A, B, C, and D.

- **Event O** is the **reference event,** the firing of the starting gun, which we take to locate zero position in space and the zero of time. For our own convenience, we place point O at the origin of the spacetime map and measure space and time locations of all other events with respect to it.

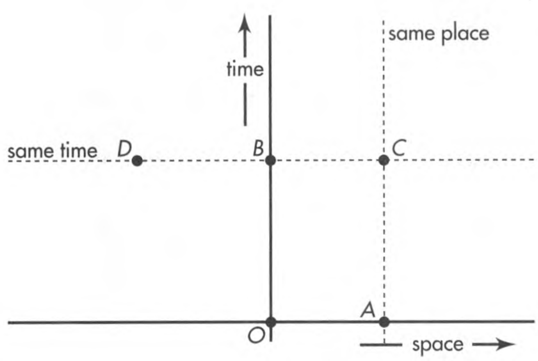


FIGURE 5-1. *Laboratory spacetime map, showing the reference event* O, *other events* A, B, C, *and* D, *a horizontal dashed line of simultaneity in time, and a vertical dashed line of equal position in space.*

- **Event B** stands on the vertical time axis, directly above reference event O. Therefore event B occurs at a later time than event O. Event B lies neither to the right of the reference event nor to the left; its horizontal (space) location is zero. Therefore it occurs at the same place as the reference event O in the laboratory but later in time.
- **Event A** lies on the horizontal space axis, directly to the right of reference event O. Therefore event A occurs at a different space location than event O. It is neither above nor below event O; its vertical (time) location is zero. Therefore it occurs at the same time as reference event O as observed in the laboratory.
- **Event C** rests above and to the right of the reference event. Standing higher than the reference event on the map, event C occurs later in time than O in this frame. Since it lies to the right, event C occurs at a positive space location with respect to event O in this frame.
- **Event D** reposes above and to the left of the reference event. It also occurs later in time than reference event O but at a negative space location with respect to event O as observed in the laboratory.

Scatter other event points on the spacetime map. Each event point can represent an important happening. Then a single glance at the spacetime map gives us, in principle, a global picture of *all* significant events that have occurred along *one* line in space and as far back in time as we wish to look. The spacetime map puts all this history at our fingertips!

In exploring history, we may want to know which events occurred at the same time as others in the laboratory free-float frame. Two events that occur at the same time have the same vertical (time) location on the spacetime map. A horizontal line drawn through one event point passes through all events simultaneous with that event in the given frame. In Figure 5-1, the dashed horizontal line shows that events B and D are simultaneous as observed in the laboratory frame, although they occur at different locations in space. Similarly, events O and A are simultaneous as observed in this frame.

Horizontal line on spacetime diagram picks out events that are simultaneous in this frame

When we wish to "retell history," we draw a sequence of horizontal lines above one another on the spacetime map. We mimic the advance of time by stepping in imagination from one horizontal line to the next horizontal line above it, noting which events occur at each time.

Vertical lines on the spacetime map indicate which events occur at the same place along the single line in space. Events A and C in Figure 5-1 occur at the same space location as measured in the laboratory, but at different times as measured in this frame. Similarly, events O and B occur at the same place as one another in the laboratory.

5.2 SAME EVENTS; DIFFERENT FREE-FLOAT FRAMES

different frames: different points for an event on their spacetime maps, but same spacetime interval between two events

Figure 5-1 demonstrates two great payoffs of the spacetime map: (1) It places space and time on an equal footing, thus recognizing a basic symmetry of nature. (2) It allows us to review at a single glance the whole history of events and motions that have occurred along the given line in space.

Same events, different frames: Different spacetime maps

We want to take advantage of a third payoff of the spacetime map: Plot the same events on two, three, or more spacetime maps based on two, three, or more different free-float frames in uniform relative motion. Compare. In this way analyze the various space and time relations among these events as measured in different frames. Why do this? In order to find out what is different in the different frames and what remains the same.

Figure 5-3 shows three spacetime maps — for laboratory, rocket, and super-rocket free-float frames. The super-rocket moves faster than the rocket with respect to the laboratory (but not faster than light!). On each of the three spacetime maps we plot the same two events: the events of emission E and reception R of a light flash. These are the two events analyzed in Chapter 3 to derive the expression for the spacetime interval. As a reminder of the physical phenomena behind events E and R, refer to Figure 5-2.

The light flash is emitted (event E) from a sparkplug attached to the reference clock of the first rocket. Take event E as the reference event, called event O in Figure 5-1. By prearrangement the sparkplug fires at the instant when both the rocket reference clock and the super-rocket reference clock pass the laboratory reference clock. All three

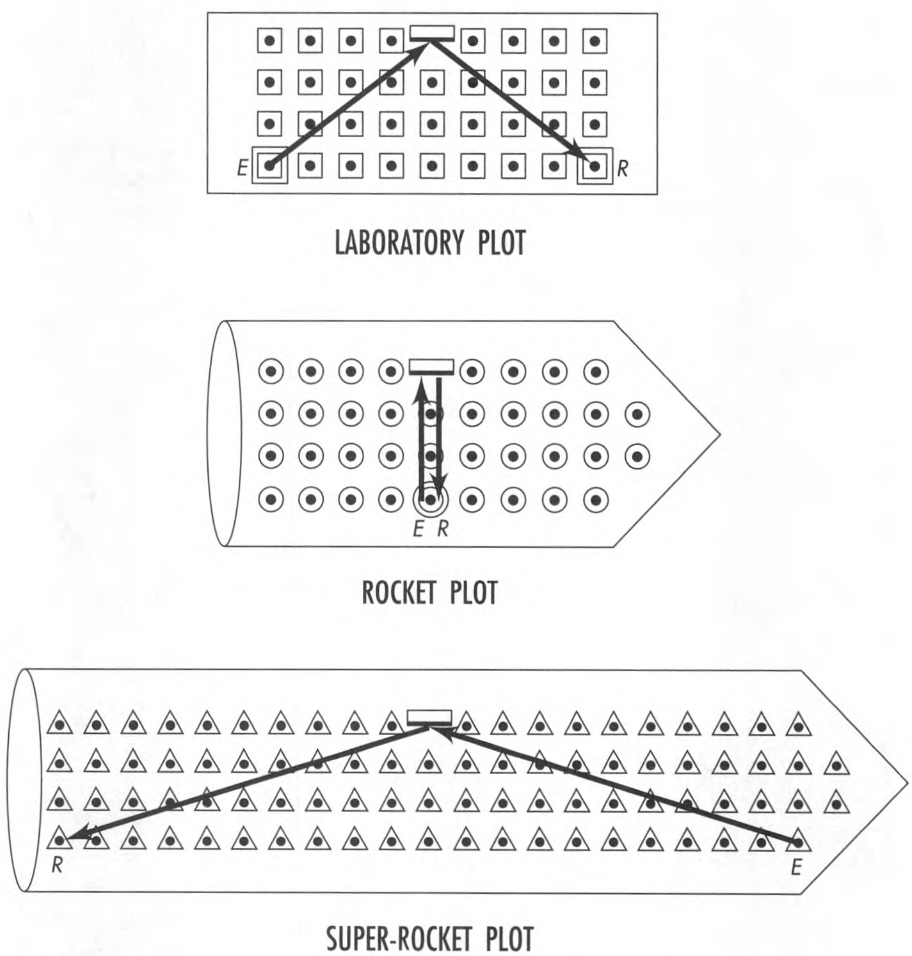


FIGURE 5-2 *(Figure 3-5 repeated).* ***The flash path as recorded in three different frames, showing event* E*, emission of the flash, and event* R*, its reception after reflection.*** *Squares, circles, and triangles represent the latticework of recording clocks in laboratory, rocket, and super-rocket frames, respectively. The super-rocket frame moves to the right with respect to the rocket, so that the event of reception,* R*, occurs to the left of the event of emission,* E*, as measured in the super-rocket frame. The reflecting mirror is fixed in the rocket, hence appears to move from left to right in the laboratory and from right to left in the super-rocket.*

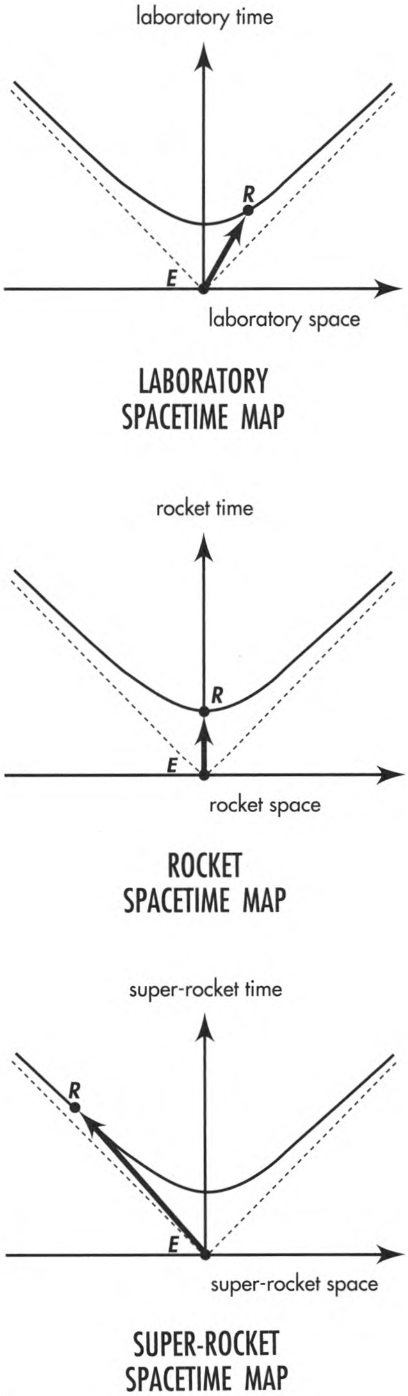


FIGURE 5-3. ***Spacetime maps for three frames, showing emission of the reference flash and its reception after reflection.*** *The hyperbola drawn in each map satisfies the equation for the invariant interval (or proper time), which has the same value in all three frames:* $(interval)^2 = (time)^2 - (space)^2$.

reference clocks are set to read zero at this reference event, whose event point is placed at the origin of all three spacetime maps.

Now use the latticework of meter sticks and clocks in each free-float frame (clocks pictured in Figure 5-2) to measure the position and time of every other event with respect to the reference event. In particular, record the position and time of the reception (event *R*) of the flash in each of the three frames.

The reception of the light ray (event *R*) occurs at different locations and at different times as measured in the three frames. In the rocket the reception of the reflected flash occurs back at the reference clock (the zero of position) and 6 meters of time later, as seen in Figure 5-2 and more directly in Figure 5-3 (center):

Same events, different frames: Different space and time coordinates

Rocket: (position of reception, event *R*) = 0
Rocket: (time of reception, event *R*) = 6 meters

Emission and reception occur at the same place in the rocket frame. Therefore the rocket time, 6 meters, is just equal to the interval, or proper time, between these two events:

$$(\text{proper time})^2 = \overset{\textbf{Rocket}}{\begin{pmatrix}\text{time of}\\ \text{reception}\end{pmatrix}^2} - \overset{\textbf{Rocket}}{\begin{pmatrix}\text{position of}\\ \text{reception}\end{pmatrix}^2}$$

$$= \overset{\textbf{Rocket}}{\begin{pmatrix}\text{time of}\\ \text{reception}\end{pmatrix}^2} - \overset{\textbf{Rocket}}{(\text{zero})^2} = (6\text{ meters})^2$$

In the laboratory the reception event R occurs at a time greater than 6 meters, as can be seen from the expression for interval:

$$\overset{\textbf{Laboratory}}{\begin{pmatrix}\text{time of}\\ \text{reception}\end{pmatrix}^2} - \overset{\textbf{Laboratory}}{\begin{pmatrix}\text{position of}\\ \text{reception}\end{pmatrix}^2} = (6\text{ meters})^2$$

In this equation the square of 6 meters results from subtracting a positive quantity from the square of the laboratory time of reception. Therefore the laboratory time of reception itself must be greater than 6 meters:

Laboratory: (position of reception, event *R*) = 8 meters
Laboratory: (time of reception, event *R*) = 10 meters

In the laboratory frame, reception appears to the right of the emission, as seen in Figure 5-2. Hence it is plotted to the right of the origin in the laboratory map (Figure 5-3, top).

Same events, different frames: Same spacetime interval

In the super-rocket frame, moving faster than the rocket with respect to the laboratory, the event of reception appears to the left of the emission (Figure 5-2). Therefore the space separation is called negative and plotted to the left of the origin in the super-rocket map (Figure 5-3, bottom). The time separation in the super-rocket is greater than 6 meters, by the same argument used for the time of reception in the laboratory frame:

$$\overset{\textbf{Super-rocket}}{\begin{pmatrix}\text{time of}\\ \text{reception}\end{pmatrix}^2} - \overset{\textbf{Super-rocket}}{\begin{pmatrix}\text{position of}\\ \text{reception}\end{pmatrix}^2} = (6\text{ meters})^2$$

In this equation, the space separation is a negative quantity. Nevertheless its square is a positive quantity. So the equation says that the square of 6 meters results from subtracting a positive quantity from the square of the super-rocket time of reception. Therefore the super-rocket time separation must also be greater than 6 meters:

Super-rocket: (position of reception, event *R*) = −20 meters
Super-rocket: (time of reception, event *R*) = 20.88 meters

5.3 INVARIANT HYPERBOLA

all observers agree: "event point lies somewhere on this hyperbola"

Different reception points marked R in different spacetime maps all refer to the same event. What do these different separations of the same event from the reference event have in common? They all satisfy invariance of the interval, reflected in the equation

$$(\text{time separation})^2 - (\text{space separation})^2 = (\text{interval})^2 = \text{constant}$$

Constant? Constant with respect to what?

With respect to free-float frame. Record different space and time measurements in different frames, but figure out from them always the same interval.

Curves drawn on the three maps conform to this equation. This kind of curve, in which the difference of two squares equals a constant, is called a **hyperbola.** Somewhere on this hyperbola is recorded the time and position of one and the same reception event as measured in every possible rocket and super-rocket frame. Same reception event, different frames, all summarized in one hyperbola, the **invariant hyperbola.**

Invariant hyperbola: Locus of same event in all rocket frames

Spacetime arrows in all three maps connect the same pair of events. They imply the identical invariant interval. They embody the same spacetime reality. In a deep sense these three arrows on the page represent the same arrow in spacetime. Spacetime maps of different observers show different projections—different perspectives—of the same arrow in spacetime.

The same arrow? The same magnitude for the spacetime arrow pictured in all three maps of Figure 5-3? Then why do the three arrows have obviously different lengths in the three maps?

Because the paper picture of spacetime is a lie! The length of an arrow on a piece of paper is Euclidean, related to the sum of squares of the space separations of the endpoints in two perpendicular directions. Euclidean geometry works fine if what is being represented is flat space, for example the map of a township. But Euclidean geometry is the wrong geometry and betrays us when we try to lay out time along one direction on the page. Instead we need to use Lorentz geometry of spacetime. In Lorentz geometry, time must be combined with space through a *difference* of squares to find the correct magnitude of the resulting spacetime vector—the interval. That is why the arrows in the different spacetime maps of Figure 5-3 seem to be of different lengths. The reality that these lengths represent, however—the value of the interval between two events—is the same in all three spacetime maps.

5.4 WORLDLINE

the moving particle traces out a line—its *worldline*—on the spacetime diagram

We describe the world by listing events and showing how they relate to one another. Until now we have focused on pairs of events and spacetime intervals between them. Now we turn to a whole chain of events, events that track the passage of a particle through spacetime. Think of a speeding sparkplug that emits a spark every meter of time read on its own wristwatch. Each spark is an event; the collection of spark events forms a chain that threads through spacetime, like pearls. String the pearls together. The thread connecting the pearl events, tracing out the path of a particle through spacetime, has a wonderfully evocative name: **worldline.** The sparkplug travels through spacetime trailing its worldline behind it.

String of event pearls: Worldline!

The speeding sparkplug is only an example. Every particle has a worldline that connects events along its spacetime path, events such as collisions or near-collisions (close calls) with other particles.

Events — pearls in spacetime — exist independent of any reference frame we may choose to describe them. A worldline strings these event pearls together. The worldline, too, exists independent of any reference frame. A particle traverses spacetime — follows a worldline — totally oblivious to our poor efforts to describe its motion using one or another free-float frame. Yet we are accustomed to using a free-float frame and its associated latticework of rods and clocks. One clock after another records its encounter with the particle. The worldline of the particle connects this chain of encounter events.

Worldline versus line on spacetime map

We can draw this worldline of a particle on the spacetime map for this reference frame. Such worldlines are shown in Figure 5-5 and in later figures of this chapter. Strictly speaking, the line drawn on the spacetime map is not the worldline itself. It is an image of the worldline — a strand of ink printed on a piece of paper. When we use a highway map, we often refer to a line drawn on the paper as "the highway." Yet is not the highway itself, but an image. Ordinarily this causes no confusion; no one tries to drive a car across a highway map! Similarly, we loosely refer to the line drawn on the spacetime map as the worldline, even though the worldline in spacetime stands above and beyond all our images of it.

Examples of worldlines

The worldline is seen in no way more clearly than through example. Particle 1 starts at the laboratory reference clock at zero time and moves to the right with constant speed (Figure 5-4). As particle 1 zooms along a line of laboratory latticework of clocks, each clock it encounters records the time at which the particle passes. Each clock record shows where the clock is located and the time at which particle 1 coincides with the clock. "Where and when" determines an event, the **event of coincidence** of particle and recording clock. Afterwards the chief observer travels throughout the lattice of clocks, collecting the records of these coincidence events. She plots these events as points on her spacetime map. She then draws a line through event points in sequence — the worldline of particle 1 (Figure 5-5).

Particle 1 moves with constant speed along a single direction in space. The distance it covers is equal for each tick of the laboratory clocks. The worldline of particle 1 shows equal changes in space during equal lapses of time by being straight on the spacetime map.

Particle 2 moves to the right faster than particle 1 and so covers a greater distance in the same time lapse (Figure 5-4). Lattice clocks record their events of coincidence with particle 2, and the observer collects these records and plots the worldline of particle 2 on the same spacetime map (worldline shown in Figure 5-5).

And so it goes: Particle 3 is a light flash and moves to the right in space (Figure 5-4) with maximum speed: one meter of distance per meter of time. With horizontal and vertical axes calibrated in meters, the light-flash worldline rises at an angle of 45 degrees (Figure 5-5).

Particle 4 does not move at all in laboratory space; it rests quietly next to the laboratory reference clock. Like you sitting in your chair, it moves only along the time dimension; in the laboratory spacetime map its worldline is vertical (Figure 5-5).

Particle 5 moves not to the right but to the left in space according to the laboratory observer (Figure 5-4), so its worldline angles up and leftward in the laboratory spacetime map (Figure 5-5).

Each of these particles moves with constant speed, so each traces out a straight worldline. After 3 meters of time as measured in the laboratory frame, different

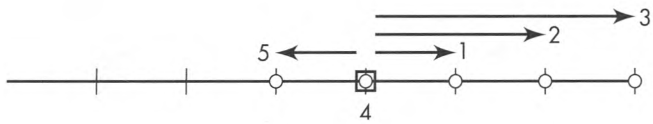


FIGURE 5-4. ***Trajectories*** **in space *(not in spacetime!) of particles 1 through 5 during 3 meters of time.*** *Each particle starts at the reference clock (the square) at zero of time and moves with a constant velocity.*

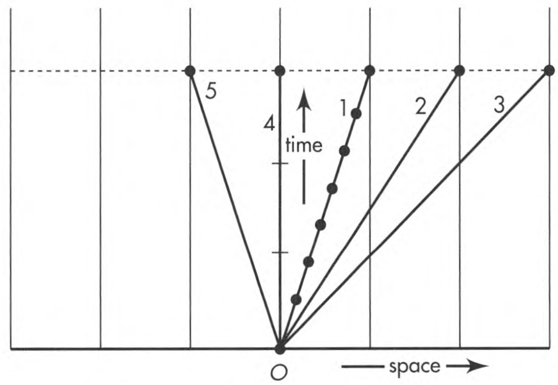


FIGURE 5-5. ***Worldlines*** **in spacetime *of the particles shown in Figure 5-4, plotted for the laboratory frame.*** *Only the worldline for particle 1 includes a sample set of event points that are connected to make up the worldline.*

particles have moved different distances from the starting point (Figure 5-4). In the laboratory spacetime map their space positions after 3 meters of time lie along the upper horizontal **line of simultaneity,** shown dashed in Figure 5-5.

Particle 4 is not the only object stationary in space. Every laboratory clock lies at rest in the laboratory frame; it moves neither right nor left as time passes. Nevertheless each laboratory clock moves forward in time, tracing out its own vertical worldline in the laboratory spacetime map. The background vertical lines in Figure 5-5 are worldlines of the row of laboratory clocks.

What is the difference between a "path in space" and a "worldline in spacetime"?

Path in space versus worldline in spacetime

The transcontinental airplane leaves a jet trail in still air. That trail is the plane's path in space. Take a picture of that trail and you have a *space map* of the motion. From that space map alone you cannot tell how fast the jet is moving at this or that different point on its path. The space map is an incomplete record of the motion.

The plane moves not only in space but also in time. Its beacon flashes. Plot those emissions as events on a spacetime map. This spacetime map has not only a horizontal space axis but also a vertical time axis. Now connect those event points with a worldline. The worldline gives a *complete* description of the motion of the jet as recorded in that frame. For example, from the worldline we can reckon the speed of the plane at every event along its path.

Worldline gives spacetime map of the journey of the jet. Likewise a worldline drawn on a spacetime map images the journey of any particle through spacetime. A worldline is not a physical path, not a trajectory, not a line in *space.* An object at rest in your frame has, for you, no path at all through space; it stays always at one space point. Yet this stationary particle traces out a "vertical" worldline in your spacetime map (such as line 4 in Figure 5-5). A particle *always* has a worldline in *spacetime.* As you sit quietly in your chair reading this book, you glide through spacetime on your own unique worldline. Every stationary object lying near you also traces out a worldline, parallel to your own on your spacetime map.

Not all particles move with constant speed. When a particle changes speed with respect to a free-float frame, we know why: A force acts on it. Think of a train moving on a straight stretch of track. A force applied by the locomotive speeds up all the cars. *Small speed:* small distance covered in a given time lapse; worldline inclined slightly to the vertical in the spacetime map. *Great speed:* great distance covered in the same stretch of time; worldline inclined at a greater angle to the vertical in the spacetime map. *Changing speed:* changing distances covered in equal time periods; worldline that changes inclination as it ascends on the spacetime map—a curved worldline!

Changing speed means curving worldline

Wait a minute! The train moves along a straight track. Yet you say its worldline is curved. Straight or curved? Make up your mind!

Straight in space does not necessarily mean straight in spacetime. Place your finger on the straight edge of a table near you. Now move your finger rapidly back and forth along this edge. Clearly this motion lies along a straight line. As your fingertip changes speed and direction, however, it travels different spans of distance in equal time periods. During a spell in which it is at rest on the table edge, your fingertip traces out a vertical portion of its worldline on the spacetime map. When it moves slowly to the right on the table, it traces out a worldline inclined slightly to the right of vertical on the map. When it moves rapidly to the left, your fingertip leaves a spacetime trail inclined significantly to the left on the map. Changing inclination of the worldline from point to point results in a curved worldline. Your finger moves straight in space but follows a curved worldline in spacetime!

Figure 5-6 shows a curved worldline, not for a locomotive, but for a particle constrained to travel down the straight track of a linear accelerator. The particle starts at the reference clock at the time of the reference event (*O* on the map). Initially the particle moves slowly to the right along the track. As time passes—advancing upward on the spacetime map—the particle speed increases to a large fraction of the speed of light. Then the particle slows down again, comes to rest at event *Z*, with a vertical tangent to its worldline at that event. Thereafter the particle accelerates to the left in space until it arrives at event *P*.

Limit on worldline slope: speed of light

What possible worldlines are available to the particle that has arrived at event *P*? A material particle must move at less than the speed of light. In other words, it travels less than one meter of distance in one meter of time. Its future worldline makes an "angle with the vertical" somewhere between plus 45 degrees and minus 45 degrees when space and time are measured in the same units and plotted to the same scale along horizontal and vertical axes on the graph. These limits of slope—which apply to every point on a particle worldline—are shown as dashed lines emerging from event *P* in Figure 5-6 (and also from event *O*).

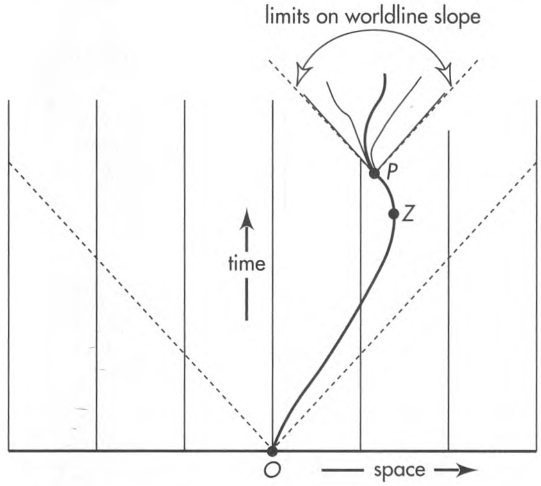


FIGURE 5-6. ***Curved laboratory worldline of a particle that changes speed as it moves back and forth along a straight line in space.*** *Some possible worldlines available to the particle after event* P.

The worldline gives a *complete* description of particle motion in spacetime. As drawn in the spacetime map for any frame, the worldline tells position and velocity of the particle at every event along its trail. In contrast, the trajectory or orbit or path shape of a particle in space does not give a complete description of the motion. To complete the description we need to know when the particle occupies each location on that trajectory. A worldline in a spacetime map automatically displays all of this information.

The spacetime map provides a tool for retrospective study of events that have already taken place and have been reported to the free-float observer who plots them. Once she plots these event points, this analyst can trace already plotted worldlines backward in time. She can examine at a single glance event points that may have occurred light-years apart in space. These features of the spacetime map do not violate our experience that time moves only forward or that nothing moves faster than light. Everything plotted on a spacetime map is history; it can be scanned rapidly back and forth in the space dimension or the time dimension or both. The spacetime map supplies a comprehensive tool for recognizing patterns of events and teasing out laws of nature, but it is useless for influencing the events it represents.

Spacetime map displays only already detected events

5.5 LENGTH ALONG A PATH

straight line has *shortest* length between two given *points* in *space*

Distance is a central idea in all applications of Euclidean geometry. For instance, using a flexible tape measure it is easy to quantify the total distance along a winding path that starts at one point (point O in Figure 5-7) and ends at another point (point B). Another way to measure distance along the curved path is to lay a series of short straight sticks end to end along the path. Provided the straight sticks are short enough to conform to the gently curving path, total distance along the path equals the sum of lengths of the sticks.

Measure length of curved path with tape measure . . .

The length of a short stick laid between any two nearby points on the path—for instance, points 3 and 4 in Figure 5-7—can also be calculated using the northward separation and the eastward separation between the two ends of the stick as measured by a surveyor.

. . . or with short straight sticks laid end to end along path

$$(\text{length})^2 = (\text{northward separation})^2 + (\text{eastward separation})^2$$

Distance is invariant for surveyors. Therefore the length of this stick is the same when calculated by any surveyor, even though the northward and eastward separations between two ends of the stick have different values, respectively, for different surveyors. The length of another stick laid elsewhere along the path is also agreed on by all surveyors despite their use of different northward directions. Therefore the sum of the lengths of all short sticks laid along the path has the same value for all surveyors. This sum equals the value of the total length of the path, on which all surveyors agree. And this total length is just the length measured using the flexible tape.

All surveyors agree on length of path

It is possible to proceed from O to B along quite another path—for example along straight line OB in Figure 5-7. The length of this alternative path is evidently different from that of the original curved path. This feature of Euclidean geometry is so well known as to occasion hardly any comment and certainly no surprise: In Euclidean geometry a curved path between two specified points is longer than a straight path between them. The existence of this difference of length between two paths violates no law. No one would claim that a tape measure fails to perform properly when laid along a curved path.

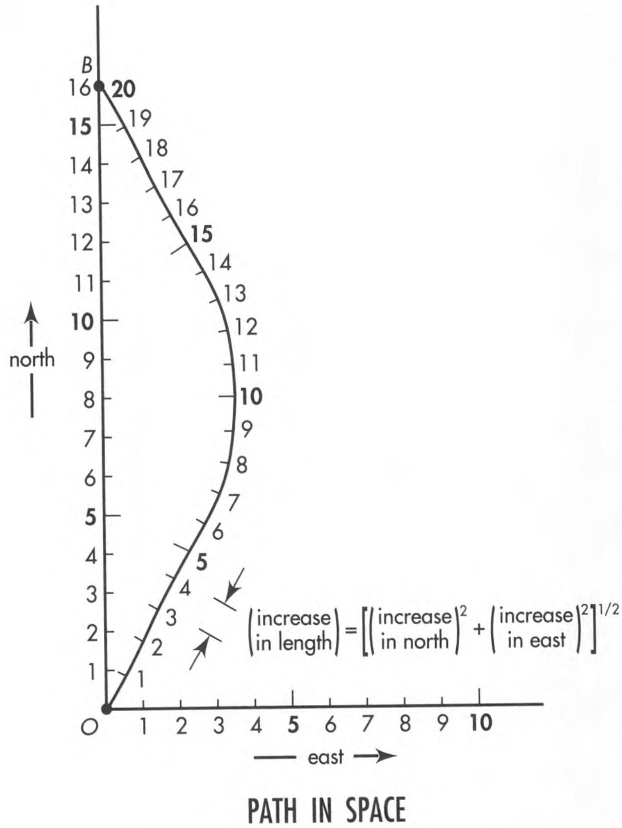


FIGURE 5-7. ***Length along a winding path starting at the town square.*** *Notice that the total length along the winding path from point* O *to point* B *is* greater *than the length along the straight northward axis from* O *to* B.

Straight path in space has *shortest* length

Among all possible paths between two points in space, the straight-line path is unique. All surveyors agree that this path has the shortest length. When we speak of "the distance between two points," we ordinarily mean the length of this straight path.

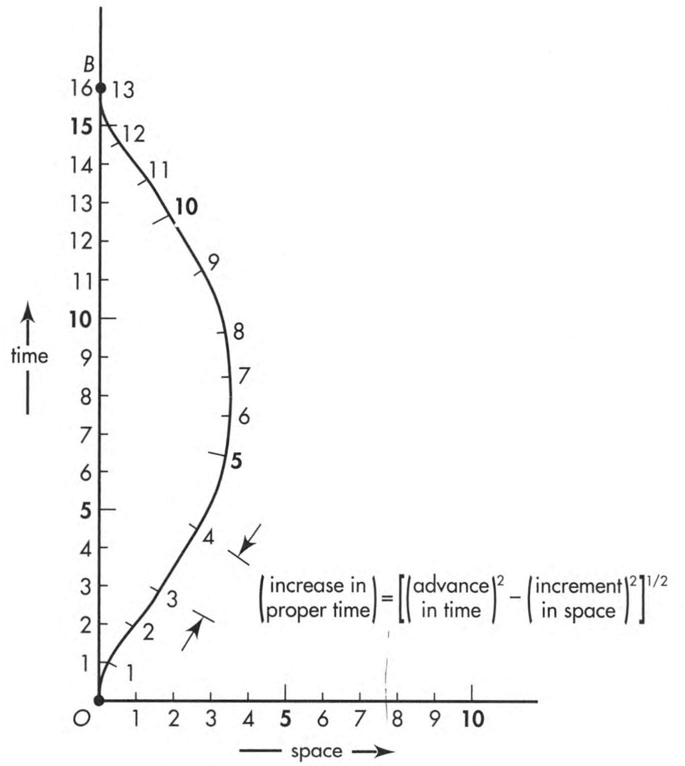
5.6 WRISTWATCH TIME ALONG A WORLDLINE

straight worldline has *longest proper time* between two given *events* in *spacetime*

Measure proper time along curved worldline with wristwatch . . .

A curved path in Euclidean space is determined by laying down a flexible tape measure and recording distance along the path's length. A curved worldline in Lorentz spacetime is measured by carrying a wristwatch along the worldline and recording what it shows for the elapsed time. The summed spacetime interval—the proper time read directly on the wristwatch—measures the worldline in Lorentz geometry in the same way that distance measures path length in Euclidean geometry.

A particle moves along the worldline in Figure 5-8. This particle carries a wristwatch and a sparkplug; the sparkplug fires every meter of time (1, 2, 3, 4, . . .) as read off the particle's wristwatch. The laboratory observer notes which of his clocks the



WORLDLINE IN SPACETIME

FIGURE 5-8. ***Proper time along a curved worldline.*** *Notice that the total proper time along the curved worldline from event* O *to event* B *is* smaller *than the proper time along the straight line from* O *to* B.

traveling particle is near every time the sparkplug fires. He plots that location and that lattice clock time on his spacetime map, tracing out the worldline of the particle. He numbers spark points sequentially on the resulting worldline, as shown in Figure 5-8, knowing that these numbers register meters of time recorded on the moving wristwatch.

. . . or as sum of intervals between adjacent events

Consider the spacetime interval between two sequential numbered flashes of the sparkplug, for instance those marked 3 and 4 in the figure. In the laboratory frame these two sparks are separated by a difference in position and also by a difference in time (the time between them). The squared interval—the proper time squared—between the sparks is given by the familiar spacetime relation:

$$(\text{proper time})^2 = (\text{difference in time})^2 - (\text{difference in position})^2$$

What about the proper time between sparks 3 and 4 calculated from measurements made in the sparkplug frame? In this frame, both sparks occur at the same place, namely at the position of the sparkplug. The difference in position between the sparks equals zero in this frame. As a result, the time difference in the sparkplug frame—the "wristwatch time"—is equal to the proper time between these two events:

$$(\text{proper time})^2 = (1\ \text{meter})^2 - (\text{zero})^2 = (1\ \text{meter})^2 \quad \textbf{[recorded on traveling wristwatch]}$$

This analysis assumes that sparks are close together in both space and time. For sparks close enough together, the velocity of the emitting particle does not change much from one spark to the next; the particle velocity is effectively constant between sparks; the piece of curved worldline can be replaced with a short straight segment. Along this straight segment the particle acts like a free-float rocket. The proper time is invariant in free-float rocket and free-float laboratory frames. Thus the laboratory observer can compute the value of the proper time between events 3 and 4 and predict the time lapse — one meter — on the traveling wristwatch, which measures the proper time directly.

Elsewhere along the worldline the particle moves with a different speed. Nevertheless the proper time between each consecutive pair of sparks must also be independent of the free-float frame in which that interval is reckoned. For sparks close enough together, this proper time equals the time read directly on the wristwatch.

All observers agree on proper time along worldline

All observers agree on the proper time between every sequential pair of sparks emitted by the sparkplug. Therefore the sum of of all individual proper times has the same value for all observers. This sum equals the value of the total proper time, on which all free-float observers agree. And this total proper time is just the wristwatch time measured by the traveling sparkplug.

In brief, proper time is the time registered in a rocket by its own clock, or by a person through her own wristwatch or her own aging. Like aging, proper time is cumulative. To obtain total proper time racked up along a worldline between some marked starting event and a designated final event, we first divide up the worldline into segments so short that each is essentially "straight" or "free-float." For each segment we determine the interval, that is, the lapse of proper time, the measurement of aging experienced on that segment. Then we add up the aging, the proper time for each segment, to get total aging, total wristwatch time, total lapse of proper time.

An automobile may travel the most complicated route over an entire continent, but the odometer adds it all up and gives a well-understood number. The traveler through the greater world of spacetime, no matter how many changes of speed or direction she undergoes, has the equivalent of the odometer with her on her journey. It is her wristwatch and her body — her aging. Your own wristwatch and your biological clock automatically add up the bits of proper time traced out on all successive segments of your worldline.

Straight worldline has *longest* proper time

It is possible to proceed from event O to event B along quite another worldline — for example, along the straight worldline OB in Figures 5-8 and 5-9 (bottom). The proper time from O to B along this new worldline can be measured directly by a flashing clock that follows this new worldline. It can also be calculated from records of flashes emitted by the clock as recorded in any laboratory or rocket frame.

Total proper time along this alternative worldline has a different value than total proper time along the original worldline. In Lorentz geometry a curved worldline between two specified events is *shorter* than the direct worldline between them — shorter in terms of total proper time, total wristwatch time, total aging.

Total proper time, the aging along any given worldline, straight or curved, is an *invariant:* it has the same value as reckoned by observers in all overlapping free-float frames. This value correctly predicts elapsed time recorded directly on the wristwatch of the particle that travels this worldline. It correctly predicts the aging of a person or a mouse that travels this worldline. A different worldline between the same two events typically leads to a different value of aging — a new value also agreed on by all free-float observers: Aging is maximal along the straight worldline between two events. This uniqueness of the straight worldline is also a matter of complete agreement among all free-float observers. All agree also on this: The straight worldline is the one actually followed by a free particle. Conclusion: Between two fixed events, a free particle follows the worldline of maximal aging. This more general prediction of the worldline of a free particle is called the **Principle of Maximal Aging.** It is true not only for "straight" particle worldlines in the limited regions of spacetime described by special relativity but also, with minor modification, for the motion of free particles in wider spacetime regions in the vicinity of gravitating mass. The Principle of Maximal Aging provides one bridge between special relativity and general relativity.

Principle of Maximal Aging predicts motion of free particle

The stark contrast between Euclidean geometry and Lorentz geometry is shown in Figure 5-9. In Euclidean geometry distance between nearby points along a curved

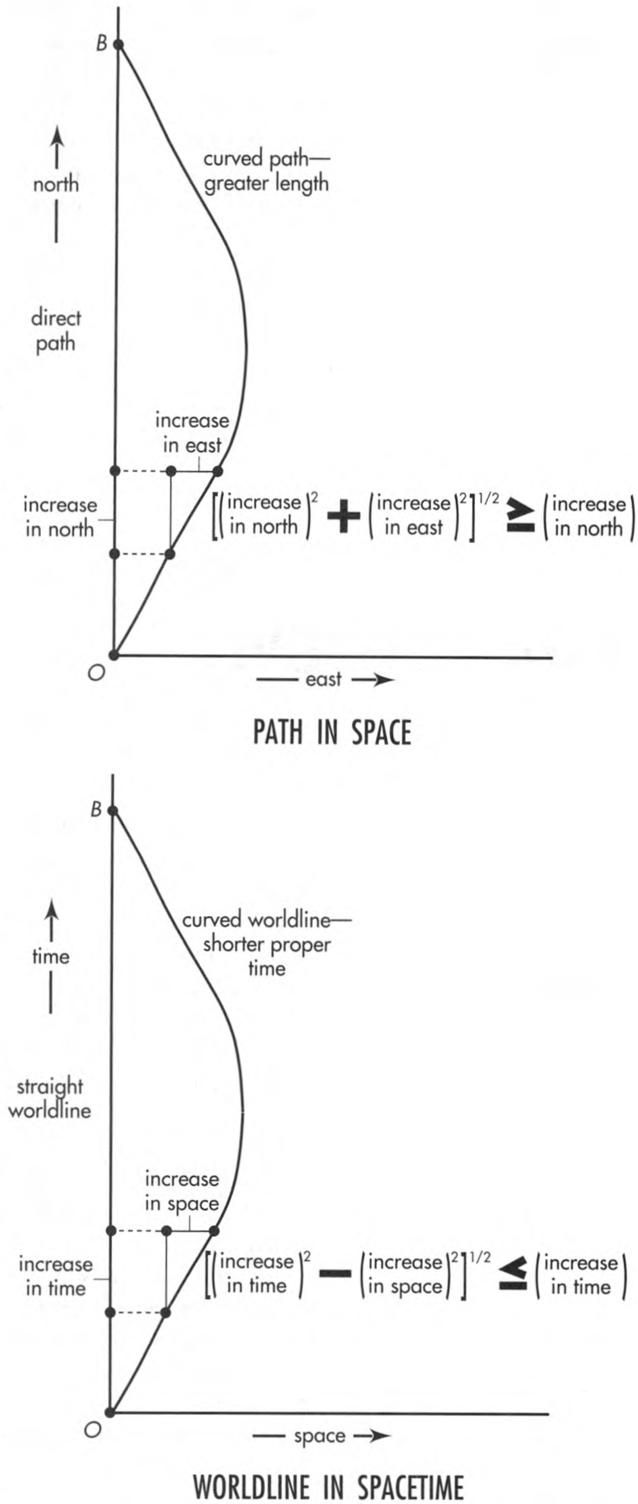


FIGURE 5-9. ***Path in space:*** *In* Euclidean *geometry the* curved *path has* greater *length.* ***Worldline in spacetime:*** *In* Lorentz *geometry the* curved *worldline is traversed in* shorter *proper time.*

path is always equal to or *greater* than the northward separation between those two points. In contrast, proper time between nearby events along a curved worldline is always equal to or *less* than the corresponding time along the direct worldline as measured in that frame.

Stark contrast between Euclidean and Lorentz geometries

The difference of proper time between two alternative worldlines in spacetime violates no law, just as the difference of length between two alternative paths in space violates no law. There is nothing wrong with a wristwatch that reads different proper times when carried along different worldlines between events O and B in spacetime, just as there is nothing wrong with a tape measure that records different lengths for different paths between points O and B in space. In both cases the measuring device is simply giving evidence of the appropriate geometry: Euclidean geometry for space, Lorentz geometry for spacetime.

Proper times compare worldlines

In brief, the determination of cumulative interval, proper time, wristwatch time, aging along a worldline between two events is a fundamental method of comparing different worldlines that connect the same two events.

Among all possible worldlines between two events, the straight worldline is unique. All observers agree that this worldline is straight and has the longest proper time — greatest aging — of any possible worldline connecting these events.

5.7 KINKED WORLDLINE

kink in the worldline decreases aging along that worldline

The change in slope of the worldline from event to event in Figures 5-8 and 5-9 (bottom) means that the clock being carried along this worldline changes velocity: It accelerates. Different clocks behave differently when accelerated. Typically a clock can withstand a great acceleration only when it is small and compact. A pendulum clock is not an accurate timepiece when carried by car through stop-and-go traffic; a wristwatch is fine. A wristwatch is destroyed by being slammed against a wall; a radioactive nucleus is fine. Typically, the smaller the clock, the more acceleration it can withstand and still register properly, and the sharper can be the curves and kinks on its worldline. In all figures like Figures 5-8 and 5-9 (bottom), we assume the ideal limit of small (acceleration-proof) clocks.

Acceleration-proof clocks

We are now free to analyze a motion in which particle and clock are subject to a great acceleration. In particular, consider the simple special case of the worldline of Figure 5-8. That worldline gradually changes slope as the particle speeds up and slows down. Now make the period of speeding up shorter and shorter (great driving force!); also make the period of slowing down shorter and shorter. In this way come eventually to the limiting case in which episodes of acceleration and deceleration — curved portions of the worldline — are too short even to show up on the scale of the spacetime map (worldline OQB in Figure 5-10). In this simple limiting case the whole history of motion is specified by (1) initial event O, (2) final event B, and (3) turnaround event Q, halfway in time between O and B. In this case it is particularly easy to see how the lapse of proper time between O and B depends on the location of the halfway event — and thus to compare three worldlines, OPB, OQB, and ORB.

Simplify: Worldlines with straight segments

Path OPB is the worldline of a particle that does not move in space; it stays next to the reference-frame clock. Proper time from O to B by way of P is evidently equal to time as measured in the free-float frame of this reference clock:

$$(\text{total proper time along } OPB) = 10 \text{ meters of time}$$

In contrast, on the way from O to B via R, for each segment the space separation equals the time separation, so the proper time has the value zero:

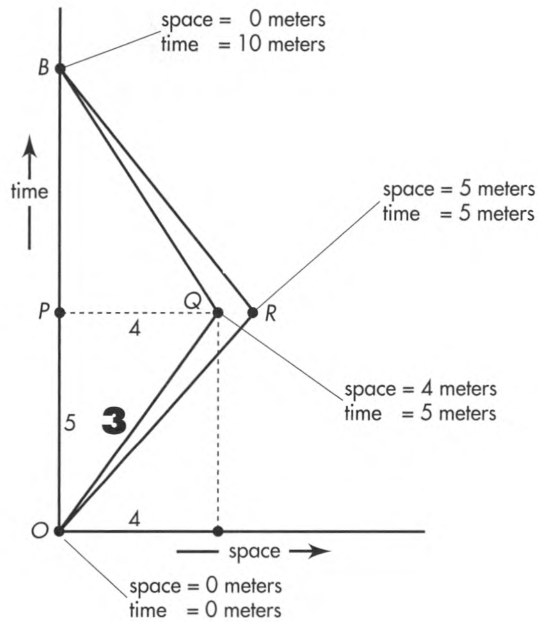


FIGURE 5-10. ***Three alternative worldlines connecting events O and B.*** *The sharp changes of velocity at events Q and R have been drawn for the ideal limit of small clocks that tolerate great acceleration. The bold-face number 3 is the proper time along the segment OQ, reckoned from the* difference *between the squared time separation and the squared space separation:* $3^2 = 5^2 - 4^2$.

Zero proper time for light

$$\begin{aligned}
(\text{proper time along leg } OR)^2 &= (\text{time})^2 - (\text{space})^2 \\
&= (5 \text{ meters})^2 - (5 \text{ meters})^2 \\
&= 0 \\
(\text{total proper time along } ORB) &= 2 \times (\text{proper time along } OR) \\
&= 0
\end{aligned}$$

As far as we know, only three things can travel 5 meters of distance in 5 meters of time: light (photons), neutrinos, and gravitons (see Box 8-1). No material clock can travel at light speed. Therefore the worldline *ORB* is not actually attainable by a material particle. However, it can be approached arbitrarily closely. One can find a speed sufficiently close to light speed—and yet less than light speed—so that a trip with this speed first one way then the other will bring an ideal clock back to the reference clock with a lapse of proper time that is as short as one pleases. In the same way we can, in principle, go to the star Canopus and back in as short a round-trip rocket time as we choose (Section 4.8).

As distinguished from the limiting case *ORB,* worldline *OQB* demands an amount of proper time that is greater than zero but still less than the 10 meters of proper time along the direct worldline *OPB:*

Reduced proper time along kinked worldline

$$\begin{aligned}
(\text{proper time along leg } OQ)^2 &= (5 \text{ meters})^2 - (4 \text{ meters})^2 \\
&= 25 \text{ (meters)}^2 - 16 \text{ (meters)}^2 \\
&= 9 \text{ (meters)}^2 \\
&= (3 \text{ meters})^2
\end{aligned}$$

so

$$(\text{proper time along leg } OQ) = 3 \text{ meters}$$

and

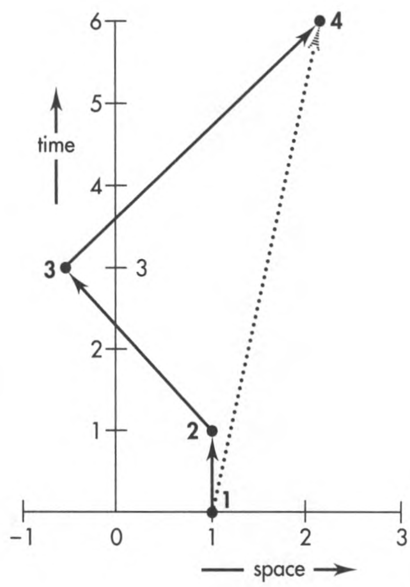
$$\begin{aligned}
(\text{total proper time along both legs } OQB) &= 2 \times (\text{proper time along } OQ) \\
&= 6 \text{ meters}
\end{aligned}$$

This is less proper time than (proper time along *OPB*) = 10 meters that characterized the "direct" worldline *OPB*. Our trip to Canopus and back described in Chapter 4 follows a worldline similar to *OQB.*

SAMPLE PROBLEM 5-1

MORE IS LESS

In the spacetime map shown, time and space are measured in years. A table shows space and time locations of numbered events in this frame.



Two alternative worldlines between events 1 and 4

SPACE AND TIME LOCATIONS OF EVENTS

	Space (years)	Time (years)
Event **1**	1	0
Event **2**	1	1
Event **3**	−0.5	3
Event **4**	2	6

a. One traveler moves along the solid straight worldline segments from event 1 to events 2, 3, and 4. Calculate the time increase on her clock between event 1 and event 2; between event 2 and event 3; between event 3 and event 4. Calculate total proper time—her aging—along worldline 1, 2, 3, 4.

b. Another traveler, her twin brother, moves along the straight dotted worldline from event 1 directly to event 4. Calculate the time increase on his clock along the direct worldline 1, 4.

c. Which twin (solid-line traveler or dotted-line traveler) is younger when they rejoin at event 4?

SOLUTION

a. From the table next to the map, space separation between events 1 and 2 equals 0. Time separation equals 1 year. Therefore the interval is reckoned from $(\text{interval})^2 = 1^2 - 0^2 = 1^2$. Thus the proper time lapse on a clock carried between events 1 and 2 equals 1 year.

Space separation between event 2 and event 3 equals $1 - (-0.5) = 1.5$ light-years. Time separation equals 2 years. Therefore the square of the interval is $2^2 - (1.5)^2 = 4 - 2.25 = 1.75$ (years)^2 and the advance of proper time equals the square root of this, or 1.32 years.

Between event 3 and event 4 space separation equals 2.5 light-years and time separation 3 years. The square of the interval has the value $3^2 - (2.5)^2 = 9 - 6.25 = 2.75$ (years)^2 and proper time between these two events equals the square root of this, or 1.66 years.

Total proper time—aging—along worldline 1, 2, 3, 4 equals the sum of proper times along individual segments: $1 + 1.32 + 1.66 = 3.98$ years.

b. Space separation between events 1 and 4 equals 1 light-year. Time separation is 6 years. The squared interval between them equals $6^2 - 1^2 = 36 - 1 = 35$ (years)^2. A traveler who moves along the direct worldline from event 1 to event 4 records a span of proper time equal to the square root of this value, or 5.92 years.

c. The brother who moves along straight worldline 1, 4 ages 5.92 years during the trip. The sister who moves along segmented worldline 1, 2, 3, 4 ages less: 3.98 years. As always in Lorentz geometry, the direct worldline (shown dotted) is longer—that is, it has more elapsed proper time, greater aging—than the indirect worldline (shown solid).

5.8 STRETCH FACTOR

ratio of frame-clock time to wristwatch time

A speeding beacon emits two flashes, F and S, in quick succession. These two flashes, as recorded in the rocket that carries the beacon, occur with a 6-meter separation in time but a zero separation in space. Zero space separation? Then 6 meters is the value of the interval, the proper time, the wristwatch time between F and S. As registered in the laboratory, in contrast, the second flash S occurs 10 meters of time later than the first flash F. The ratio between this frame time, 10 meters, and the proper time, 6 meters, between the two events we call the time stretch factor, or simply **stretch factor.** Some authors use the lowercase Greek letter gamma, γ, for the stretch factor, as we do occasionally. We will also use the Greek letter tau, τ, for proper time.

Different reference frames: different times between two events

The same two events register in the super-rocket frame that overtakes and passes the beacon—register with a separation in time of 20.88 meters. In this frame, the time stretch factor between the two events is $(20.88)/6 = 3.48$. In the beacon frame the stretch factor is unity: $6/6 = 1$. Why? Because in this beacon frame flashes F and S occur at the same place, so beacon-frame clocks record the proper time directly. This proper time is less than the time between the two flashes as measured in either laboratory or super-rocket frame. The larger value of time observed in laboratory and super-rocket frames shows up in Figure 5-11 (center and right). Among all conceivable frames, the separation in time between the two flashes evidently takes on its minimum value in the beacon frame itself, the value of the proper time τ.

Time lapse minimum for frame in which events occur at same place

Hold it! In Sections 5.6 and 5.7 you insisted that the time along a straight worldline is a MAXIMUM. Now you show us a straight worldline along which the time is—you say—a MINIMUM. Maximum or minimum? Please make up your mind!

The worldline taken by the beacon wristwatch from F to S is straight. It is straight whether mapped in the beacon frame itself or in the rocket or super-rocket frame. The beacon racks up 6 meters of *proper* time regardless of the frame in which we reckon this time. When we turn from this wristwatch time to what different

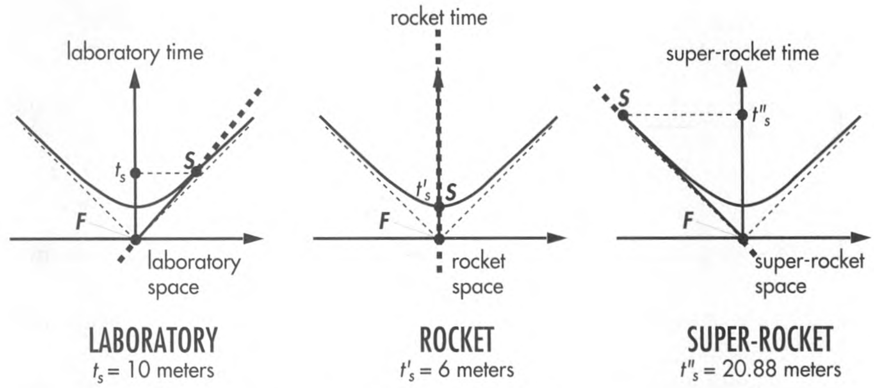


FIGURE 5-11. ***Spacetime maps of Figure 5-3, modified to show the worldline of the speeding beacon (heavy dashed line) and the segment of this line between emission* F *of the first flash and the second flash* S *(solid section of worldline).*** *Emission* F *is taken as the zero of space and time. Time* t_S *of the second emission* S *is different as recorded in different frames. The shortest time is recorded in that frame in which the two events occur at the same place—in this case the rocket frame.*

free-float frames show for the separation in map time (latticework time, frame time) between the two flashes, however, the record displays a minimal value for that separation in time only in the beacon frame itself.

In contrast, Figure 5-12 (Figure 5-10 in simplified form) shows two *different* worldlines that join events *O* and *B* mapped in the same reference frame. In this case we compare two different proper times: a proper time of 10 meters racked up by a wristwatch carried along the direct course from *O* to *B*, and a proper time of 6 meters recorded by the wristwatch carried along on the kinked worldline *OQB*. In every such comparison made in the context of flat spacetime, the direct worldline displays maximum proper time. Caution: Conditions can be different in curved spacetime (Chapter 9).

In summary, two points come to the fore in these comparisons of the time between two events. (1) Are we comparing map time (frame time, latticework time) between those two events, pure and simple, free of any talk about any worldline that might connect those events? Then separation in time between those events is least as mapped in the free-float frame that shows them happening at the same place. (2) Or are we directing our attention to a worldline that connects the two events? More specifically, to the time racked up by a wristwatch toted along that worldline? Then we have to ask, is that worldline straight? Then it registers maximal passage of proper time. Or does it have a kink? Then the proper time racked up is not maximal.

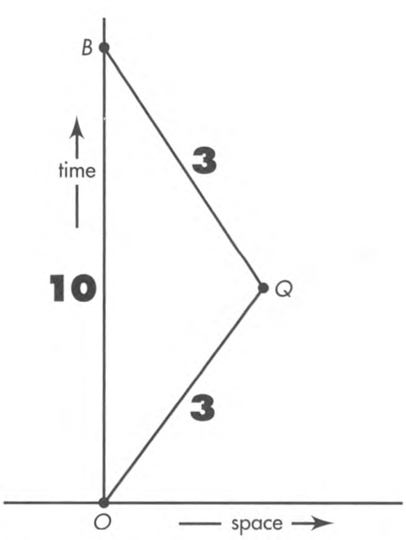


FIGURE 5-12. ***Figure 5-10 stripped down to emphasize total proper time (wristwatch time), printed boldface along two different worldlines between the same two events* O *and* B *in a given reference frame.*** *Among all possible worldlines connecting events* O *and* B, *the straight worldline registers maximal lapse of proper time.*

When we find ourselves in a free-float frame and see a beacon zooming past in a straight line with speed v, how much is the factor by which our frame-clock time is stretched relative to the beacon wristwatch time? Answer: The stretch factor is

$$\text{(stretch factor)} = \gamma = \frac{1}{(1 - v^2)^{1/2}} \tag{5-1}$$

How can we derive this famous formula? If you do not cover up the following lines and derive this answer on your own, here is the reasoning: Start with measurements in the laboratory frame. We know that for this rocket

$$\text{(advance in proper time)}^2 = \text{(advance in lab time)}^2 - \text{(lab distance covered)}^2$$

However, we want to compare lapses in laboratory time and proper time; laboratory distance covered is not of interest. For the laboratory observer the proper clock moving along a straight worldline covers the distance between the two events in the time between the events. Therefore this distance and time are related by particle speed:

Stretch factor
= frame time/proper time

$$\text{(lab distance covered)} = \text{(speed)} \times \text{(advance in lab time)}$$

Substitute this expression into the equation for proper time:

$$\begin{aligned}\text{(proper time)}^2 &= \text{(lab time)}^2 - \text{(speed)}^2 \times \text{(lab time)}^2 \\ &= \text{(lab time)}^2\,[1 - \text{(speed)}^2]\end{aligned}$$

This leads to an expression for the square of the stretch factor:

$$\frac{\text{(lab time)}^2}{\text{(proper time)}^2} = \text{(stretch factor)}^2 = \frac{1}{1 - \text{(speed)}^2} = \frac{1}{1 - v^2}$$

where we use the symbol $v = v_{\text{conv}}/c$ for speed. The equation for the stretch factor becomes

Stretch factor derived

$$\text{(stretch factor)} = \gamma = \frac{1}{(1 - v^2)^{1/2}} \tag{5-1}$$

The stretch factor has the value unity when $v = 0$. For all other values of v the stretch factor is greater than unity. For very high relative speeds, speeds close to that of light ($v \longrightarrow 1$), the value of the stretch factor increases without limit.

The value of the stretch factor does not depend on the direction of motion of the rocket that moves from first event to second event: The speed is squared in equation (5-1), so any negative sign is lost.

The stretch factor is the ratio of frame time to proper time between events, where speed (= v) is the steady speed necessary for the proper clock to pass along a straight worldline from one event to the other in that frame.

Lorentz-contraction by same "stretch" factor

The stretch factor also describes the Lorentz contraction, the measured shortening of a moving object along its direction of motion when the observer determines the distance between the two ends *at the same time*. For example, suppose you travel at speed v between Earth and a star that lies distance L away as measured in the Earth frame. Your trip takes time $t = L/v$ in the Earth-linked frame. Proper time τ—your wristwatch time—is smaller than this by the stretch factor: $\tau = L/[v \times \text{(stretch factor)}] = (L/v)\,(1 - v^2)^{1/2}$. Now think of a very long rod that reaches from Earth to star and is at rest in the Earth frame. How long is that rod in your rocket frame? In your frame the rod is moving at speed v. One end of the rod, at the position of Earth, passes at speed v. A time τ later in your frame the other end of the rod arrives—along with the star—also moving at speed v according to your rocket measurements. From these data you calculate that the length of the rod in your rocket frame—call it L'—is equal to $L' = v\tau = v(L/v)\,(1 - v^2)^{1/2} = L\,(1 - v^2)^{1/2}$. This is a valid measure of length. By this method the rod is measured to be shorter.

Stretch factor as a measure of speed

Finally, the stretch factor is often used as an alternative measure of particle speed: A particle moves with a speed such that the stretch factor is 10. This statement assumes that the particle is moving with constant speed, so that the separation between any pair of events on the particle worldline has the same stretch factor as the separation between any other pair. This way of describing particle speed can be both convenient and powerful. We will see (Chapter 7) that the total energy of a particle is proportional to the stretch factor.

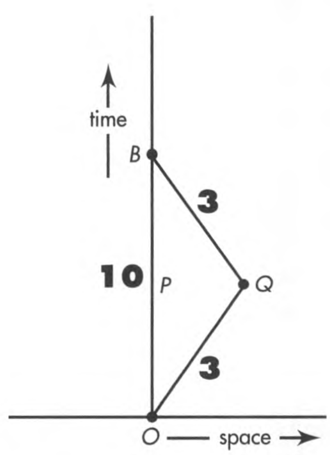
SAMPLE PROBLEM 5-2

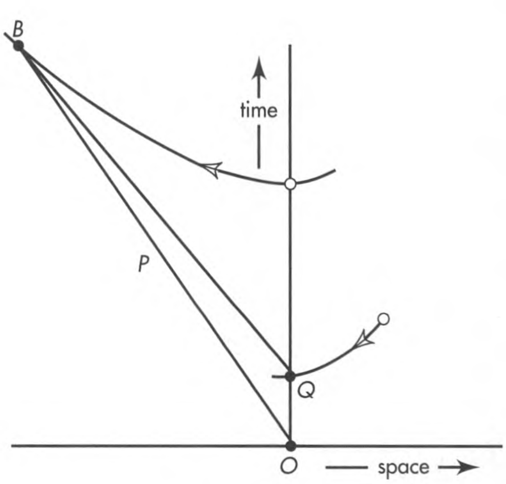
ROUND TRIP OBSERVED IN A DIFFERENT FRAME

Return to the alternative worldlines between events O and B, shown in Figure 5-10 and the spacetime maps in this sample problem. Measure these worldlines from a rocket frame that moves outward with the particle from O to Q and keeps on going forever at the same constant velocity. Show that an observer in this outward-rocket frame predicts the same proper time—wristwatch time—for worldline OQB as that predicted in the laboratory frame. Similarly show that this outward-rocket-frame observer predicts the same proper time along the *direct* worldline OPB as does the laboratory observer. Finally, show that both observers predict the elapsed wristwatch time along OQB to be less than along OPB.

SOLUTION

Here are laboratory and rocket spacetime maps for these round trips, simplified and drawn to reduced scale.





Laboratory and outward-rocket spacetime maps, each showing alternative worldlines (direct* OPB *and indirect* OQB*) between events* O *and* B. *Laboratory spacetime map: *Figure 5-10, redrawn to a different scale. Proper times are shown on the laboratory spacetime map.* ***Outward-rocket spacetime map:*** *The rocket in which the outgoing particle is at rest. Portions of two invariant hyperbolas show how events* Q *and* B *transform. The direct worldline* OPB *has longer total proper time—greater aging—as computed using measurements from either frame.*

Find x'_Q and t'_Q: First compute space and time locations of events Q and B in the outgoing rocket frame—right-hand map. (Event O is the reference event, $x = 0$ and $t = 0$ in all frames by convention.) We choose the rocket frame so that the worldline segment OQ lies vertical and the outbound rocket does not move in this frame. As a result, event Q occurs at rocket space origin: $x'_Q = 0$. (Primes refer to measurements in the outward-rocket frame.) The rocket time t'_Q for this event is just the wristwatch time between O and Q, because the wristwatch is at rest in this frame: $t'_Q = 3$ meters.

In summary, using a prime for rocket measurements:

$$x'_Q = 0$$
$$t'_Q = 3 \text{ meters}$$

Find x'_B and t'_B: In the *laboratory* frame, the particle moves to the *right* from event O to event Q, covering 4 meters of distance in 5 meters of time. Therefore its speed is the fraction $v = 4/5 = 0.8$ of light speed. As measured in the rocket frame, the laboratory frame moves to the *left* with speed $v = 0.8$, by symmetry. Use equation (5-1) with $v = 0.8$ to compute the value of the stretch factor:

$$\frac{1}{[1-v^2]^{1/2}} = \frac{1}{[1-(0.8)^2]^{1/2}} = \frac{1}{[1-0.64]^{1/2}} = \frac{1}{[0.36]^{1/2}} = \frac{1}{0.6} = \frac{10}{6} = \frac{5}{3}$$

This equals the ratio of rocket time period t'_B to proper time τ_B along the direct path OPB. Hence elapsed rocket time $t'_B = (5/3) \times 10$ meters $= 50/3$ meters of time. In this time, the laboratory moves to the left in the rocket frame by the distance $x'_B = -vt'_B = -(4/5)\,(50/3) = -200/15 = -40/3$ meters. In summary for outgoing rocket:

$$x'_B = -\frac{40}{3} \text{ meters} = -13\frac{1}{3} \text{ meters}$$

$$t'_B = \frac{50}{3} \text{ meters} = 16\frac{2}{3} \text{ meters of time}$$

Events Q and B are plotted on the rocket spacetime map.

Compare Wristwatch Times: Now compute the total proper time—wristwatch time, aging—along alternative worldlines OPB and OQB using rocket measurements. Direct worldline OB has proper time τ_{OB} given by the regular expression for interval:

$$(\tau_{OB})^2 = (t'_{OB})^2 - (x'_{OB})^2 = \left(\frac{50}{3}\right)^2 - \left(-\frac{40}{3}\right)^2$$

$$= \frac{2500}{9} - \frac{1600}{9} = \frac{900}{9} = 100 \text{ (meters)}^2$$

whence $\tau_{OB} = 10$ meters computed from rocket measurements. This is the same value as computed in the laboratory frame (in which proper time equals laboratory time, since laboratory separation in space is zero).

Worldline OQB has two segments. On the first segment, OQ, proper time lapse is just equal to the rocket time span, 3 meters, since the space separation equals zero in the rocket frame. For the second segment of this worldline, QB, we need to compute elapsed time in this frame:

$$t'_{QB} = t'_B - t'_Q = \frac{50}{3} - 3 = \frac{50}{3} - \frac{9}{3} = \frac{41}{3} \text{ meters}$$

$$x'_{QB} = -\frac{40}{3} \text{ meters}$$

Therefore,

$$(t_{QB})^2 = (t'_{QB})^2 - (x'_{QB})^2 = \left(\frac{41}{3}\right)^2 - \left(\frac{40}{3}\right)^2$$

$$= \frac{1681}{9} - \frac{1600}{9} = \frac{81}{9} = 9 \text{ (meters)}^2$$

whence $\tau_{QB} = 3$ meters. So the total increase in proper time—the total aging—along worldline OQB sums to $3 + 3 = 6$ meters as reckoned from outward-rocket measurements. This is the same as figured from laboratory measurements.

How can these weird results be true? In our everyday lives why don't we have to take account of clocks that record different elapsed times between events, and rods that we measure to be contracted as they speed by us?

In answer, consider two events that occur at the same place in our frame. The proper clock moving in spacetime between these two events has speed zero for us. In this case the stretch factor has the value unity: the frame clock is the proper clock. The same is *approximately* true for events that are much closer together in space (measured in meters) than the time between them (also measured in meters). In these cases the proper clock moving between them has speed v—measured in meters/meter—that is very much less than unity. That is, the proper clock moves very much slower than the speed of light. For such slow speeds, the stretch factor has a value that approaches unity; the proper clock records *very nearly* the same time lapse between two events as frame clocks. This is the situation for all motions on earth that we can follow by eye. For all such "ordinary-speed" motions, moving clocks and stationary clocks record essentially the same time lapses. This is the assumption of Newtonian mechanics: "Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external . . ."

A similar argument leads to the conclusion that Lorentz contraction is negligible for objects moving at everyday speeds. Newton's mechanics—with its unique measured time between events and its unique measured length for an object whether or not it moves—gives correct results for objects moving at everyday speeds. In contrast, for particle speeds approaching light speed (approaching one meter of distance traveled per meter of elapsed time in the laboratory frame), the denominator on the right of equation (5-1) approaches zero and the stretch factor increases without limit. Increased without limit, also, is the laboratory time between ticks of the zooming particle's wristwatch. This is the case for high-speed particles in accelerators and for cosmic rays, very high-energy particles (mostly protons) that continually pour into our atmosphere from space. Newton's mechanics gives results wildly in error when applied to these particles and their interactions; the laws of relativistic mechanics must be used.

More than one cosmic ray has been detected (indirectly by the resulting shower of particles in the atmosphere) moving so fast that it could cross our galaxy in 30 seconds as recorded on its own wristwatch. During this trip a thousand centuries pass as recorded by clocks on Earth! (See Exercise 7-7.)

5.9 TOURING SPACETIME WITHOUT A REFERENCE FRAME

all you need is worldlines and events

An explosion is an explosion. Your birth was your birth. An event is an event. Every event has a concreteness, an existence, a reality independent of any reference frame. So, too, does a worldline that connects the trail of event points left by a high-speed sparkplug that flashes as it streaks along. Events mark worldlines, independent of any reference frame.

Events and worldlines exist independent of any reference frame

Worldlines also locate events. The intersection of two worldlines locates an event as clearly and sharply as the intersection of two straws specifies the place of a dust speck in a great barn full of hay (Figure 5-13). To say that an event marks a collision between two particles is identification enough. The worldlines of those two particles are rooted in the past and stretch out into the future. They have a rich texture of connections with nearby worldlines. The nearby worldlines in turn are linked in a hundred ways with

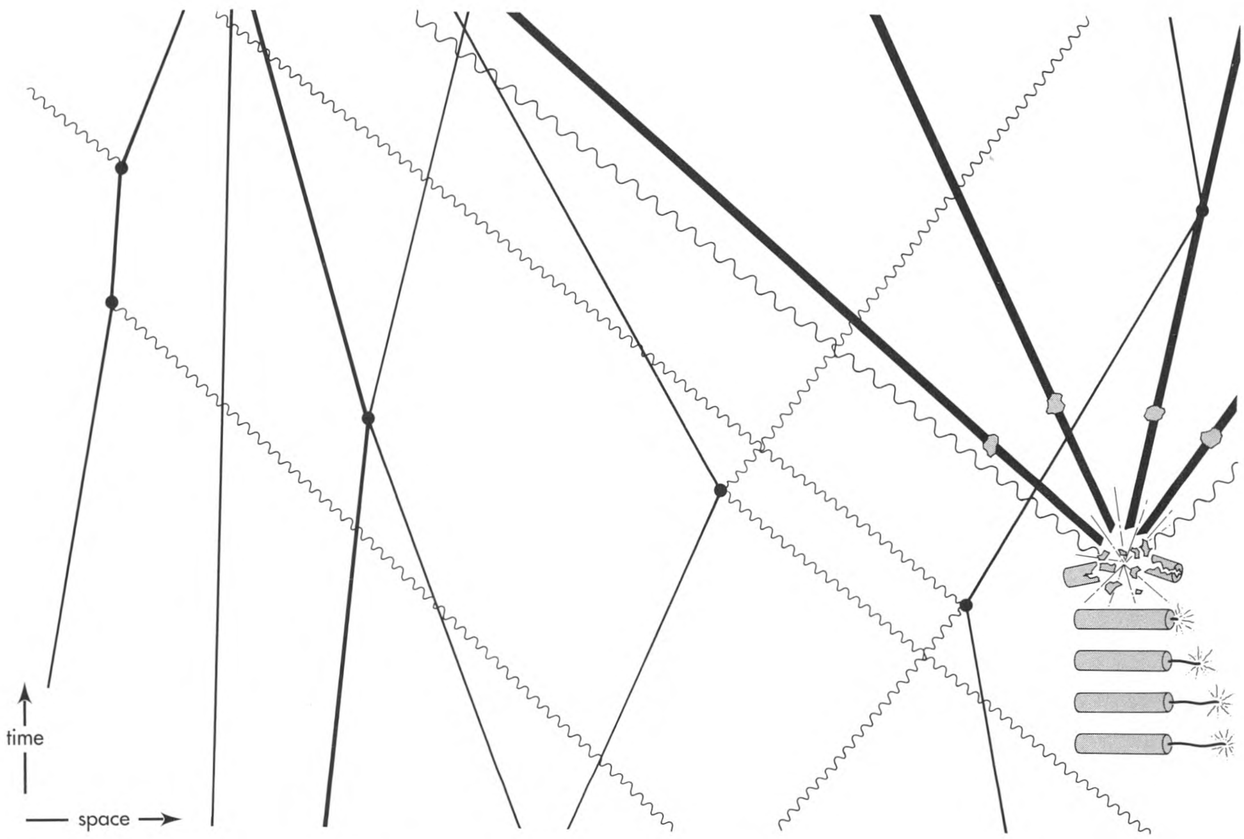


FIGURE 5-13. *The crossing of straws in a barn full of hay is a symbol for the worldlines that fill up spacetime. By their crossings and jogs, these worldlines mark events with a uniqueness beyond all need of reference frames. Straight worldlines track particles with mass; wiggly worldlines trace photons. Typical events symbolized in the map (black dots) from left to right: absorption of a photon; reemission of a photon; collision between a particle and a particle; collision between a photon and another particle; another collision between a photon and a particle; explosion of a firecracker; collision of a particle from outside with one of the fragments of that firecracker.*

worldlines more remote. How then does one tell the location of an event? Tell first what worldlines thread the event. Next follow each of these worldlines. Name additional events that they encounter. These events pick out further worldlines. Eventually the whole barn of hay is cataloged. Each event is named. One can find one's way as surely to a given intersection as the London dweller can pick her path to the meeting of St. James's Street and Piccadilly. No numbers giving space and time location of an event in a given reference frame. No reference frame at all!

Locate house at intersection of streets

Most streets in Japan have no names and most houses no numbers. Yet mail is delivered just the same. Each house is named after its senior occupant, and everyone knows how the streets interconnect these named houses. Now print the map of Japanese streets on a rubber sheet and stretch the sheet this way and that. The postal carrier is not fooled. Each house has its unique name and the same interconnections with neighbor houses as on the unstretched map. So dispense with all maps! Replace them with a catalog or directory that lists each house by name, notes streets passing the house, and tabulates the distance to each neighboring house along the streets.

Locate event at intersection of worldlines

Similarly, the visual pattern of event dots on a spacetime map (spacetime diagram) and the apparent lengths of worldlines that connect them depend on the reference frame from which they are observed (for example, compare alternative spacetime maps of the same worldline shown in the figure in Sample Problem 5-2). However, each named event is the same for every observer: the event of your birth is unique to you and to everyone connected with you. Moreover, the segment of a worldline that connects one event with the next has a unique magnitude — the interval or proper time — also the same for every observer. Therefore dispense with reference frames altogether! Replace them with a catalog or directory that lists each event by name, notes each worldline that threads the event, and tabulates the interval that connects the event with the next event along each worldline. With this directory in hand we can say precisely how all events are interconnected with each other and which events caused which other events. That is the essence of science; in principle we need no reference frames.

Events and worldlines alone can describe Nature

But reference frames are convenient. We are accustomed to them. Most of us prefer to live on named streets with numbered houses. Similarly, most of us speak easily of space separations between events and time separations between the same events as if space and time separations were unconnected. In this way we enjoy the concreteness of using our latticework of rods and clocks while suffering the provinciality of a single reference frame. So be it! Nevertheless, with worldlines Nature gives us power to relate events — to do science — without reference frames at all.

5.10 SUMMARY

straighter worldline? greater aging!

Events? Yes. Each event endowed with its own location in that great fabric we call spacetime? Yes. But time? No point in all that fabric displays any trace of anything we can identify with any such thing as the "time" of that event. Label that event with a "time" anyway? Sure. No one can stop us. Moreover, such labeling often proves quite useful. But it is *our* labeling! A different reference frame, a different wristwatch brought to that event along a different worldline yields a different time label for that event.

For our own convenience, then, we plot events on a **spacetime map (spacetime diagram)** for a particular free-float frame and its latticework of rods and clocks. This map can be printed on the page of a book if events are limited to one line in space. Distance along this line is plotted horizontally on the spacetime map, with time of the event plotted vertically (Section 5.1). The time and space values of an event are measured with respect to a common **reference event,** plotted at the origin of the spacetime map. The invariance of the interval: $(\text{interval})^2 = (\text{time})^2 - (\text{distance})^2$ between an event and the reference event corresponds to the equation of a **hyperbola,** the same hyperbola as plotted on the spacetime map of every overlapping free-float frame. The event point lies somewhere on the same **invariant hyperbola** as plotted on every one of these spacetime maps (Sections 5.2 and 5.3).

Billions of events sparkle like sand grains scattered over the spacetime map. A given event is unconnected to most other events on the map. Here we pay attention to particular strings of events that are connected. The **worldline** of a particle connects in sequence events that occur at the particle (Section 5.4). The "length" of a worldline between an initial and a final event is the elapsed time measured on a clock carried along the worldline between the two events (Section 5.6). This is called the proper time, wristwatch time, or aging along this worldline. The lapse of proper time is given the symbol τ, in contrast to the symbol t for the frame time read on the latticework clocks in a given free-float frame.

Carry a wristwatch (or grow old!) along a worldline: This is one way to measure the total proper time along it from some initial event (such as the birth of a person or a particle) to some final event (such as death of a person or annihilation of a particle). This method is direct, experimental, simple. A second method? Calculate the interval between each pair of adjacent events that make up the worldline, and then add up all these intervals, assuming that each tiny segment is short enough to be considered straight. This method seems more bothersome and detailed, but it can be carried out by the observer in *any* free-float frame. All such observers will agree with one another — and with the clock-carrier — on the value of the total proper time from the initial event to the final event on the worldline (Section 5.6).

Among all possible worldlines between two given events, the straight line is the worldline of **maximal aging.** This is the actual worldline followed by a free particle that travels from one of these two events to the other (Section 5.6).

As measured in a given free-float frame, the **stretch factor** $= 1/(1-v^2)^{1/2}$ equals the ratio of elapsed frame time t to elapsed proper time τ along a segment of worldline in which the particle moves with speed v in that frame. The stretch factor is also the Lorentz contraction factor (Section 5.8): Locate, at the same time, the front and back ends of an object moving in a given free-float frame. These end locations will be $(1-v^2)^{1/2}$ as far apart in that frame as they are in a frame in which the object is at rest.

Worldlines connect events. Like events, they exist independent of any reference frame. In principle, worldlines allow us to relate events to one another — to do science — without using reference frames at all (Section 5.9).

REFERENCES

Newton quotation toward the end of Section 5.8: Sir Isaac Newton, *Mathematical Principles of Natural Philosophy and His System of the World (Philosophiae Naturalis Principia Mathematica),* Joseph Streater, London, July 5, 1686; translated from Latin — the scholarly language of Newton's time — by Andrew Motte in 1729, revised and edited by Florian Cajori and published in two paperback volumes (University of California Press, Berkeley, 1962).

Section 5.9 uses slightly modified passages from Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler, *Gravitation* (W.H. Freeman, New York, 1973), pages 5–8. Figure 5-13 is taken directly from this reference, its caption slightly altered from the original.

CHAPTER 5 EXERCISES

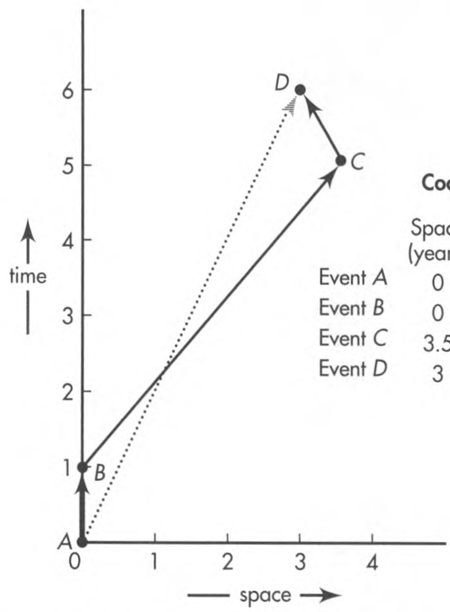
PRACTICE

5-1 more is less

The spacetime diagram shows two alternative worldlines from event A to event D. The table shows coordinates of numbered events in this frame. Time and space are measured in years.

a One traveler moves along the solid segmented worldline from event A to events B, C, and D. Calculate the time increase on his wristwatch (proper clock)

(1) between event A and event B.
(2) between event B and event C.
(3) between event C and event D.
(4) Also calculate the total proper time along worldline A, B, C, D.



	Space (years)	Time (years)
Event *A*	0	0
Event *B*	0	1
Event *C*	3.5	5
Event *D*	3	6

EXERCISE 5-1. *Two alternative worldlines between initial event* A *and final event* D.

b His twin sister moves along the straight dotted worldline from event *A* directly to event *D*. Calculate the time increase on her wristwatch between events *A* and *D*.

c Which twin (solid-line or dotted-line traveler) is younger when they rejoin at event *D*?

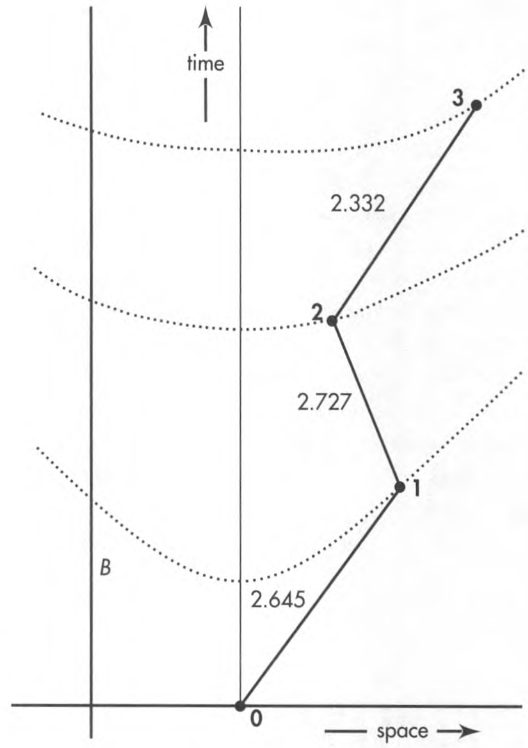
5-2 transforming worldlines

The laboratory spacetime diagram in the figure shows two worldlines. One, the vertical line labeled *B*, is the worldline of an object that is at rest in this frame. The other, the segmented line that connects events 0, 1, 2, and 3, is the worldline of an object that moves at different speeds at different times in this frame. The proper time is written on each segment and invariant hyperbolas are drawn through events 1, 2, and 3. The event table shows the space and time locations in this frame of the four events 0, 1, 2, and 3.

a Trace the axes and hyperbolas onto a blank piece of paper. Sketch a qualitatively correct spacetime diagram for the same pair of worldlines observed in a frame in which the particle on the segmented worldline has zero velocity between event 1 and event 2.

b What is the velocity, in this new frame, of the particle moving along worldline *B*?

c On each straight portion of the segmented worldline for this new frame write the numerical value of the interval between the two connected events.



Event Coordinates in the Laboratory Frame

Event 0	$x = 0$	$t = 0$
Event 1	$x = 3.000$	$t = 4.000$
Event 2	$x = 1.750$	$t = 7.000$
Event 3	$x = 5.000$	$t = 11.000$

EXERCISE 5-2. *Two worldlines as recorded in the laboratory frame. Numbers on the segmented worldline are proper times along each straight segment.*

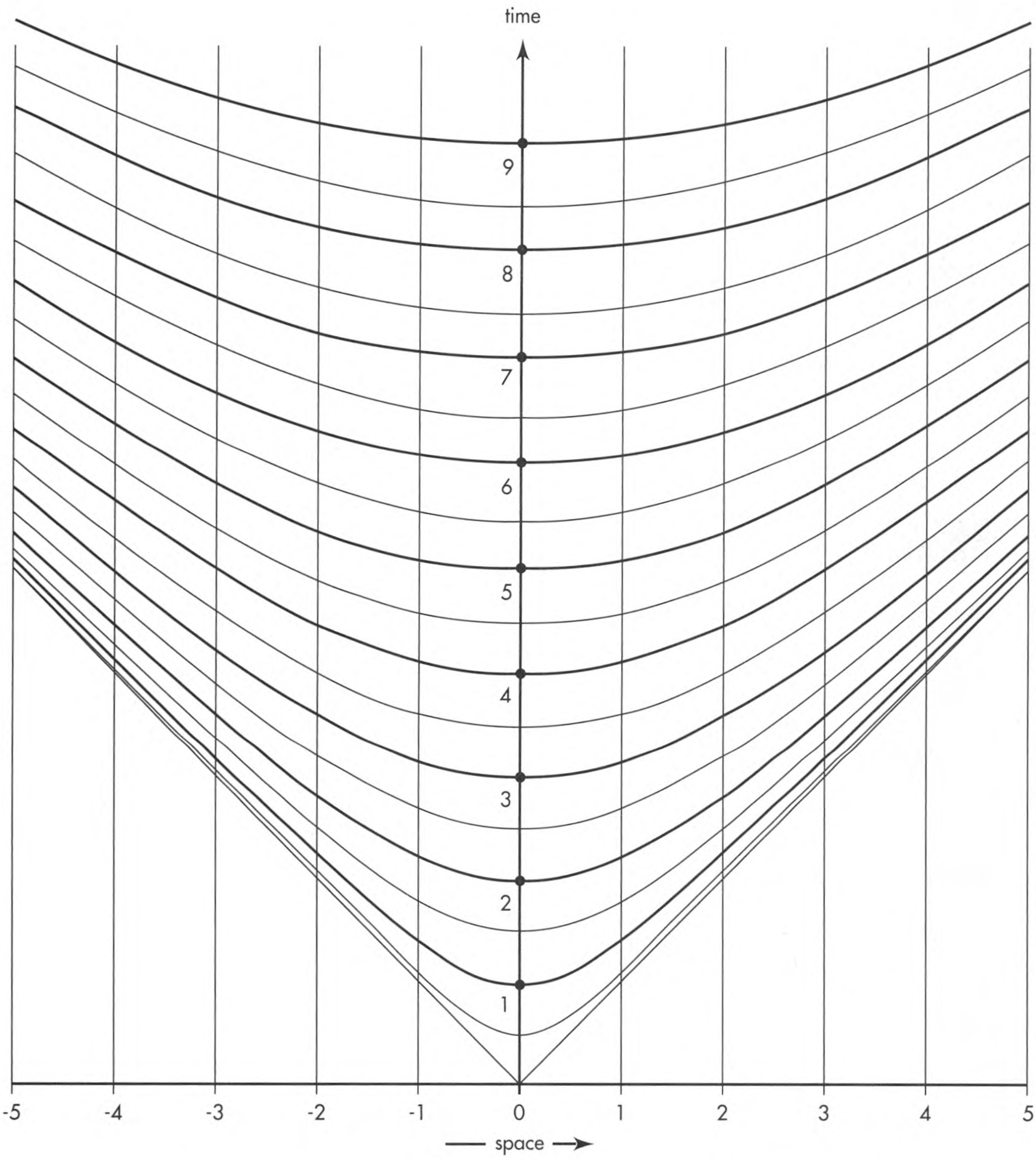
5-3 mapmaking in spacetime

Note: Recall Exercise 1-6, the corresponding mapmaking exercise in Chapter 1.

Here is a table of timelike intervals between events, in meters. The events occur in the time sequence *ABCD* in all frames and along a single line in space in all frames. (They do *not* occur along a single line on the spacetime map.)

INTERVAL to event / from event	*A*	*B*	*C*	*D*
A	0	1.0	3.161	5.196
B		0	2.0	4.0
C			0	2.0
D				0

a Use a ruler and the hyperbola graph to construct a spacetime map of these events. Draw this map



EXERCISE 5-3. *Template of hyperbolas for converting intervals into a spacetime map.*

on thin paper so you can lay it over the hyperbola graph and see the hyperbolas.

Discussion: How to start? With three arbitrary decisions! (1) Choose event A to be at the origin of the spacetime map. (2) Choose event B to occur at the same place as event A. That is, event point B is located on the positive time axis with respect to event point A. After plotting B, use your ruler to draw this straight time axis through event points A and B. Keep this line parallel to the vertical lines on the hyperbola graph in all later constructions. (3) Even with these choices, there are two spacetime locations (x, t) at which you can locate the event point C; choose either of these two spacetime locations arbitrarily. Then go on to plot event D.

Analogy to surveying: In surveying (using Euclidean geometry) you locate all points a given distance from some stake by using that stake as origin and drawing a circle of radius equal to the desired distance. In a spacetime map (using Lorentz geometry) you locate all event points a given interval from some event by using that event point as origin and drawing a hyperbola with nearest point equal to the desired interval.

b Now take a new piece of paper and draw a spacetime map for another reference frame. Choose event D to be at the origin of the spacetime map. This means that all other events occur before D. Hence turn the hyperbola plot upside down, so that the hyperbolas open downward. Choose event B to occur at the same place as D. Now find the locations of A and C using the same strategy as in part **a**.

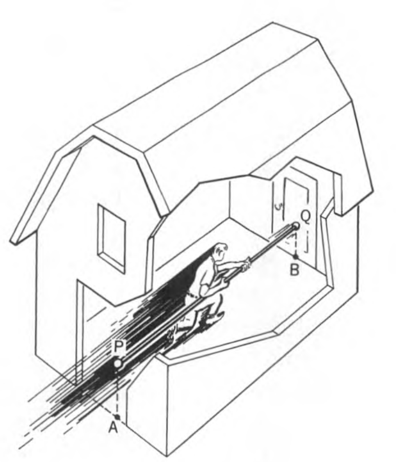
c Find an approximate value for the relative speed of the two frames for which you have made spacetime plots.

d Hold one of your spacetime maps up to the light with the marks on the side of the paper facing the light. Does the map you see from the back also satisfy the table entries?

PROBLEMS

5-4 the pole and barn paradox

A worried student writes, "Relativity must be *wrong*. Consider a 20-meter pole carried so fast in the direction of its length that it appears to be only 10 meters long in the laboratory frame of reference. Let the runner who carries the pole enter a barn 10 meters long, as shown in the figure. At some instant the farmer can close the front door and the pole will be entirely enclosed in the barn. However, look at the same situation from the frame of reference of the runner. To him the barn appears to be contracted to half its length. How can a 20-meter pole possibly fit into a 5-meter barn? Does not this unbelievable conclusion prove that relativity contains somewhere a fundamental logical inconsistency?"



EXERCISE 5-4. *Fast runner with "20-meter" pole enclosed in a "10-meter" barn. In the next instant he will burst through the back door, which is made of paper.*

Write a reply to the worried student explaining clearly and carefully how the pole and barn are treated by relativity without internal contradiction. Use the following outline or some other method.

a Make two carefully labeled spacetime diagrams, one an xt diagram for the barn rest frame, the other an $x't'$ diagram for the runner rest frame. Referring to the figure, take the event "Q coincides with A" to be at the origin of both diagrams. In both plot the worldlines of A, B, P, and Q. Pay attention to the scale of both diagrams. Label both diagrams with the time (in meters) of the event "Q coincides with B" (derived from Lorentz transformation equations or otherwise). Do the same for the times of events "P coincides with A" and "P coincides with B."

b **Discussion question:** Suppose the barn has no back door but rather a back wall of steel-reinforced concrete. What happens after the farmer closes the front door on the pole?

c Replace the pole with a line of ten tennis balls the same length as the pole and moving together with the same velocity as the pole. The farmer's ten children line up inside the barn, and each catches and stops one tennis ball at the same time as the farmer closes the front door of the barn. Describe the stopping events as recorded by the observer riding on the last tennis ball. Plot them on your two diagrams.

5-5 radar speed trap

A highway patrolman aims a stationary radar transmitter backward along the highway toward oncoming traffic. A detector mounted next to the transmitter analyzes the radar wave reflected from an approaching car. An internal computer uses the shift in frequency of the reflected wave to reckon and display the car's speed. Analyze this shift in frequency as in parts **a**–**e** or with some other method. Treat the car as a simple mirror and assume that the radar signals move back and forth along one line on the highway. Radar is an electromagnetic wave that moves with the speed of light.

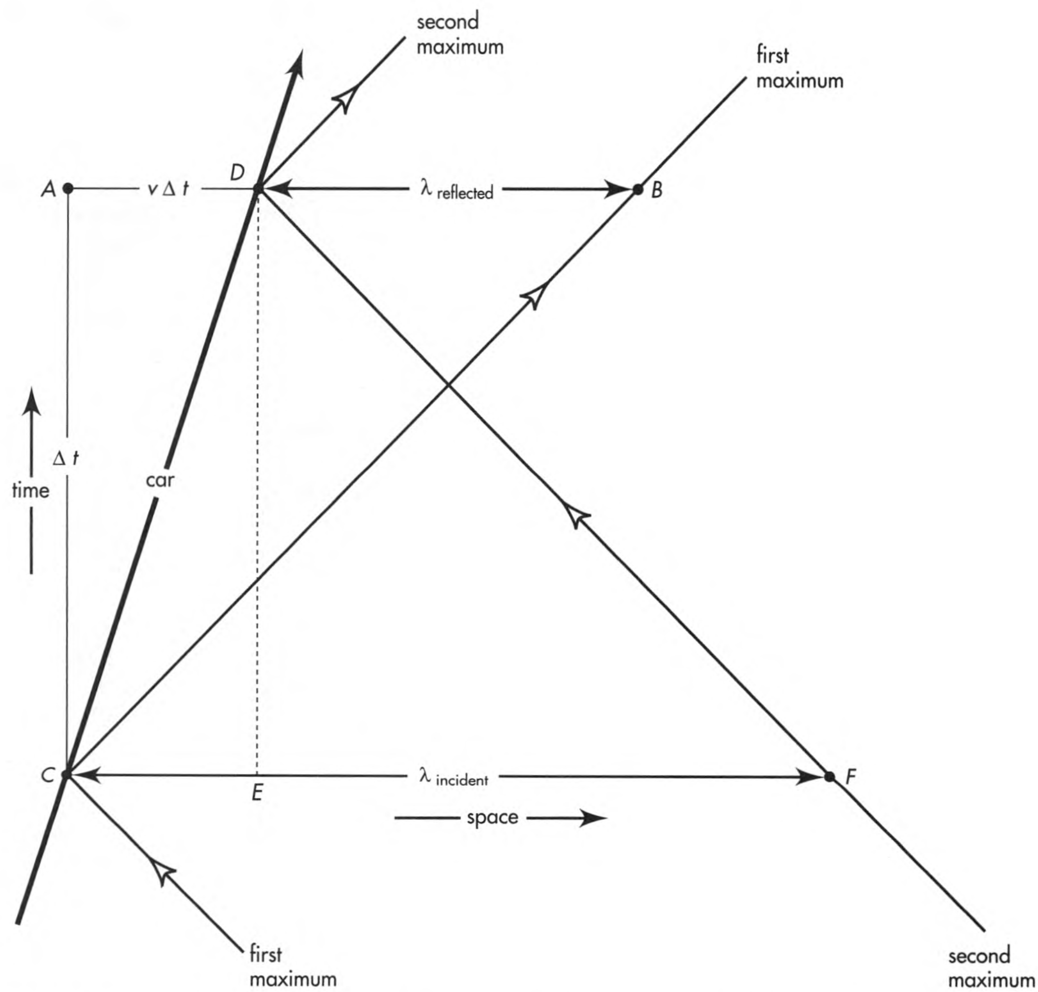
The figure shows the worldline of the car, worldlines of two adjacent maxima of the radar wave, and the wavelength λ of incident and reflected waves.

a From the 45-degree right triangle ABC, show that

$$\Delta t = v\Delta t + \lambda_{\text{reflected}}$$

From the 45-degree right triangle DEF, show that

$$\Delta t = \lambda_{\text{incident}} - v\Delta t$$



EXERCISE 5-5. *Worldlines of approaching car and two radar wave maxima that reflect from the car. The speed of the car is greatly exaggerated.*

Eliminate Δt from these two equations to find an expression for $\lambda_{\text{reflected}}$ in terms of $\lambda_{\text{incident}}$ and the automobile speed v.

b The frequency f of radar (in cycles/second) is related to its wavelength λ in a vacuum by the formula $f = c/\lambda$, where c is the speed of light ($\approx$ the speed of radar waves in air). Derive an expression or frequency $f_{\text{reflected}}$ of the reflected radar signal in terms of frequency f_{incident} of the incident wave and the speed v of the oncoming automobile. Show that the result is

$$f_{\text{reflected}} = \left(\frac{1+v}{1-v}\right) f_{\text{incident}}$$

c For an automobile moving at a speed $v = v_{\text{conv}}/c$ that is a small fraction of the speed of light, assume that the fractional change in frequency of reflected radar is small. Under this assumption, use the first two terms of the binomial expansion

$$(1 - z)^n \approx 1 - nz \text{ for } |z| << 1$$

to show that the fractional change of frequency is given by the approximate expression

$$\frac{\Delta f}{f} \approx 2v$$

Substitute the speed of a car moving at 100 kilometers/hour ($= 27.8$ meters/second ≈ 60 miles/hour) and show that your assumption about the small fractional change is justified.

d One radar gun used by the Massachusetts Highway Patrol operates at a frequency of 10.525×10^9 cycles/second. By how many cycles/

second is the reflected beam shifted in frequency when reflected from a car approaching at 100 kilometers/hour?

e What discrimination between different frequency shifts must the unit have if it can distinguish the speed of a car moving at 100 kilometers/hour from the speed of one moving at 101 kilometers/hour?

Reference: T. M. Kalotas and A. R. Lee, *American Journal of Physics*, Volume 58, pages 187–188 (February 1990).

5-6 a summer evening's fantasy

You are standing alone outdoors at dusk on the first day of summer. You see Sun setting due west and the planet Venus in the same direction. On the opposite horizon the full Moon is rising due east. An alien ship approaches from the east and lands beside you. The occupants inform you that they are from Proxima Centauri, which lies due east beyond the rising Moon. They say they have been traveling straight to Earth and that their reduced approach speed within the solar system was such that the time stretch factor gamma during the approach was 5/3.

At the same instant that the aliens land, you see Sun explode. The aliens admit to you that earlier, on their way to Earth, they shot a laser light pulse at Sun, which caused this explosion. They warn that Sun's explosion emitted an immense pulse of particles moving at half the speed of light that will blow away Earth's atmosphere. In confirmation, shortly after the aliens land you notice that the planet Venus, lying in the direction of Sun, suddenly changes color.

You grab a passing human of the opposite sex and plead with the aliens to take you both away from Earth in order to establish the human gene pool elsewhere. They agree and set the dials to flee in an easterly direction away from Sun at top speed, with time stretch factor gamma of 25/7. The takeoff is to be 7 minutes after the alien landing on Earth.

Do you make it?

Draw a detailed Earth spacetime diagram showing the events and worldlines of this story. Use the following information.

- Sun is 8 light-minutes from Earth.
- Venus is 2 light-minutes from Earth.
- Assume that Sun, Venus, Earth, and Moon all lie along a single direction in space and are relatively at rest during this short story. The incoming and outgoing paths of the alien ship lie along this same line in space.
- All takeoffs and landings involve instantaneous changes from initial to final speed.
- $5^2 - 3^2 = 4^2$ and $(25)^2 - (7)^2 = (24)^2$

a Plot EVENTS labeled with the following NUMBERS.

0. your location when the aliens land (at the origin)
1. Sun explodes
2. light from Sun explosion reaches you
3. Venus's atmosphere blown away
4. light from event 3 reaches you
5. you and aliens depart Earth (you hope!)
6. Earth atmosphere blown away

b Plot WORLDLINES labeled with the following CAPITAL LETTERS.

A. your worldline
B. worldline of Earth
C. aliens' worldline
D. worldline of Sun
E. worldline of Venus
F. worldline of light from Sun's explosion
G. worldline of the "speed-one-half" pulse of particles from Sun's explosion
H. worldline of light emitted when Venus loses atmosphere
J. terminal part of the worldline of the laser cannon pulse fired at Sun by the aliens

c Write numerical values for the speed $v = v_{\text{conv}}/c$ on every segment of all worldlines.

5-7 the runner on the train paradox

A letter sent to the Massachusetts Institute of Technology by Hsien-Yen Tsao of Los Angeles poses the following paradox, which he asserts disproves the theory of relativity. The Chairman of the Physics Department sends the inquiry along to you, asking you to respond to Mr. Tsao. You determine to make the answer clear, concise, decisive, and polite—a personal test of your diplomacy and grasp of relativity.

The setting: A train travels at high speed. A runner on the train sprints toward the back of the train with the same speed (with respect to the train) as the train moves forward (with respect to Earth). Therefore the runner is not moving with respect to Earth.

The paradox: We know that, crudely speaking, clocks on the train run "slow" compared to the Earth clock. We also know that the runner's clock runs "slow" compared to the train clocks. Therefore the runner's clock should run "doubly slow" with respect to the Earth clock. *But the runner is not moving with respect to Earth!* Therefore the runner's clock must run at the same rate as the Earth clock. How can it possibly be that the runner's clock runs *"doubly slow"* with respect to the Earth clock and also runs *at the same rate* as the Earth clock?

5-8 the twin paradox put to rest—a worked example

Motto: The swinging line of simultaneity tells all!

Combine the Lorentz transformation with the spacetime diagram to clear up—once and for all!—the solution to the Twin Paradox. An astronaut travels from Earth to Canopus (Chapter 4) at speed $v_{rel} = 99/101$, arriving at Canopus $t' = 20$ years later according to her rocket clock, $t = 101$ years later according to Earth-linked clocks—which means that the stretch factor γ has the value $101/20$.

The key idea is "lines of simultaneity" (boxed labels in the figure). A line of simultaneity connects events that occur "at the same time." But events simultaneous in the Earth ("laboratory") frame are typically not simultaneous in the rocket frame (Section 3.4). *Horizontal* is the line of simultaneity on the Earth ("laboratory") spacetime map that connects events occurring at the same time in the Earth frame. *Totally different*—not a horizontal line!—is a line of simultaneity on the Earth spacetime map that connects events simultaneous in the outgoing astronaut frame. To draw this line of outgoing-astronaut simultaneity, start with the inverse Lorentz transformation equation for time:

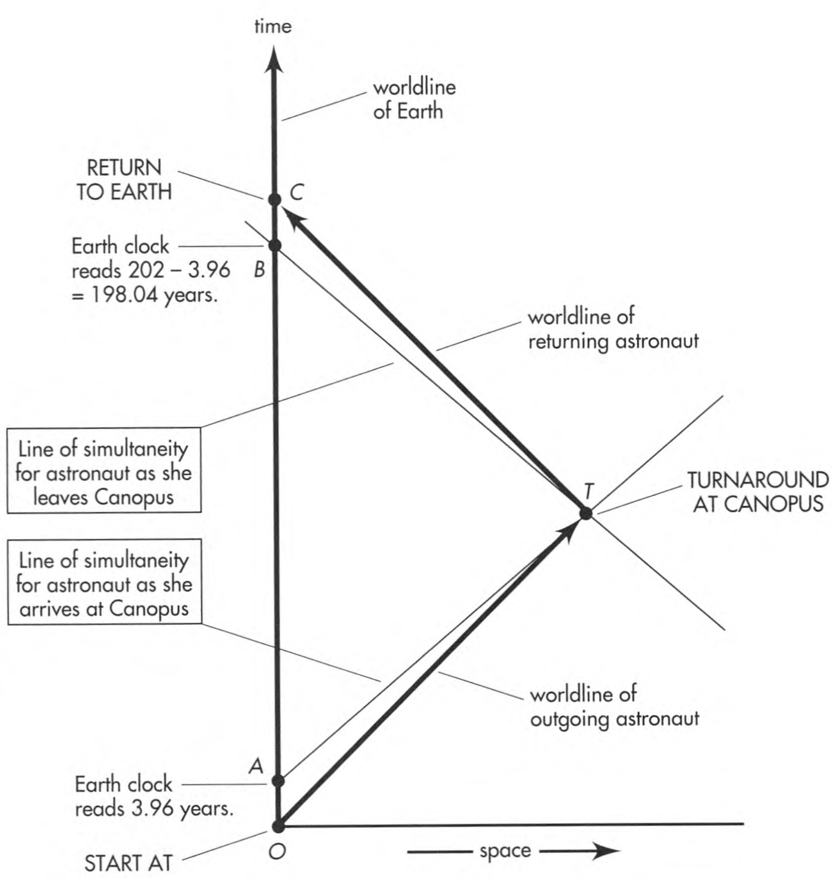
$$t' = -v_{rel}\gamma x + \gamma t$$

For the outgoing astronaut, $v_{rel} = 99/101$ and $\gamma = 101/20$. We want the line of simultaneity that passes through turnaround event T. So let $t' = 20$ years. Then:

$$20 = -(99/101)(101/20)x + (101/20)\,t$$

Multiply through by $20/101$:

$$400/101 = -(99/101)x + t$$



EXERCISE 5-8. *Earth spacetime map of the trip to Canopus and back. As the astronaut arrives at Canopus, her colleagues in her outgoing reference frame record along line* AT *events simultaneous with this arrival, including Earth-clock reading of 3.96 years at* A. *At Canopus the astronaut changes frames, thus changing the line of simultaneity, which swings to* BT. *As she leaves Canopus, her new colleagues take an Earth-clock reading of 198.04 years at* B. *At turnaround, the ticks on the Earth clock along worldline segment* AB *go from the outward-moving astronaut's future to the incoming astronaut's past.*

which yields

$$t = 0.980\,x + 3.96$$

This is the equation for a straight line passing through event points A and T in the spacetime diagram. It is the line of simultaneity for the outgoing astronaut, connecting all events simultaneous with the arrival of the rocket at Canopus (simultaneous in that frame). Among these events is event A, the Earth clock reading of 3.96 years, which occurs at Earth position $x = 0$. In brief, at the moment the rocket arrives at Canopus, the Earth clock reads 3.96 years as observed in the outgoing rocket frame.

Now the astronaut jumps to the incoming rocket frame. This reverses the velocity of the astronaut with respect to the Earth-linked frame—*and so reverses the slope of the line of astronaut simultaneity.* This new line of astronaut simultaneity passes through event points B and T in the figure. Event B is the Earth clock reading of $202 - 3.96 = 198.04$ years.

To go back over the astronaut trip while looking at the spacetime map is (finally!) to solve the Twin Paradox. As the astronaut travels outward toward Canopus, many colleagues follow her at the same speed, with clocks synchronized in her frame. As they whiz past Earth, each records the reading on the Earth clock. Later analysis leads them to agree that the time between ticks of Earth's clock is longer than the time between ticks of their own outward-moving clocks. (They say, "The Earth clock runs slow.") At any event point on her outward worldline, the astronaut's line of simultaneity slopes upward to the right in the Earth spacetime diagram, as shown in the figure. Simultaneous with astronaut arrival at Canopus (event T, when *all* outward-moving clocks read 20 years), one of her colleagues reads a time 3.96 years on the Earth clock (event A).

Now the astronaut jumps from the outward-moving rocket to a returning rocket. She inherits a *completely new set of colleagues,* with a new set of synchronized clocks. The astronaut's new line of simultaneity slopes upward to the left in the Earth spacetime diagram. Simultaneous with her departure from Canopus (event T, when *all* inward-moving clocks read 20 years), one of her new colleagues reads a time $202 - 3.96 = 198.04$ years on the passing Earth clock (event B). Thereafter new colleague after new colleague streaks past Earth, recording the fact that Earth clock ticks are farther apart in time than the ticks on their own clocks. (They say, "The Earth clock runs slow.").

The analysis so far accounts for the short time segments OA and BC recorded by the Earth clock on its vertical worldline AC. What about the omitted time lapse AB? This is recorded, sure enough, by the Earth clock plowing forward along worldline OC in its comfortable single free-float frame. However, the story of time AB is quite different for the turn-around astronaut. Before she reaches turnaround at T, events on line AB are *in her future.* All those Earth clock ticks are yet to be recorded by her outgoing colleagues. These events lie *above* her line of simultaneity AT as she arrives at Canopus at T. However, as she turns around, her line of simultaneity also slews forward, swinging from line AT to line BT. Suddenly the events on line AB—all those intermediate ticks of the Earth clock—are in the astronaut's *past.* These events lie *below* the line of simultaneity BT as she starts back at T. Her outward-moving colleague reads 3.96 years on the Earth clock as she reaches Canopus; an instant later on her clock, her new inward-moving colleague reads 198.04 on the Earth clock.

Shall we say that the Earth clock "jumps ahead" as the astronaut turns around? No! Utterly ridiculous! For what single observer does it jump ahead? Not for the Earth observer. Not for the outgoing set of clock-readers. Not for the returning set of clock readers. For whom then? Nobody! *At the same time as she reaches Canopus*—old meaning of simultaneous!—the astronaut's outgoing colleague records 3.96 years for the Earth clock. *At the same time as she leaves Canopus*—new meaning of simultaneous!—her new ingoing colleague records 198.04 years on the Earth clock. The astronaut has nobody but herself to blame for her misperception of a "jump" in the Earth clock reading.

The "lost Earth time" AB in the figure makes consistent the story each observer tells about the clocks. Simple is the story told by the Earth observer: "My clock ticked along steadily at the 'proper' rate from astronaut departure to astronaut return. In contrast, ticks on the astronaut clock were far apart in time on both the outgoing and incoming legs of her trip. We agree that her total ticks are less than my total ticks: she is younger than I when we meet again." More complicated is the astronaut account of clock behavior: "Ticks on the Earth clock were far apart in time as I traveled to Canopus; ticks on the Earth clock were also far apart as I traveled home again. But as I turned around, a whole bunch of Earth clock ticks went from my future to my past. This accounts for the larger number of total ticks on the Earth clock than on my clock during the trip. We agree that I am younger when we meet again."

So saying, the astronaut renounces her profession and becomes a stand-up comedian.

Reference: E. Lowry, *American Journal of Physics,* Volume 31, page 59 (1963).